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k-contact Lie systems: Theory and applications

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This paper introduces a new class of Lie systems that are Hamiltonian relative to a k-contact manifold. We show that a recent distributional approach to k-contact manifolds along with a related k-contact Hamiltonian vector field notion allows us to understand relevant Lie systems as Hamiltonian relative to a k-contact manifold. Our procedure is more general than previously known methods with this aim. As a result, we find that a plethora of Lie systems related to control and physical problems can be considered in a natural manner as k-contact Lie systems. We study their t-dependent and t-independent constants of motion, master symmetries of higher order, and other properties of interest. Finally, we use our new techniques and findings to study PDE Lie systems with a compatible k-contact manifold, some of which become Hamilton–De Donder–Weyl equations.

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1. Introduction

Toward the end of the 19th century, there was much interest in determining t-dependent systems of first-order ordinary differential equations whose solutions could be described as a t-independent function of a generic family of particular solutions and some constants: the *Lie systems* [9, 13, 24, 47]. In particular, Lie proved that Lie systems are related to a certain curve in a finite-dimensional Lie algebra of vector fields: a *Vessiot–Guldberg Lie algebra.*

Lie systems have been studied along with different compatible geometric structures as this allowed one to study more easily their properties (see [24] for a survey on the topic). In particular, *Lie–Hamilton systems* are Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a Poisson bivector [10, 13]. Lie–Hamilton systems were the first class of Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a geometric structure that was studied. In this case, the Poisson bivector enabled the derivation of their superposition rules and constants of motion in a simpler manner than previous methods via, for instance, the Poisson coalgebra method [2, 10, 13, 24]. Nevertheless, Lie–Hamilton systems were insufficient for studying many types of Lie systems [24]. Several types of Hamiltonian Lie systems relative to different geometric structures have been posteriorly analyzed. They not only aided in comprehending the characteristics of various Lie systems but also facilitated the development of innovative results and techniques in differential geometry. For instance, k-symplectic Lie systems introduced Poisson brackets of certain families of functions in k-symplectic geometry to obtain superposition rules [25], or, quite relevantly, Gràcia, de Lucas, Román-Roy, Muñoz-Lecanda, Rivas and Vilariño introduced multisymplectic Lie systems, which were used to study different types of multisymplectic reduction and tensor invariants for Lie systems [20, 33].

Contact Lie systems [7, 8, 22] were recently introduced to study Lie systems of physical relevance which, in some cases, cannot be studied via Lie–Hamilton systems. These works were also motivated by the interest of contact geometry in the study of dynamical systems, specially in those with a dissipative behavior [16, 18]. Contact Lie systems appeared for the first time in [22]. Recently, contact Lie systems have also appeared in the study of the reduction of Lie–Hamilton systems related to thermodynamic systems [7], while the characterization of contact Lie systems on low-dimensional manifolds was also related to the geometric analysis of hydrodynamic equations via quasi-rectifiable families of vector fields in [36].

Recently k-contact geometry appeared as a generalization of contact geometry to study field theories [58]. In particular, k-contact geometry allows for the description of the Hamilton–De Donder–Weyl equations for field theories. The initial idea behind k-contact geometry was to analyze the properties of a generalization of contact forms to a certain type of $\mathbb{R}^k$-valued differential one-forms: the *k-contact forms*. Moreover, many other mathematical and physical related features of k-contact forms appeared over the years [23, 32, 34, 59, 62]. This work introduces *k-contact Lie systems*, namely Lie systems admitting a Vessiot–Guldberg Lie algebra of $\boldsymbol{\eta}$-Hamiltonian vector fields relative to a k-contact form, which retrieves contact Lie systems as a particular case for $k = 1$ [7, 22]. Then, we use k-contact geometry to investigate k-contact Lie systems.

k-contact geometry was, at first, mainly aimed at studying systems of partial differential equations. Then, the authors of [23] used k-contact forms to define $\boldsymbol{\eta}$-Hamiltonian vector fields relative to a k-contact form $\boldsymbol{\eta}$, which were used to analyze ordinary differential equations (see Sec. 4 for a review on the results in [23] on this matter). This allows for our definition of k-contact Lie systems. Relevantly, the authors of [23] introduced k-contact distributions, namely distributions given locally as the kernel of a k-contact form, which allowed for a deeper insight into k-contact geometry. In particular, a k-contact distribution is a generalization of contact distributions, which are recovered for $k = 1$. This work shows that this k-contact distributional approach is very practical for determining k-contact Lie systems with practical applications. The idea is that obtaining a k-contact form $\boldsymbol{\eta}$ when turning a Lie system into an $\boldsymbol{\eta}$-Hamiltonian system is difficult, but it is simpler by using k-contact distributions. In fact, we show that many control systems, e.g. appearing in [55], can be studied via k-contact Lie systems by using a quite powerful method developed in Sec. 8. In particular, our ideas can also be used to understand Lie systems as contact Lie systems in a simple manner. We also consider the relation of k-contact Lie systems with other Lie systems admitting a compatible Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a k-symplectic or presymplectic form.

It is remarkable that the use of k-contact geometry techniques for the study of k-contact Lie systems shows new uses of features appearing in k-contact geometry. In particular, $\boldsymbol{\eta}$-Hamiltonian k-functions play a role in the study of quantities that are constants of the motion, generators of constants of motion of order m, or master symmetries for k-contact Lie systems. It is worth noting that it is the first time that generators of constants of motion and associated master symmetries are analyzed for Lie systems (see [11, 27] for works on such notions on symplectic and Poisson geometry). As a byproduct, we obtain an extension of contact dissipated quantities to the k-contact realm [48].

As an application, we show several Lie systems that admit compatible k-contact manifolds turning them into k-contact Lie systems. These examples include extensions of classical equations like the Riccati equation, and novel systems such as the complex Schwarz equation, higher-order generalizations of Brockett systems, a front-wheel driven kinematic car, and so on. Theoretical results on k-contact Lie systems are used to analyze their generators of order m of constants of motion, constants of motion, and our results that are applied to the examined examples. It is worth noting that the study of master symmetries and generalized constants of motion of order m are introduced for the first time in the realm of Lie systems.

We prove that the diagonal prolongation of a k-contact Lie system on N to N^s is a ks-contact Lie system. This solves a problem appearing in the Poisson coalgebra method for contact manifolds, where it was observed that the diagonal prolongations of contact Lie systems are not, in general, contact Lie systems as they may be defined on even-dimensional manifolds. The fact that the diagonal prolongation of a contact Lie system may not be a contact Lie system posed a problem to develop a contact Poisson coalgebra method to derive superposition rules and constants of motion for contact Lie systems, which was partially solved using Jacobi geometry. In the case of k-contact Lie systems, the use of Jacobi manifolds is neither needed nor available as k-contact manifolds are not in general related to Jacobi manifolds.

In the last part of the work, we briefly recall the theory of PDE Lie systems [9] and show that a Lie algebra of $\boldsymbol{\eta}$-Hamiltonian vector fields relative to a co-oriented k-contact manifold $(M, \boldsymbol{\eta})$ allows us to construct PDE Lie systems with a compatible k-contact manifold, the so-called *k-contact PDE Lie systems*. Some of these PDE Lie systems can be understood as Hamilton–De Donder–Weyl equations in the k-contact realm, which is illustrated by examples considered in Sec. 11 and Theorem 11.9. It is worth recalling that there are not many applications of PDE Lie systems in the literature [35, 36, 55] and our new applications are specially interesting due to this fact. These new applications are concerned with Floquet theory, $\mathfrak{g}$-structures, and Lax pairs, while PDE Lie systems on Lie groups and Wess–Zumin–Witten–Novikov system were previously studied [12, 35]. We briefly describe the potential extensions of our techniques for k-contact Lie systems to k-contact PDE Lie systems.

The structure of this work is as follows. Section 2 presents the basics on Lie systems and related notions. k-vector fields and their integral submanifolds are presented in Sec. 3, while k-contact geometry is briefly presented in Sec. 4. Then, a theory of k-contact Lie systems is derived in Sec. 5, while a theory on t-dependent constants of motion for k-contact Lie systems and other related notions is developed in Sec. 6. The relation of k-contact Lie systems and other geometric structures is given in Sec. 7. Methods to construct k-contact Lie systems from Lie systems are introduced in Sec. 8. In Sec. 9, we show that the diagonal prolongation to N^s of a

k-contact Lie system on N is a ks-contact Lie system. Afterward, applications of k-contact Lie systems appear in Sec. 10. The theory of PDE Lie systems, some new and known examples, as well as PDE Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a k-contact form are introduced in Sec. 11. Examples of these k-contact PDE Lie systems with potential applications are presented. Finally, the conclusions and outlook of our work are presented in Sec. 12.

2. Basics on Lie Systems

Let us establish some basic definitions about Lie systems and other related concepts to be used in this work [9]. Natural numbers start at one. Hereafter, manifolds are assumed to be smooth, Hausdorff, connected, and finite-dimensional, unless otherwise stated. In particular, M is an m-dimensional manifold a U is always an open subset of a manifold. Our further considerations are mainly local at generic points, and the problem of establishing whether structures are globally defined will be generally skipped. We will also focus on smooth functions. Moreover, $\{e_1, \ldots, e_k\}$ is a basis for $\mathbb{R}^k$. From now on, $\pi : TM \to M$ stands for the canonical tangent bundle projection and we set $\pi_2 : \mathbb{R} \times M \ni (t, x) \mapsto x \in M$. Given two subsets A_1, A_2 of a Lie algebra $\mathfrak{g}$, we represent by $[A_1, A_2]$ the vector space spanned by the Lie brackets between the elements of A_1 and A_2, respectively. We say that vector fields $X_1, \ldots, X_r$ are *linearly independent at a generic point* if $\sum_{\alpha=1}^{r} f_\alpha X_\alpha = 0$ on some U only for functions $f_1 = \cdots = f_r = 0$ on U.

It is essential in the theory of Lie systems to describe t-dependent systems of first-order ordinary differential equations in normal form as t-dependent vector fields.

A *t-dependent vector field* is a map $X : \mathbb{R} \times M \ni (t, x) \mapsto X(t, x) \in TM$ such that $\pi \circ X = \pi_2$. The latter amounts to the fact that each t-dependent vector field X on a manifold M is equivalent to a family $\{X_t\}_{t\in\mathbb{R}}$ of vector fields on M.

We call *smallest Lie algebra* of a t-dependent vector field X the smallest Lie algebra (in the sense of inclusion) containing all the vector fields $\{X_t\}_{t\in\mathbb{R}}$. We denote the smallest Lie algebra of X by V^X. Every t-dependent vector field X on M gives rise to a generalized distribution $\mathcal{D}^X_x = \{Y_x \mid Y \in V^X\}$, for every $x \in M$. For simplicity, we call generalized distributions. Given a vector field Z on M, we write $[Z, \mathcal{D}]$ for the distribution spanned by the Lie brackets of Z with vector fields taking values in $\mathcal{D}$. Although this notation conveys a little abuse of notation, its meaning is clear and simplifies the presentation.

Given a t-dependent vector field X on M, its *associated system* is the system of first-order ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x), \tag{2.1}$$

whose solution $x : \mathbb{R} \to M$ with $x(0) = x_0$ is the *integral curve* of X with initial condition x_0 at $t = 0$. Every t-dependent vector field has an associated t-dependent system of differential equations in normal form (2.1) and vice versa. It is convenient then to use X to refer both to a t-dependent vector field and the t-dependent system of ordinary differential equations determining its integral curves.

In order to introduce Lie systems, consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b_1(t) + b_2(t)x + b_3(t)x^2, \tag{2.2}$$

where $x \in \mathbb{R}, t \in \mathbb{R}$ and $b_1(t), b_2(t), b_3(t)$ are arbitrary t-dependent functions. Differential equations of this type are called *Riccati equations.*[a] The general solution of (2.2) can be written as

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) - kx_{(2)}(t)(x_{(3)}(t) - x_{(1)}(t))}{x_{(3)}(t) - x_{(2)}(t) - k(x_{(3)}(t) - x_{(1)}(t))}, \tag{2.3}$$

where $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ are three different particular solutions of (2.2) and $k \in \mathbb{R}$.

More generally, a t-dependent system of first-order ordinary differential equations on M of the form (2.1) for a t-dependent vector field X on M is called a *Lie system*, if it admits a *superposition rule*, namely a t-independent map $\Phi : M^\ell \times M \to M$ of the form $x = \Phi(x_{(1)}, \ldots, x_{(\ell)}; k)$, such that every generic solution of (2.1) can be written as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(\ell)}(t); k), \tag{2.4}$$

where $x_{(1)}(t), \ldots, x_{(\ell)}(t)$ is any generic family of particular solutions of system (2.1) and $k \in M$.

There are some subtle problems related to the above definition and the meaning of being a "generic solution" or a "generic family". They are mostly technical (see [5, 9]), therefore we will omit discussing them in detail. We will simply say that the domain of Φ is, in reality, some open subset of $M^\ell \times M$, and, for every generic family of particular solutions, one can recover locally almost every solution [9, 24].

The necessary and sufficient conditions to characterize Lie systems were provided by Sophus Lie in the *Lie Theorem* [45–47].

Theorem 2.1 (Lie Theorem). *A t-dependent system X on a manifold M admits a superposition rule* (2.4) *if and only if X can be written as*

$$X(t, x) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x), \quad t \in \mathbb{R}, \quad x \in M, \tag{2.5}$$

where $X_1, \ldots, X_r$ is a family of vector fields on M spanning an r-dimensional real Lie algebra of vector fields V, the so-called Vessiot–Guldberg Lie algebra of X on M, and $b_1(t), \ldots, b_r(t)$ are arbitrary t-dependent functions.

[a]It is sometimes assumed that $b_1(t)b_3(t) \neq 0$, but this condition is frequently ignored in Lie systems theory [9, 24].

Lie proved that the existence of a superposition rule depending on ℓ particular solutions implies the existence of a Vessiot–Guldberg Lie algebra V with a basis $X_1, \ldots, X_r$ such that $r \leq \dim M \cdot \ell$, and, conversely, given a decomposition (2.5), there exists a superposition rule depending on ℓ particular solutions with $r \leq \dim M \cdot \ell$. The inequality $r \leq \ell \cdot \dim M$ is referred to as *Lie's condition* [9].

A Riccati equation describes the integral curves of a t-dependent vector field on $\mathbb{R}$ of the form

$$X(t,x) = (b_1(t) + b_2(t)x + b_3(t)x^2)\frac{\partial}{\partial x}. \tag{2.6}$$

One can present X as a linear combination of three vector fields on $\mathbb{R}$ given by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = x^2\frac{\partial}{\partial x}, \tag{2.7}$$

which satisfy the commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \tag{2.8}$$

By the Lie Theorem, it follows that Riccati equations admit a superposition rule. Conversely, one can also say that the existence of a superposition rule (2.3) implies that (2.6) is such that $\{X_t\}_{t\in\mathbb{R}}$ are included in some Lie algebra of dimension three or less. In view of (2.8), the vector fields (2.7) span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, which is three-dimensional [15]. Since the superposition rule (2.3) depends on three particular solutions, Riccati equations satisfy Lie's condition.

Let us recall the analysis of Lie systems via contact geometry, which gives rise to contact Lie systems [22].

Definition 2.2. A *contact Lie system* is a triple (M, η, X), where η is a contact form on M and X is a Lie system on M admitting a Vessiot–Guldberg Lie algebra of contact Hamiltonian vector fields relative to η. A *contact Lie system* is called *conservative* or of *Liouville type* if the Reeb vector field of (M, η) is a Lie symmetry of X.

The term "conservative" or of "Liouville type" is coined because such contact Lie systems preserve a volume form like in the Liouville theorem in symplectic geometry [22].

The contact Hamiltonian function of a contact Hamiltonian vector field is a first integral of the Reeb vector field if and only if the contact Hamiltonian vector field commutes with the Reeb vector field. Hence, a contact Lie system of Liouville type amounts to a Lie system X on M admitting a smallest Lie algebra V^X of contact Hamiltonian vector fields relative to a contact form on M that commute with the Reeb vector field, R, of the contact form, namely R is a Lie symmetry of V^X.

In view of Lie Theorem, a Lie system X can be considered as a curve in V^X parametrized by t. Every contact form on a manifold M gives rise to an isomorphism mapping each contact Hamiltonian vector field on M to a contact Hamiltonian function on M and vice versa (see Corollary 4.11 for $k = 1$). Therefore, the Lie algebra of contact Hamiltonian vector fields V^X gives rise to an isomorphic Lie algebra of functions $\mathfrak{W}$ relative to the bracket on $\mathscr{C}^\infty(M)$ induced by η. Hence, X also defines a curve in $\mathfrak{W}$ parametrized by t. This suggests us the following definition.

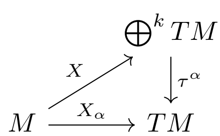
Definition 2.3. A *contact Lie–Hamiltonian system* is a triple $(M, \eta, h : \mathbb{R} \times M \to \mathbb{R})$, where (M, η) is a co-oriented contact manifold and h gives rise to a t-dependent family of functions $h_t : x \in M \mapsto h(t, x) \in \mathbb{R}$, with $t \in \mathbb{R}$, contained in a finite-dimensional Lie algebra of functions relative to the Lie bracket in $\mathscr{C}^\infty_\eta(M, \mathbb{R})$ induced by (M, η): a *contact Lie–Hamilton algebra*. The function h is called a *contact Lie–Hamiltonian* relative to η.

It is worth noting that every contact Lie system related to η gives rise to a unique contact Lie–Hamiltonian system related to the same η and vice versa.

3. k-Vector Fields and Its Integral Sections

k-vector fields are of great use in the geometric study of systems of partial differential equations [19, 57]. Consider the Whitney sum $\bigoplus^k_M TM := TM \oplus_M \overset{(k)}{\cdots} \oplus_M TM$, with the natural projections[b] $\tau^\alpha\colon \bigoplus^k TM \to TM$, $\tau^k_M\colon \bigoplus^k TM \to M$, $\alpha = 1, \ldots, k$, where τ^α denotes the projection onto the αth component of the Whitney sum $\bigoplus^k_M TM$, and τ^k_M is the projection onto the base manifold M.

A *k-vector field* on M is a section $\mathbf{X} : M \to \bigoplus^k TM$ of the projection τ^k_M. The space of k-vector fields on M is denoted by $\mathfrak{X}^k(M)$. Taking into account the diagram aside, a k-vector field $\mathbf{X} \in \mathfrak{X}^k(M)$ amounts to a family of vector fields $X_1, \ldots, X_k \in \mathfrak{X}(M)$ given by $X_\alpha = \tau^\alpha \circ \mathbf{X}$ with



$\alpha = 1, \ldots, k$. With this in mind, one can denote $\mathbf{X} = (X_1, \ldots, X_k)$. A k-vector field $\mathbf{X}$ induces a decomposable contravariant totally skew-symmetric tensor field, $X_1 \wedge \cdots \wedge X_k$, which is a decomposable section of the bundle $\bigsqcup_{x\in M} \Lambda^k T_x M = \Lambda^k TM \to M$, where $\bigsqcup_{x\in M}$ is the disjoint sum over $x \in M$.

Given a mapping $\phi\colon U \subset \mathbb{R}^k \to M$, its *first prolongation* to $\bigoplus^k TM$ is the map $\phi'\colon U \subset \mathbb{R}^k \to \bigoplus^k TM$ defined as follows:

$$\phi'(t) = \left(\phi(t); T_t\phi\left(\left.\frac{\partial}{\partial t^1}\right|_t\right), \ldots, T_t\phi\left(\left.\frac{\partial}{\partial t^k}\right|_t\right)\right) := (\phi(t); \phi'_1(t), \ldots, \phi'_k(t)), \quad t \in \mathbb{R}^k,$$

where $t = (t^1, \ldots, t^k)$, and $\{t^1, \ldots, t^k\}$ are the canonical coordinates on $\mathbb{R}^k$.

[b]From now on, the subindex in $\oplus_M$ in the Whitney sum will be skipped.

As for integral curves of vector fields, one can define integral sections of a k-vector field as follows. Let $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$ be a k-vector field. An *integral section* of $\mathbf{X}$ is a map $\phi\colon U \subset \mathbb{R}^k \to M$ such that $\phi' = \mathbf{X} \circ \phi$, namely $T\phi\left(\frac{\partial}{\partial t^\alpha}\right) = X_\alpha \circ \phi$ for $\alpha = 1, \ldots, k$. A k-vector field $\mathbf{X} \in \mathfrak{X}^k(M)$ is said to be *integrable* if every point of M lies in the image of an integral section of $\mathbf{X}$.

Let $\mathbf{X} = (X_1, \ldots, X_k)$ be a k-vector field on M with local expression $X_\alpha = \sum_{i=1}^m X_\alpha^i \frac{\partial}{\partial x^i}$ for $\alpha = 1, \ldots, k$. Then, $\phi\colon U \subset \mathbb{R}^k \to M$, with local expression $\phi(t) = (\phi^i(t))$, is an integral section of $\mathbf{X}$ if and only if its components satisfy the following system of PDEs:

$$\frac{\partial \phi^i}{\partial t^\alpha} = X_\alpha^i \circ \phi, \quad i = 1, \ldots, m, \quad \alpha = 1, \ldots, k. \tag{3.1}$$

Then, $\mathbf{X}$ is integrable if and only if $[X_\alpha, X_\beta] = 0$ for $\alpha, \beta = 1, \ldots, k$. These are necessary and sufficient conditions for the integrability of the system of PDEs (3.1) (see [43, 53] for details).

Every k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ on M defines a distribution $\mathcal{D}^{\mathbf{X}} \subset TM$ given by $\mathcal{D}^{\mathbf{X}}_x = \mathcal{D}^{\mathbf{X}} \cap T_xM = \langle X_1(x), \ldots, X_k(x)\rangle$. However, the notion of an integral submanifold of the k-vector field $\mathbf{X}$ is stronger than the notion of an integral section of the distribution $\mathcal{D}^{\mathbf{X}}$. The distribution $\mathcal{D}^{\mathbf{X}}$ is integrable if and only if $[X_\alpha, X_\beta] = \sum_{\gamma=1}^k f_{\alpha\beta}^\gamma X_\gamma$, with $\alpha, \beta = 1, \ldots, k$ for certain functions $f_{\alpha\beta}^\gamma \in \mathscr{C}^\infty(M)$ with $\alpha, \beta, \gamma = 1, \ldots, k$, and $\mathcal{D}^{\mathbf{X}}$ is invariant relative to the one-parameter group of diffeomorphisms of any vector field taking values in $\mathcal{D}^{\mathbf{X}}$ [42, 60, 61]. On the other hand, the k-vector field $\boldsymbol{X}$ is integrable if and only if $X_1, \ldots, X_k$ commute with each other, which is a stronger condition.

4. *k*-Contact Geometry

This section summarizes previous known results about k-contact manifolds [58] and a new approach, via distributions, devised in the preprint [23]. We have included just some proofs of our previous results in [23] for completeness.

Let us start by the classical definition of a k-contact form on an open subset [58], whose motivation was to generalize the concept of a contact form.

Definition 4.1. A *k-contact form on an open subset* $U \subset M$ is a differential one-form on U taking values in $\mathbb{R}^k$, let us say $\boldsymbol{\eta} \in \Omega^1(U, \mathbb{R}^k)$, such that

(1) $\ker \boldsymbol{\eta} \subset TU$ is a regular non-zero distribution of corank k,
(2) $\ker \mathrm{d}\boldsymbol{\eta} \subset TU$ is a regular distribution of rank k,
(3) $\ker \boldsymbol{\eta} \cap \ker \mathrm{d}\boldsymbol{\eta} = 0$.

If the k-contact form $\boldsymbol{\eta}$ is defined on M, the pair $(M, \boldsymbol{\eta})$ is called a *co-oriented k-contact manifold* and $\ker \mathrm{d}\boldsymbol{\eta}$ is called the *Reeb distribution* of $(M, \boldsymbol{\eta})$.

Nowadays, the only approach to k-contact Hamiltonian dynamics is constructed exclusively for co-oriented k-contact manifolds [23, 58]. Reeb vector fields for k-contact forms are the natural analogue of Reeb vector fields for contact forms, which play a relevant role in contact geometry and the study of periodic orbits of Hamiltonian systems [17, 54].

Theorem 4.2 (Reeb vector fields [31]). *Let* $(M, \boldsymbol{\eta} = \sum_{\alpha=1}^{k} \eta^\alpha \otimes e_\alpha)$ *be a co-oriented k-contact manifold. There exists a unique family of vector fields* $R_1, \ldots, R_k \in \mathfrak{X}(M)$ *such that*

$$\iota_{R_\alpha} \eta^\beta = \delta_\alpha^\beta, \quad \iota_{R_\alpha} \mathrm{d}\eta^\beta = 0,$$

for $\alpha, \beta = 1, \ldots, k$. *The vector fields* $R_1, \ldots, R_k$ *commute between themselves, i.e.* $[R_\alpha, R_\beta] = 0$ *for* $\alpha, \beta = 1, \ldots, k$, *and* $\ker \mathrm{d}\boldsymbol{\eta} = \langle R_1, \ldots, R_k \rangle$.

As shown later, it is generally difficult to find k-contact forms that transform the vector fields of a Vessiot–Guldberg Lie algebra for a Lie system into k-contact Hamiltonian vector fields, thereby enabling the application of various methods to study these vector fields (cf. [23, 58]). Instead, we will devise new methods to obtain such k-contact forms via k-contact distributions to be defined next (see [23] for details). It is worth noting that k-contact distribution plays the role of contact distributions in contact geometry, and they are easy to employ, practically and theoretically, to study k-contact manifolds [23]. In particular, we will show in this paper how k-contact distributions are simpler to use to study dynamical systems via k-contact geometry.

Definition 4.3. A *k-contact distribution* on M is a distribution $\mathcal{D} \subset TM$ such that, for each point $x \in M$, there is an open neighborhood $U \ni x$ and a k-contact form $\boldsymbol{\eta}$ on U such that $\mathcal{D}|_U = \ker \boldsymbol{\eta}$. We say that $(M, \mathcal{D})$ is a *k-contact manifold.*

Note that k-contact distributions are regular because the kernel of k-contact forms is regular too. Let us now develop the notions of $\boldsymbol{\eta}$-Hamiltonian vector fields and $\boldsymbol{\eta}$-Hamiltonian k-functions, which will be very useful in the description of Lie systems in Sec. 10.

The contraction of a vector field X with a differential one-form taking values in $\mathbb{R}^k$, e.g. $\boldsymbol{\eta} = \sum_{\alpha=1}^{k} \eta^\alpha \otimes e_\alpha$, is defined component-wise, namely $\iota_X \boldsymbol{\eta} = \sum_{\alpha=1}^{k} \iota_X \eta^\alpha \otimes e_\alpha$. The Lie bracket or contraction of X with a differential s-form taking values in $\mathbb{R}^k$ is defined similarly. In order to characterize k-distributions without determining a k-contact form, which can be challenging, we use Lie symmetries of distributions to be defined below. Moreover, the concept gives a natural manner to generalize to k-contact geometry the theory of contact Hamiltonian vector fields.

Definition 4.4. A *Lie symmetry* of a distribution $\mathcal{D}$ on M is a vector field X on M such that $[X, \mathcal{D}] \subset \mathcal{D}$, where $[X, \mathcal{D}]$ stands for the distribution generated by the Lie brackets of X with vector fields taking values in $\mathcal{D}$. A *k-contact vector field* relative to a k-contact manifold $(M, \mathcal{D})$ is a Lie symmetry X of $\mathcal{D}$. If additionally $\mathcal{D} = \ker \boldsymbol{\eta}$ for a k-contact form $\boldsymbol{\eta}$, then X is called an *$\boldsymbol{\eta}$-Hamiltonian vector field* and $-\iota_X \boldsymbol{\eta}$ is its *$\boldsymbol{\eta}$-Hamiltonian k-function.*

Denote by $\mathfrak{X}_{\mathcal{D}}(M)$ the space of k-contact vector fields on M relative to $(M,\mathcal{D})$ and we write $\mathfrak{X}_{\boldsymbol{\eta}}(M)$ for the space of $\boldsymbol{\eta}$-Hamiltonian vector fields relative to a co-oriented k-contact manifold $(M,\boldsymbol{\eta})$, respectively. Meanwhile, $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M,\mathbb{R}^k)$ stands for the space of $\boldsymbol{\eta}$-Hamiltonian k-functions relative to a k-contact form $\boldsymbol{\eta}$.

The following proposition explains why $-\iota_X\boldsymbol{\eta}$ plays the role of an $\boldsymbol{\eta}$-Hamiltonian k-function for an $\boldsymbol{\eta}$-Hamiltonian vector field X.

Proposition 4.5. *A vector field X on M is an $\boldsymbol{\eta}$-Hamiltonian vector field relative to $(M,\mathcal{D}=\ker\boldsymbol{\eta})$ if and only if*

$$\iota_X\eta^\alpha=-h^\alpha,\quad \iota_X\mathrm{d}\eta^\alpha=\mathrm{d}h^\alpha-\sum_{\beta=1}^k(R_\beta h^\alpha)\eta^\beta,\quad \alpha=1,\ldots,k, \tag{4.1}$$

for some k-function $\boldsymbol{h}=\sum_{\alpha=1}^k h^\alpha e_\alpha\in\mathscr{C}^\infty(M,\mathbb{R}^k)$.

Similarly, k-contact Hamiltonian k-vector fields can be defined as follows. It is worth noting that $\sum_{\beta=1}^k(R_\beta h^\alpha)\eta^\beta$, for every α, is a linear combination of the one-forms $\eta^1,\ldots,\eta^k$ with coefficients given by the functions $R_\beta h^\alpha$.

Definition 4.6. Given $h\in\mathscr{C}^\infty(M)$ on a co-oriented k-contact manifold $(M,\boldsymbol{\eta})$, called a *$\boldsymbol{\eta}$-Hamiltonian function*, a k-vector field $\mathbf{X}^c_h=(X_\alpha)\in\mathfrak{X}^k(M)$ satisfying the equations

$$\sum_{\alpha=1}^k\iota_{X_\alpha}\mathrm{d}\eta^\alpha=\mathrm{d}h-\sum_{\alpha=1}^k(R_\alpha h)\eta^\alpha,\quad \sum_{\alpha=1}^k\iota_{X_\alpha}\eta^\alpha=-h, \tag{4.2}$$

is called an *$\boldsymbol{\eta}$-Hamiltonian k-vector field.* Equations (4.2) can be rewritten as

$$\sum_{\alpha=1}^k\mathscr{L}_{X_\alpha}\eta^\alpha=-\sum_{\alpha=1}^k(R_\alpha h)\eta^\alpha,\quad \sum_{\alpha=1}^k\iota_{X_\alpha}\eta^\alpha=-h.$$

We call $(M,\boldsymbol{\eta},h)$ a *k-contact Hamiltonian system.*

Not every element in $\mathscr{C}^\infty(M,\mathbb{R}^k)$ is related to an $\boldsymbol{\eta}$-Hamiltonian vector field, e.g. $\boldsymbol{\eta}=(\mathrm{d}z-p\mathrm{d}x)\otimes e_1+\mathrm{d}w\otimes e_2$ on $\mathbb{R}^4$ is not a $\boldsymbol{\eta}$-Hamiltonian two-function of the form pe_1+pe_2. But every function $h\in\mathscr{C}^\infty(M)$ gives rise to a one-form $\mathrm{d}h-\sum_{\alpha=1}^k(R_\alpha h)\eta^\alpha$ belonging to the annihilator of the Reeb distribution. If the codistribution $\langle\mathrm{d}\eta^1,\ldots,\mathrm{d}\eta^k\rangle$ is not equal to the annihilator of the Reeb distribution, then $\ker\mathrm{d}\boldsymbol{\eta}\cap\ker\boldsymbol{\eta}\neq 0$, which is a contradiction. Hence, there exists a series of vector fields $X_1,\ldots,X_k$ such that $\sum_{\alpha=1}^r\iota_{X_\alpha}\mathrm{d}\eta^\alpha=\mathrm{d}h-\sum_{\alpha=1}^k(R_\alpha h)\eta^\alpha$. If $g=\sum_{\alpha=1}^k\iota_{X_\alpha}\eta^\alpha$, then, $(X_1-(h+g)R_1,X_2,\ldots,X_k)$ is an $\boldsymbol{\eta}$-Hamiltonian k-vector field related to h.

Let us now provide some notation [23].

Definition 4.7. Given a co-oriented k-contact manifold $(M,\boldsymbol{\eta})$, every k-contact Hamiltonian k-function $\boldsymbol{h}\in\mathscr{C}^\infty_{\boldsymbol{\eta}}(M,\mathbb{R}^k)$ is related to its *Reeb derivation*, namely the vector field on M of the form

$$R_{\boldsymbol{h}} = \sum_{\alpha=1}^{k} h^\alpha R_\alpha.$$

Moreover, let us define the so-called *Reeb tensor field* $\mathfrak{R}_{\boldsymbol{\eta}} : \mathrm{T}^*M \to \mathrm{T}^*M$ satisfying

$$\mathfrak{R}_{\boldsymbol{\eta}}\theta = \sum_{\alpha=1}^{k} \eta^\alpha \iota_{R_\alpha}\theta, \quad \forall \theta \in \mathrm{T}^*M.$$

The advantages of the previous notation are illustrated by the conciseness of the expressions in following propositions. In particular, the following proposition will be useful to demonstrate some properties of k-contact Lie systems (we refer to [23] for details).

Proposition 4.8. *Let $X_{\boldsymbol{f}}$ be the $\boldsymbol{\eta}$-Hamiltonian vector field of $\boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$. Then,*

(i) $\mathscr{L}_{X_{\boldsymbol{f}}}\boldsymbol{\eta} = -\sum_{\alpha,\beta=1}^{k}(R_\beta f^\alpha)\eta^\beta \otimes e_\alpha = -\mathfrak{R}_{\boldsymbol{\eta}}\mathrm{d}\boldsymbol{f}$,
(ii) $X_{\boldsymbol{f}}\boldsymbol{f} = -\sum_{\alpha,\beta=1}^{k} f^\beta(R_\beta f^\alpha)e_\alpha = -R_{\boldsymbol{f}}\boldsymbol{f}$,
(iii) $[R_\beta, X_{\boldsymbol{f}}] = X_{R_\beta \boldsymbol{f}}$,

where $R_\beta \boldsymbol{f} = \sum_{\alpha=1}^{k} R_\beta f^\alpha e_\alpha$ and $\beta = 1, \ldots, k$.

The following proposition follows from expression (4.1) and the fact that an $\boldsymbol{\eta}$-Hamiltonian vector field is a vector field that leaves invariant a k-contact distribution.

Proposition 4.9. *The space of $\boldsymbol{\eta}$-Hamiltonian vector fields $\mathfrak{X}_{\boldsymbol{\eta}}(M)$ relative to a k-contact form $\boldsymbol{\eta}$ is a Lie algebra and, for $\boldsymbol{h}, \boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$, one has that $[X_{\boldsymbol{h}}, X_{\boldsymbol{g}}]$ is the $\boldsymbol{\eta}$-Hamiltonian vector field related to the $\boldsymbol{\eta}$-Hamiltonian k-function $-\iota_{[X_{\boldsymbol{h}}, X_{\boldsymbol{g}}]}\boldsymbol{\eta}$.*

In a view of Proposition 4.9, one can define a following Lie bracket on $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$.

Definition 4.10. Given a co-oriented k-contact manifold $(M, \boldsymbol{\eta})$, we define a bracket on $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$ of the form

$$\{\boldsymbol{h}_1, \boldsymbol{h}_2\}_{\boldsymbol{\eta}} = \boldsymbol{\eta}([X_{\boldsymbol{h}_1}, X_{\boldsymbol{h}_2}]), \quad \forall \boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k). \tag{4.3}$$

The result below immediately follows from Proposition 4.9 (see also [23]).

Corollary 4.11. *The bracket* (4.3) *induces a Lie algebra isomorphism $\phi : \boldsymbol{h} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k) \mapsto -X_{\boldsymbol{h}} \in \mathfrak{X}_{\boldsymbol{\eta}}(M)$ and an exact sequence of Lie algebra morphisms*

$$0 \longrightarrow \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k) \stackrel{\phi}{\longrightarrow} \mathfrak{X}_{\boldsymbol{\eta}}(M) \longrightarrow 0,$$

where $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$ is endowed with the Lie bracket (4.3).

Note that

$$\begin{aligned}\{\boldsymbol{h}_1, \boldsymbol{h}_2\}_{\boldsymbol{\eta}} &= \boldsymbol{\eta}([X_{\boldsymbol{h}_1}, X_{\boldsymbol{h}_2}]) = (\mathscr{L}_{X_{\boldsymbol{h}_1}} \iota_{X_{\boldsymbol{h}_2}} - \iota_{X_{\boldsymbol{h}_2}} \mathscr{L}_{X_{\boldsymbol{h}_1}})\boldsymbol{\eta} \\ &= -\mathscr{L}_{X_{\boldsymbol{h}_1}} \boldsymbol{h}_2 - R_{\boldsymbol{h}_2} \boldsymbol{h}_1,\end{aligned} \tag{4.4}$$

for all $\boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$ and

$$X_{\boldsymbol{f}} \boldsymbol{f} = -R_{\boldsymbol{f}} \boldsymbol{f}, \quad \forall \boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k).$$

It is remarkable that, in the contact case, a dissipated quantity (see [48] and references therein) is a function such that $X_h f = -f(Rh)$. Hence, the expression above can be used to define an analogue of dissipated quantities in k-contact geometry.

Definition 4.12. A *dissipated k-contact k-function* relative to a k-contact Hamiltonian system $(M, \boldsymbol{\eta}, \boldsymbol{h})$ is an $\boldsymbol{\eta}$-contact k-function $\boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$ such that

$$X_{\boldsymbol{h}} \boldsymbol{f} = -R_{\boldsymbol{f}} \boldsymbol{h}.$$

In particular every $\boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$ is dissipated relative to itself, namely $X_{\boldsymbol{f}} \boldsymbol{f} = -R_{\boldsymbol{f}} \boldsymbol{f}$. Moreover,

$$\{\boldsymbol{h}, \boldsymbol{f}\} = 0 \quad \Longleftrightarrow \quad X_{\boldsymbol{h}} \boldsymbol{f} = -R_{\boldsymbol{f}} \boldsymbol{h}.$$

In other words, dissipated quantities are those that $\boldsymbol{\eta}$-commute with $\boldsymbol{h}$. Moreover, an $\boldsymbol{\eta}$-contact k-function is dissipated if and only if its associated $\boldsymbol{\eta}$-Hamiltonian vector field is a Lie symmetry of $X_{\boldsymbol{h}}$.

On the other hand, note that

$$\iota_{X_{\boldsymbol{h}}} \mathrm{d}\boldsymbol{\eta} = \mathrm{d}\boldsymbol{h} - \mathfrak{R}_{\boldsymbol{\eta}} \mathrm{d}\boldsymbol{h} = (\mathrm{Id}_{\mathrm{T}^*M} - \mathfrak{R}_{\boldsymbol{\eta}})(\mathrm{d}\boldsymbol{h}), \quad \iota_{X_{\boldsymbol{h}}} \boldsymbol{\eta} = -\boldsymbol{h}, \quad \mathscr{L}_{X_{\boldsymbol{h}}} \boldsymbol{\eta} = -\mathfrak{R}_{\boldsymbol{\eta}} \mathrm{d}\boldsymbol{h}.$$

It is relevant that (4.3) is not a Poisson bracket and $\boldsymbol{h}_2$ may be a constant function having a non-vanishing Lie bracket with other functions. Examples of this will be shown in Sec. 10.

Let us now study the relation of k-contact distributions with the maximal non-integrability notion defined via distributions [63].

Definition 4.13. Let $\mathcal{D}$ be a regular distribution on M and let $\pi\colon TM \to TM/\mathcal{D}$ be the natural vector bundle projection. Then, $\mathcal{D}$ is *maximally non-integrable* in a *distributional sense* if $\mathcal{D} \neq 0$ and the vector bundle mapping $\rho\colon \mathcal{D} \times_M \mathcal{D} \to TM/\mathcal{D}$ over M given by

$$\rho(v, v') = \pi([X, X']_x), \quad \forall v, v' \in \mathcal{D}_x, \quad \forall x \in M, \tag{4.5}$$

where X, X' are vector fields taking values in $\mathcal{D}$ locally defined around x such that $X_x = v$ and $X'_x = v'$, is non-degenerate.

The mapping ρ is well defined as shown in Proposition A.1 or [23]. It is worth noting that the terms "maximally non-integrable" and "completely non-integrable" have been used in the literature with equal, different, or even related meanings [1, 3, 23].

Proposition 4.14. *If* $(M, \boldsymbol{\eta})$ *is a co-oriented* k*-contact manifold, then* $\mathrm{d}\boldsymbol{\eta}$ *is non-degenerate when restricted to* $\ker \boldsymbol{\eta}$*. In other words,* $\ker \boldsymbol{\eta}$ *is maximally non-integrable.*

Let us recall the main result in [23] used in this work.

Theorem 4.15. *A distribution* $\mathcal{D}$ *on* M *is a* k*-contact distribution if and only if it is maximally non-integrable and, around an open neighborhood* U *of every* $x \in M$*, admits an integrable* k*-vector field* $\mathbf{S} = (S_1, \ldots, S_k)$ *of Lie symmetries of* $\mathcal{D}|_U$ *such that*

$$\langle S_1, \ldots, S_k \rangle \oplus \mathcal{D}|_U = TU. \tag{4.6}$$

The proof of the previous theorem is indeed the method used in our applications to obtain a k-contact form $\boldsymbol{\eta}$ turning out the vector fields of a Vessiot–Guldberg Lie algebra into $\boldsymbol{\eta}$-Hamiltonian vector fields. The reason for the use of the previous theorem is that it is difficult to obtain a k-contact form compatible with a Lie system, but it is pretty much simpler in many cases to obtain a compatible k-contact distribution. This will be the key in this paper to obtain the applications in control theory shown in [55]. Indeed, the difference between the abundance of examples in this work and [22] illustrates that it is difficult to obtain Lie systems with a compatible contact or, even more difficult, k-contact form.

5. k-Contact Lie Systems

Let us introduce k-contact Lie systems and determine their fundamental properties. It is worth noting that the notion is a generalization of contact Lie systems [22], and it follows the general idea of Lie systems admitting compatible geometric structures [24, 33, 38].

Definition 5.1. A *k-contact Lie system* is a triple $(M, \boldsymbol{\eta}, X)$, where $\boldsymbol{\eta}$ is a k-contact form on M and X is a Lie system on M whose smallest Lie algebra, V^X, consists of $\boldsymbol{\eta}$-Hamiltonian vector fields. A *k-contact Lie system* $(M, \boldsymbol{\eta}, X)$ is called *projectable* if the $\boldsymbol{\eta}$-Hamiltonian k-functions associated with the vector fields in V^X are first integrals of the Reeb vector fields of $(M, \boldsymbol{\eta})$.

k-contact Lie systems are called *projectable* because they can be projected onto k-symplectic Lie systems [25], as it will be shown soon. Sometimes, X is simply said to be a k-contact Lie system if there exists some k-contact form turning it into a k-contact Lie system $(M, \boldsymbol{\eta}, X)$. Moreover, $\boldsymbol{\eta}$-Hamiltonian vector fields, and other names of notions containing $\boldsymbol{\eta}$ like $\boldsymbol{\eta}$-Hamiltonian k-functions, may be called k-contact Hamiltonian vector fields or k-contact Hamiltonian k-functions when the knowledge of the particular $\boldsymbol{\eta}$ is not important.

Proposition 4.8 yields that an $\boldsymbol{\eta}$-Hamiltonian k-function of an $\boldsymbol{\eta}$-Hamiltonian vector field is a first integral of all the Reeb vector fields of $\boldsymbol{\eta}$ if and only if the $\boldsymbol{\eta}$-Hamiltonian vector field commutes with all the Reeb vector fields of $\boldsymbol{\eta}$. Hence, a projectable k-contact Lie system $(M, \boldsymbol{\eta}, X)$ amounts to a Lie system X on a manifold M such that each element of $V^X \subset \mathfrak{X}_{\boldsymbol{\eta}}(M)$ is invariant relative to the flows of all the Reeb vector fields of $\boldsymbol{\eta}$. In other words, $R_1, \ldots, R_k$ are Lie symmetries of V^X and, therefore, are Lie symmetries of the k-contact Lie system itself.

Again due to the isomorphism of Lie algebras

$$\mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k) \ni \boldsymbol{f} \longmapsto -X_{\boldsymbol{f}} \in \mathfrak{X}_{\boldsymbol{\eta}}(M), \tag{5.1}$$

every k-contact Lie system $(M, \boldsymbol{\eta}, X)$ is related to a unique t-dependent $\boldsymbol{\eta}$-Hamiltonian k-function $\boldsymbol{h} : \mathbb{R} \times M \to \mathbb{R}^k$ whose $\{\boldsymbol{h}_t\}_{t \in \mathbb{R}}$ are contained in a finite-dimensional Lie algebra of $\boldsymbol{\eta}$-Hamiltonian k-functions, and vice versa. This motivates the following definition.

Definition 5.2. A *k-contact Lie–Hamiltonian system* is a triple $(M, \boldsymbol{\eta}, \boldsymbol{h} : \mathbb{R} \times M \to \mathbb{R}^k)$, where $(M, \boldsymbol{\eta})$ is a k-contact manifold and $\boldsymbol{h}$ is a t-dependent family of k-functions $\boldsymbol{h}_t : x \in M \mapsto \boldsymbol{h}(t, x) \in \mathbb{R}^k$, with $t \in \mathbb{R}$, contained in a finite-dimensional Lie algebra $\mathfrak{W}$ of $\boldsymbol{\eta}$-Hamiltonian k-functions relative to the Lie bracket in $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$. We call $\mathfrak{W}$ an $\boldsymbol{\eta}$-Lie–Hamilton algebra.

As the function $\boldsymbol{h} : \mathbb{R} \times M \to \mathbb{R}^k$ in $(M, \boldsymbol{\eta}, \boldsymbol{h} : \mathbb{R} \times M \to \mathbb{R}^k)$ gives rise to a t-dependent family of $\boldsymbol{\eta}$-Hamiltonian k-functions contained in an $\boldsymbol{\eta}$-Lie–Hamilton algebra $\mathfrak{W}$, the isomorphism (5.1) yields that the t-dependent $\boldsymbol{\eta}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ is a Lie system with a Vessiot–Guldberg Lie algebra given by the $\boldsymbol{\eta}$-Hamiltonian vector fields of $\mathfrak{W}$. Then, one has the following proposition.

Proposition 5.3. *Every k-contact Lie system with respect to a k-contact form $\boldsymbol{\eta}$ gives rise to a unique $\boldsymbol{\eta}$-Lie–Hamiltonian system, and vice versa.*

Anyhow, it is remarkable that $\mathfrak{W}$ need not be unique, as shown in following examples.

Example 5.4. (**t-dependent frequency isotropic harmonic oscillators**). Consider the system of harmonic isotropic oscillators on $\mathbb{R}^2$ with a t-dependent frequency described by the t-dependent system of ordinary differential equations on $T\mathbb{R}^2 \simeq \mathbb{R}^4$ of the form

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = v_i, \\ \dfrac{\mathrm{d}v_i}{\mathrm{d}t} = -\nu^2(t)x_i, \end{cases} \qquad i = 1, 2, \tag{5.2}$$

for an arbitrary t-dependent frequency $\nu(t)$ and coordinates x_1, v_1, x_2, v_2. Define three vector fields on $T\mathbb{R}^2 \simeq \mathbb{R}^4$ given by

$$X_1 = \sum_{i=1}^{2} v_i \frac{\partial}{\partial x_i} \quad X_2 = \frac{1}{2}\sum_{i=1}^{2}\left(x_i \frac{\partial}{\partial x_i} - v_i \frac{\partial}{\partial v_i}\right), \quad X_3 = -\sum_{i=1}^{2} x_i \frac{\partial}{\partial v_i}.$$

The vector fields X_1, X_2, X_3 span (over the real numbers) a Lie algebra V_{is} isomorphic to $\mathfrak{sl}_2$. Indeed,

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3,$$

and one obtains a distribution $\mathcal{D}^{V_{is}}$ on $T\mathbb{R}^2$ of the form

$$\mathcal{D}^{V_{is}}_x = \langle X_1(x), X_2(x), X_3(x)\rangle, \quad \forall x \in T\mathbb{R}^2,$$

of rank three almost everywhere. Note that system (5.2) is related to the t-dependent vector field

$$X_{is} = \nu^2(t) X_3 + X_1$$

and becomes a Lie system.

Moreover, $T\mathbb{R}^2$ admits a natural symplectic form

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}v_1 + \mathrm{d}x_2 \wedge \mathrm{d}v_2,$$

and X_1, X_2, X_3 are Hamiltonian vector fields relative to ω with Hamiltonian functions $h^1, h^2, h^3 \in \mathscr{C}^\infty(T\mathbb{R}^2)$ because $\mathscr{L}_{X_\alpha}\omega = 0$ for $\alpha = 1, 2, 3$ and $T\mathbb{R}^2$ is simply connected, which implies that every closed one-form is exact. Additionally, $\mathrm{d}h^1 \wedge \mathrm{d}h^2 \wedge \mathrm{d}h^3 \neq 0$.

Moreover, there exists another vector field on $T\mathbb{R}^2$, namely

$$X_4 = \sum_{i=1}^{2}\left(x_i \frac{\partial}{\partial x_i} + v_i \frac{\partial}{\partial v_i}\right),$$

commuting with X_1, X_2, X_3. Hence, $X_1, \ldots, X_4$ span a Lie algebra V_e isomorphic to the matrix Lie algebra of 2×2 matrices. The vector field X_4 is not Hamiltonian relative to ω because $\mathscr{L}_{X_4}\omega = 2\omega \neq 0$. Let us prove that we can turn all vector fields $X_1, \ldots, X_4$ into $\boldsymbol{\eta}_{is}$-Hamiltonian relative to a two-contact form $\boldsymbol{\eta}_{is}$.

The vector fields X_1, X_2, X_3, X_4 are linearly independent at the manifold $\mathcal{O}_{is}$ of points in $T\mathbb{R}^2$ where $v_2 x_1 - x_2 v_1 \neq 0$. Moreover, X_1, X_2, X_3, X_4 are the fundamental vector fields of a locally transitive linear Lie group action of GL_2, the Lie group of invertible linear real automorphisms on $\mathbb{R}^2$, on the manifold $\mathcal{O}_{is} \subset T\mathbb{R}^2 \backslash \{(0, 0, 0, 0)\}$. As proved in [33], the space $\mathcal{O}_{is}$ is locally diffeomorphic to the Lie group GL_2 in such a manner that $X_1, \ldots, X_4$ become mapped into a basis of right-invariant vector fields. Then, there exists, at least locally around a generic point, a Lie algebra $\langle Y_1, \ldots, Y_4\rangle$ of Lie symmetries of the vector fields of V_e, i.e. $[Y_i, X_j] = 0$ for $i, j = 1, \ldots, 4$, that is isomorphic to $\mathfrak{gl}_2$ with $Y_1 \wedge \cdots \wedge Y_4 \neq 0$.

In fact, consider

$$Y_1 = x_2\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial v_1}, \quad Y_2 = \frac{1}{2}\sum_{i=1}^{2}(-1)^i\left(x_i\frac{\partial}{\partial x_i} + v_i\frac{\partial}{\partial v_i}\right),$$

$$Y_3 = -x_1\frac{\partial}{\partial x_2} - v_1\frac{\partial}{\partial v_2}$$

and $X_4 = Y_4$. The vector fields $Y_1, \ldots, Y_4$ commute with $X_1, \ldots, X_4$, while

$$[Y_1, Y_2] = -Y_1, \quad [Y_1, Y_3] = -2Y_2, \quad [Y_2, Y_3] = -Y_3.$$

Let $\Upsilon_1, \ldots, \Upsilon_4$ be the dual one-forms to $Y_1, \ldots, Y_4$, namely

$$\Upsilon^1 = \frac{v_1\mathrm{d}x_1 - x_1\mathrm{d}v_1}{v_1x_2 - x_1v_2}, \quad \Upsilon^2 = \frac{v_2\mathrm{d}x_1 - x_2\mathrm{d}v_1 + v_1\mathrm{d}x_2 - x_1\mathrm{d}v_2}{v_1x_2 - x_1v_2},$$

$$\Upsilon^3 = \frac{v_2\mathrm{d}x_2 - x_2\mathrm{d}v_2}{v_1x_2 - x_1v_2}$$

and

$$\Upsilon^4 = \frac{\mathrm{d}(v_1x_2 - x_1v_2)}{2(v_1x_2 - x_1v_2)}.$$

Then,

$$\mathrm{d}\Upsilon^2 = 2\Upsilon^1 \wedge \Upsilon^3, \quad \mathrm{d}\Upsilon^4 = 0.$$

Hence, $\boldsymbol{\eta}_{is} = \Upsilon^2 \otimes e_1 + \Upsilon^4 \otimes e_2$ defines a two-contact form that is invariant relative to $X_1, \ldots, X_4$. Indeed,

$$(\mathscr{L}_{X_i}\eta^\alpha_{is})(Y_j) = X_i(\eta^\alpha_{is}(Y_j)) - \eta^\alpha_{is}([X_i, Y_j]) = 0, \quad i = 1, \ldots, 4, \quad \alpha = 2, 4,$$

and $X_1, \ldots, X_4$ are locally $\boldsymbol{\eta}_{is}$-Hamiltonian relative to $\boldsymbol{\eta}_{is}$. Define $\boldsymbol{h}_i = -\iota_{X_i}\boldsymbol{\eta}_{is}$ for $i = 1, 2, 3, 4$, which read

$$\boldsymbol{h}_1 = \frac{-2v_2v_1e_1}{v_1x_2 - x_1v_2}, \quad \boldsymbol{h}_2 = \left(1 - \frac{2v_1x_2}{v_1x_2 - x_1v_2}\right)e_1,$$

$$h_3 = -\frac{2x_2x_1e_1}{v_1x_2 - x_1v_2}, \quad \boldsymbol{h}_4 = -e_2.$$

One has that the following non-vanishing commutation relations

$$\{\boldsymbol{h}_1, \boldsymbol{h}_2\} = -\boldsymbol{h}_1, \quad \{\boldsymbol{h}_1, \boldsymbol{h}_3\} = -2\boldsymbol{h}_2, \quad \{\boldsymbol{h}_2, \boldsymbol{h}_3\} = -\boldsymbol{h}_3$$

and $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_4$ close a finite-dimensional Lie algebra of $\boldsymbol{\eta}_{is}$-Hamiltonian functions, which is isomorphic to $\mathfrak{gl}(2, \mathbb{R})$. Note that $(\mathcal{O}_{is} \subset T\mathbb{R}^2, \boldsymbol{\eta}_{is}, \boldsymbol{h}_1 + \nu^2(t)\boldsymbol{h}_3)$ is a two-contact Lie–Hamiltonian system for the two-contact Lie system (5.2). Indeed, $\langle \boldsymbol{h}_1, \ldots, \boldsymbol{h}_4\rangle$ and $\langle \boldsymbol{h}_1, \ldots, \boldsymbol{h}_3\rangle$ are $\boldsymbol{\eta}_{is}$-Lie–Hamilton algebras for X_{is}.

Note that the Reeb vector fields are Y_2 and Y_4, which are Lie symmetries of X_1, X_2, X_3, X_4. Hence, their $\boldsymbol{\eta}_{is}$-Hamiltonian two-functions are first integrals of Y_2, Y_4. Then, $(\mathcal{O}_{is}, \boldsymbol{\eta}_{is}, X_{is})$ is a projectable two-contact Lie system and projects onto a new Lie system. This will be analyzed in detail in forthcoming sections. Note that we do not really include X_4 in the Vessiot–Guldberg Lie algebra of X_{is}: it was just a tool to transform the initial Lie system into a two-contact one.

To complete the presentation of k-contact Lie systems, let us give an example of non-projectable k-contact Lie system.

Example 5.5. Consider the manifold $\mathbb{R}^5$ equipped with linear coordinates $\{q, z_1, z_2, p_1, p_2\}$. Then, $\mathbb{R}^5$ has a natural two-contact form given by $\boldsymbol{\eta}_J = (\mathrm{d}z_1 - p_1\,\mathrm{d}q) \otimes e_1 + (\mathrm{d}z_2 - p_2\,\mathrm{d}q) \otimes e_2$. Its associated Reeb vector fields are $R_1 = \partial/\partial z_1$ and $R_2 = \partial/\partial z_2$. Then, $\mathbb{R}^5$ is diffeomorphic to the first jet bundle $J^1(\mathbb{R}, \mathbb{R}\times\mathbb{R}^2)$ associated with the fiber bundle $(q, z_1, z_2) \in \mathbb{R}\times\mathbb{R}^2 \mapsto q \in \mathbb{R}$ and $\boldsymbol{\eta}_J$ is, essentially, the natural two-contact form with the adapted variables $\{q, z_1, z_2, p_1, p_2\}$.

Consider the vector fields on $\mathbb{R}^5$ given by

$$X_1 = \frac{\partial}{\partial z_1}, \quad X_2 = \frac{\partial}{\partial z_2}, \quad X_3 = \frac{\partial}{\partial q}, \quad X_4 = q\frac{\partial}{\partial q} - p_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial p_2},$$
$$X_5 = z_1\frac{\partial}{\partial z_1} + \frac{1}{4}z_2\frac{\partial}{\partial z_2} + \frac{1}{2}\left(q\frac{\partial}{\partial q} + p_1\frac{\partial}{\partial p_1} - \frac{1}{2}p_2\frac{\partial}{\partial p_2}\right).$$

Note that $X_1 \wedge \cdots \wedge X_5$ is different from zero almost everywhere in $\mathbb{R}^5$. The vector fields $X_1, \ldots, X_5$ are $\boldsymbol{\eta}_J$-Hamiltonian relative to $(\mathbb{R}^5, \boldsymbol{\eta}_J)$ with $\boldsymbol{\eta}_J$-Hamiltonian two-functions

$$\boldsymbol{h}_1 = -e_1, \quad \boldsymbol{h}_2 = -e_2, \quad \boldsymbol{h}_3 = p_1e_1 + p_2e_2, \qquad \boldsymbol{h}_4 = p_1qe_1 + p_2qe_2,$$
$$\boldsymbol{h}_5 = (-z_1 + qp_1/2)e_1 + (-z_2/4 + qp_2/2)e_2,$$

and span a five-dimensional Vessiot–Guldberg Lie algebra with commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad [X_1, X_4] = 0, \quad [X_1, X_5] = X_1,$$
$$[X_2, X_3] = 0, \quad [X_2, X_4] = 0, \quad [X_2, X_5] = \frac{1}{4}X_2,$$
$$[X_3, X_4] = X_3, \quad [X_3, X_5] = \frac{1}{2}X_3,$$
$$[X_4, X_5] = 0.$$

This allows us to define a two-contact Lie system on $\mathbb{R}^5$ relative to $\boldsymbol{\eta}_J$ given by $(\mathbb{R}^5, \boldsymbol{\eta}_J, X_J)$ with

$$X_J = \sum_{\alpha=1}^{5} b_\alpha(t)X_\alpha, \tag{5.3}$$

where $b_1(t), \ldots, b_5(t)$ are any t-dependent functions. If $b_1(t), \ldots, b_5(t)$ are such that the vectors $(b_1(t), \ldots, b_5(t))$, with $t \in \mathbb{R}$, span $\mathbb{R}^5$, then $V^{X_J} = \langle X_1, \ldots, X_5\rangle$ and $\mathcal{D}_x^{X_J} = T_x\mathbb{R}^5$ for x in an open dense subset of $\mathbb{R}^5$.

Since the $\boldsymbol{\eta}_J$-Hamiltonian two-function of X_5 is not a first integral of the Reeb vector fields $R_1 = X_1, R_2 = X_2$, we have that $(\mathbb{R}^5, \boldsymbol{\eta}_J, X_J)$ is a not a projectable two-contact Lie system when $(b_1(t), \ldots, b_5(t))$, for $t \in \mathbb{R}$, span $\mathbb{R}^5$. Note also that

X_J is associated with the t-dependent $\boldsymbol{\eta}_J$-Hamiltonian two-function

$$\boldsymbol{h} = \sum_{\alpha=1}^{5} b_\alpha(t)\boldsymbol{h}_\alpha,$$

namely each $(X_J)_t$ is the $\boldsymbol{\eta}_J$-Hamiltonian vector field related to $\boldsymbol{h}_t$ for every $t \in \mathbb{R}$. Finally, $\langle \boldsymbol{h}_1, \ldots, \boldsymbol{h}_5 \rangle$ span an $\boldsymbol{\eta}_J$-Lie–Hamilton algebra for X_J. In fact, the non-vanishing commutation relations read

$$\{\boldsymbol{h}_1, \boldsymbol{h}_5\} = -\boldsymbol{h}_1, \quad \{\boldsymbol{h}_2, \boldsymbol{h}_5\} = -\frac{1}{4}\boldsymbol{h}_2, \quad \{\boldsymbol{h}_3, \boldsymbol{h}_4\} = -\boldsymbol{h}_3, \quad \{\boldsymbol{h}_3, \boldsymbol{h}_5\} = -\frac{1}{2}\boldsymbol{h}_3.$$

Then, $(\mathbb{R}^5, \boldsymbol{\eta}_J, \boldsymbol{h} = \sum_{\alpha=1}^{5} b_\alpha(t)\boldsymbol{h}_\alpha)$ is a two-contact Lie–Hamiltonian system for the two-contact Lie system (5.3).

6. Generalized *t*-Dependent Constants of the Motion for *k*-Contact Lie Systems

Given a k-contact Lie system, one obtains that a Vessiot–Guldberg Lie algebra of $\boldsymbol{\eta}$-Hamiltonian vector fields gives rise to a Lie algebra of $\boldsymbol{\eta}$-Hamiltonian k-functions, which in turn gives rise to a momentum map $\mathbf{J}\colon M \to \mathfrak{g}^* \otimes \mathbb{R}^k$. This mapping will be relevant so as to determine some properties of the initial k-contact Lie system. In particular, it is used to obtain generalized constants of motion [11] for k-contact Lie systems. This will illustrate some of the potential uses of k-contact geometry to analyze k-contact Lie systems. Let us start by a motivating example.

Example 6.1. Let us go back to Example 5.4 of the t-dependent frequency isotropic oscillator on $\mathcal{O}_{is}$. Consider the presymplectic form $\langle \mathrm{d}\boldsymbol{\eta}_{is}, e^1 \rangle$. This gives rise to a Poisson bracket on the space of its admissible functions, i.e. functions $f \in \mathscr{C}^\infty(\mathcal{O}_{is})$ such that $\mathrm{d}f$ belongs to the image of $\mathrm{d}\langle \boldsymbol{\eta}_{is}, e^1 \rangle$. Since $X_1, \ldots, X_4$ are invariant relative to the Reeb vector fields, they become Hamiltonian with respect to $\mathrm{d}\langle \boldsymbol{\eta}_{is}, e^1 \rangle$. This latter fact will be shown in detail in Proposition 6.2 and, for the time being, it is enough to verify this fact by a short calculation. Note that

$$h^1_\alpha = \langle \boldsymbol{h}_\alpha, e^1 \rangle, \quad \alpha = 1, 2, 3, 4,$$

are Hamiltonian functions for $X_1, \ldots, X_4$, respectively, and span a Lie algebra of admissible functions isomorphic to $\mathfrak{sl}_2$. Then, $h^1_1 h^1_3 - (h^1_2)^2$ Poisson commutes with h^1_1, h^1_2, h^1_3 and becomes a constant of motion for X_{is}.

Let us generalize the properties of the above example for every projective k-contact Lie system. It is worth noting that if $\boldsymbol{f}, \boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k) \cap \ker \boldsymbol{R}$, then

$$\begin{aligned}\{\boldsymbol{f}, \boldsymbol{g}\}_{\boldsymbol{\eta}} &= \boldsymbol{\eta}([X_{\boldsymbol{f}}, X_{\boldsymbol{g}}]) = X_{\boldsymbol{f}}\iota_{X_{\boldsymbol{g}}}\boldsymbol{\eta} - X_{\boldsymbol{g}}\iota_{X_{\boldsymbol{f}}}\boldsymbol{\eta} - \mathrm{d}\boldsymbol{\eta}(X_{\boldsymbol{f}}, X_{\boldsymbol{g}}) \\ &= -\iota_{X_{\boldsymbol{f}}}\iota_{X_{\boldsymbol{g}}}\mathrm{d}\boldsymbol{\eta} + \iota_{X_{\boldsymbol{g}}}\iota_{X_{\boldsymbol{f}}}\mathrm{d}\boldsymbol{\eta} - \mathrm{d}\boldsymbol{\eta}(X_{\boldsymbol{f}}, X_{\boldsymbol{g}}) = \mathrm{d}\boldsymbol{\eta}(X_{\boldsymbol{f}}, X_{\boldsymbol{g}}).\end{aligned} \tag{6.1}$$

This explains the relation of the bracket of functions in $\ker \boldsymbol{R}$ and the bracket of presymplectic forms defined in the following proposition.

Proposition 6.2. *If $\boldsymbol{\eta}$ is a k-contact form and $\theta \in \mathbb{R}^{k*}$, then $\omega = \langle \mathrm{d}\boldsymbol{\eta}, \theta\rangle$ is a presymplectic form. Every Lie algebra of k-contact Hamiltonian k-functions $\mathfrak{W} = \langle \boldsymbol{h}_1, \ldots, \boldsymbol{h}_r\rangle$ belonging to $\ker \boldsymbol{R}$ gives rise to a Lie algebra of Hamiltonian functions $\mathfrak{W}^\theta = \langle\langle h_1, \theta\rangle, \ldots, \langle h_r, \theta\rangle\rangle$ relative to ω. In particular, there exists a Lie algebra surjection*

$$\boldsymbol{f} \in \mathfrak{W} \mapsto f^\theta = \langle \boldsymbol{f}, \theta\rangle \in \mathfrak{W}^\theta.$$

If C is an admissible function of $\langle \mathrm{d}\boldsymbol{\eta}, \theta\rangle$ and Poisson commutes with the elements of $\mathfrak{W}^\theta$ relative to the Poisson bracket induced by ω, then C is a constant of motion of every Lie system X admitting an $\boldsymbol{\eta}$-Lie–Hamilton algebra $\mathfrak{W}$.

Proof. Since $\mathrm{d}\boldsymbol{\eta}$ is closed, so is $\langle \mathrm{d}\boldsymbol{\eta}, \theta\rangle$, which becomes presymplectic.[c] Moreover, the functions $h^\theta_\alpha = \langle \boldsymbol{h}_\alpha, \theta\rangle$ become Hamiltonian functions for $\langle \mathrm{d}\boldsymbol{\eta}, \theta\rangle$ because $R_\alpha \boldsymbol{h} = 0$. Indeed,

$$\iota_{X_{\boldsymbol{f}}}\mathrm{d}\boldsymbol{\eta} = \mathrm{d}\boldsymbol{f} \Rightarrow \iota_{X_{\boldsymbol{f}}}\mathrm{d}\langle \boldsymbol{\eta}, \theta\rangle = \mathrm{d}\langle \boldsymbol{f}, \theta\rangle.$$

Let us show that expression (6.1) shows that there exists a Lie algebra surjection from $\mathfrak{M}$ onto $\mathfrak{M}^\theta$ and $X_{\boldsymbol{f}}$ is a presymplectic Hamiltonian vector field of f^θ. We consider $X_{f^\theta} = X_{\boldsymbol{f}}$, although f^θ does not determine a Hamiltonian vector field relative to $\mathrm{d}\langle \boldsymbol{\eta}, \theta\rangle$ uniquely. More in detail,

$$\begin{aligned}\langle\{\boldsymbol{f}, \boldsymbol{g}\}_{\boldsymbol{\eta}}, \theta\rangle &= \mathrm{d}\langle \boldsymbol{\eta}, \theta\rangle(X_{\boldsymbol{f}}, X_{\boldsymbol{g}}) = \iota_{X_{\boldsymbol{g}}}\mathrm{d}f^\theta = \mathrm{d}\langle \boldsymbol{\eta}, \theta\rangle(X_{f^\theta}, X_{\boldsymbol{g}})\\ &= \mathrm{d}\langle \boldsymbol{\eta}, \theta\rangle(X_{f^\theta}, X_{g^\theta}) = \{f^\theta, g^\theta\}_\omega.\end{aligned}$$

The fact that C is a constant of motion of X follows from standard presymplectic geometry and the fact that X admits a t-dependent Hamiltonian of the form $h = \sum_{\alpha=1}^r b_\alpha(t)\langle \boldsymbol{h}_\alpha, \theta\rangle$ for certain t-dependent functions $b_1(t), \ldots, b_r(t)$. □

Theorem 6.3. *Consider the Lie algebra $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$ relative to $\{\cdot,\cdot\}_{\boldsymbol{\eta}}$ and the space $\mathscr{C}^\infty_{\boldsymbol{\eta},\theta}(M)$ of functions of the form $\langle \boldsymbol{f}, \theta\rangle$ for a certain $\boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k)$ and a fixed $\theta \in \mathbb{R}^{k*}$. The morphism*

$$\boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k) \cap \ker \boldsymbol{R} \mapsto f^\theta = -\langle \boldsymbol{f}, \theta\rangle \in \mathscr{C}^\infty_{\boldsymbol{\eta},\theta}(M) \cap \ker \boldsymbol{R} \tag{6.2}$$

is a Poisson algebra morphism relative to the product $\boldsymbol{f} \star \boldsymbol{g} = \sum_{\alpha=1}^r (f^\alpha g^\alpha) e_\alpha$ and the bracket $\{\cdot,\cdot\}_{\boldsymbol{\eta}}$ in $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M, \mathbb{R}^k) \cap \ker \boldsymbol{R}$ and the Poisson bracket of $\langle \mathrm{d}\boldsymbol{\eta}, \theta\rangle$ and the standard multiplication of functions in $\mathscr{C}^\infty_{\boldsymbol{\eta},\theta}(M) \cap \ker \boldsymbol{R}$.

[c] Note that its rank may not be constant.

Proof. To prove that $\mathscr{C}^\infty_{\boldsymbol{\eta}}(M,\mathbb{R}^k)\cap\ker\boldsymbol{R}$ is a Poisson algebra, take $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$ belonging to $\ker\boldsymbol{R}$ and see that

$$\{\boldsymbol{f},\boldsymbol{g}\star\boldsymbol{h}\} = -X_{\boldsymbol{f}}\left(\sum_{\alpha=1}^k g^\alpha h^\alpha e_\alpha\right) = -\sum_{\alpha=1}^k X_{\boldsymbol{f}}\left(g^\alpha\right)h^\alpha e_\alpha - \sum_{\beta=1}^k g^\beta X_{\boldsymbol{f}}(h^\beta)e_\beta$$
$$= \{\boldsymbol{f},\boldsymbol{g}\}\star\boldsymbol{h}+\boldsymbol{g}\star\{\boldsymbol{f},\boldsymbol{h}\}.$$

Moreover, since $\mathscr{C}^\infty_{\boldsymbol{\eta},\theta}(M)\cap\ker\boldsymbol{R}$ is an $\mathbb{R}$-algebra and the morphism (6.2) is an $\mathbb{R}$-algebra morphism, the morphism is a Poisson algebra morphism. □

One important thing about k-contact Lie systems is the determination of regions of the manifold that are invariant relative to its dynamics. Let us provide a first result about this.

Proposition 6.4. *Let X be a k-contact Lie system with an r-dimensional Vessiot–Guldberg Lie algebra $V=\langle X_1,\dots,X_r\rangle$ of $\boldsymbol{\eta}$-Hamiltonian vector fields with $\boldsymbol{\eta}$-Hamiltonian k-functions $\boldsymbol{h}_1,\dots,\boldsymbol{h}_r$ for $X_1,\dots,X_r$, respectively. Let $\mathfrak{W}=\langle \mathrm{h}_1,\dots,\boldsymbol{h}_r\rangle$ be a Lie–Hamilton algebra of $\boldsymbol{\eta}$-Hamiltonian k-functions isomorphic to an abstract r-dimensional Lie algebra $\mathfrak{g}$. Then, one obtains a map $\boldsymbol{J}: x\in M\mapsto \sum_{\alpha=1}^r\boldsymbol{h}_\alpha(x)v^\alpha\in\mathfrak{g}^*\otimes\mathbb{R}^k$, where $\{v^1,\dots,v^r\}$ is a basis for $\mathfrak{g}^*$. For every $\theta\in\mathbb{R}^{k*}$ such that $R_{\boldsymbol{h}_1},\dots,R_{\boldsymbol{h}_r}$ are tangent to the zero level of $J^\theta=\langle\boldsymbol{J},\theta\rangle$, which is assumed to be a submanifold, one has that $(J^\theta)^{-1}(0)$ is an invariant subset of the dynamics of the system, provided it is a submanifold.*

Proof. Note that $X_{\boldsymbol{h}}=\sum_{\alpha=1}^k b_\alpha(t)X_{\boldsymbol{h}_\alpha}$ and

$$\frac{\mathrm{d}\boldsymbol{J}^\theta}{\mathrm{d}t}=\frac{\partial\boldsymbol{J}^\theta}{\partial t}+X_{\boldsymbol{h}}\boldsymbol{J}^\theta=\sum_{\alpha,\beta=1}^k b_\alpha(t)\langle X_{\boldsymbol{h}_\alpha}\boldsymbol{h}_\beta,\theta\rangle=-\sum_{\alpha,\beta=1}^k b_\beta(t)\langle\{\boldsymbol{h}_\alpha,\boldsymbol{h}_\beta\}+R_{\boldsymbol{h}_\beta}\boldsymbol{h}_\alpha,\theta\rangle.$$

Recall that $\boldsymbol{h}_1,\dots,\boldsymbol{h}_r$ close a Lie algebra of functions, namely $\{\boldsymbol{h}_\alpha,\boldsymbol{h}_\beta\}=\sum_{\gamma=1}^r c_{\alpha\beta}{}^\gamma\boldsymbol{h}_\gamma$ for certain constants $c_{\alpha\beta}{}^\gamma$, and one has

$$\langle R_{\boldsymbol{h}_\beta}\boldsymbol{h}_\alpha,\theta\rangle=\sum_{\mu=1}^k h^\mu_\beta R_\mu\langle\boldsymbol{h}_\alpha,\theta\rangle.$$

Take into account that $\langle\boldsymbol{J},\theta\rangle=\sum_{\alpha=1}^r\langle\boldsymbol{h}_\alpha v^\alpha,\theta\rangle=\sum_{\alpha=1}^r\langle\boldsymbol{h}_\alpha,\theta\rangle v^\alpha=0$ and $R_1,\dots,R_k$ are tangent to the level set $(J^\theta)^{-1}(0)$. Hence, on such a submanifold, $\langle\boldsymbol{h}_\alpha,\theta\rangle=0$ and $R_{\boldsymbol{h}_\beta}\langle\boldsymbol{h}_\alpha,\theta\rangle=0$ for $\alpha,\beta=1,\dots,r$. Thus, $X_{\boldsymbol{h}}$ is tangent to the zero level set of J^θ. □

Example 6.5. Consider Example 5.5 and the t-dependent vector field $X_J=\sum_{\alpha=3}^5 b_\alpha(t)X_\alpha$. Assume $\theta=e_2$. One has the mapping

$$J^\theta=\langle\boldsymbol{J},\theta\rangle=(p_2,p_2q,-z_2/4+qp_2/2).$$

Then $(J^\theta)^{-1}(0)$ is the submanifold given by $p_2 = 0$ and $z_2 = 0$. Note that

$$R_{\boldsymbol{h}_3} = p_1\frac{\partial}{\partial z_1} + p_2\frac{\partial}{\partial z_2}, \quad R_{\boldsymbol{h}_4} = p_1 q\frac{\partial}{\partial z_1} + p_2 q\frac{\partial}{\partial z_2},$$

$$R_{\boldsymbol{h}_5} = \left(-z_1 + \frac{qp_1}{2}\right)\frac{\partial}{\partial z_1} + \left(-\frac{z_2}{4} + \frac{qp_2}{2}\right)\frac{\partial}{\partial z_2}$$

and $R_{\boldsymbol{h}_\beta}\langle \boldsymbol{h}_\alpha, \theta\rangle = 0$ for $\alpha, \beta = 3, 4, 5$ on $(J^\theta)^{-1}(0)$. Then, $(J^\theta)^{-1}(0)$ is an invariant submanifold of the dynamics of the system.

Let us now describe a theory to derive t-dependent constants of motion for k-contact Lie systems via k-contact Lie–Hamilton algebras. The main problem to extend the results of Lie–Hamilton systems [2] to this new realm is the fact that the Lie bracket on $\mathfrak{W}$ is not a Poisson bracket and describing the variation of a t-dependent function in time does not depend only on the Lie–Hamilton algebra, but it also depends on the action of the Reeb vector fields on them, which can be difficult to analyze. A possible solution is given below.

Proposition 6.6. *Let $(M, \boldsymbol{\eta}, X)$ be a k-contact Lie system such that the Reeb derivations of the elements of an $\boldsymbol{\eta}$-Lie–Hamilton algebra leave invariant the $\boldsymbol{\eta}$-Lie–Hamilton algebra, namely $R_{\boldsymbol{h}_\alpha}\boldsymbol{h}_\beta = \sum_{\gamma=1}^r \lambda_{\alpha\beta}{}^\gamma \boldsymbol{h}_\gamma$ for certain constants $\lambda_{\alpha\beta}{}^\gamma$ with $\alpha, \beta, \gamma = 1, \ldots, r$. Then, the t-dependent constants of motion of X of the form $I_\theta = \langle \sum_{\alpha=1}^r f^\alpha(t)\boldsymbol{h}_\alpha, \theta\rangle$ for some $\theta \in \mathbb{R}^{k*}$ and t-dependent functions $f_1(t), \ldots, f_r(t)$ are solutions to*

$$\frac{df^\alpha}{dt} = -\sum_{\nu,\beta=1}^r f^\nu b_\beta[c_{\nu\beta}{}^\alpha - \lambda_{\nu\beta}{}^\alpha], \tag{6.3}$$

if $\langle h_1^\theta, \ldots, h_r^\theta\rangle$ are linearly independent.

Proof. If $\boldsymbol{I} = \sum_{\alpha=1}^r f^\alpha(t)\boldsymbol{h}_\alpha$, then one has

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t}(t) = \frac{\partial \boldsymbol{I}}{\partial t}(t) + (X_{\boldsymbol{h}})_t\boldsymbol{I}_t = \sum_{\alpha=1}^r\left(\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}(t)\boldsymbol{h}_\alpha + \sum_{\beta=1}^r b_\beta(t)f^\alpha(t)X_{\boldsymbol{h}_\beta}\boldsymbol{h}_\alpha\right). \tag{6.4}$$

Hence,

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \sum_{\alpha=1}^r\left(\frac{\mathrm{d}f^\alpha}{\mathrm{d}t}\boldsymbol{h}_\alpha - \sum_{\beta=1}^r b_\beta(t)f^\alpha(\{\boldsymbol{h}_\beta, \boldsymbol{h}_\alpha\} + R_{\boldsymbol{h}_\alpha}\boldsymbol{h}_\beta)\right).$$

For t-dependent constants of the motion and composing both sides of the above expression with θ, one has that

$$\begin{aligned}\sum_{\alpha=1}^r \frac{\mathrm{d}f^\alpha}{\mathrm{d}t}h_\alpha^\theta &= \sum_{\alpha,\beta=1}^r b_\beta(t)f^\alpha(\{\boldsymbol{h}_\beta, \boldsymbol{h}_\alpha\}^\theta + R_{\boldsymbol{h}_\alpha}h_\beta^\theta)\\ &= -\sum_{\alpha,\beta=1}^r b_\beta(t)f^\alpha\sum_{\nu=1}^r(c_{\alpha\beta}{}^\nu h_\nu^\theta - \lambda_{\alpha\beta}{}^\nu h_\nu^\theta).\end{aligned}$$

If the functions h_α^θ are linearly independent, it follows (6.3). □

The main result of Proposition 6.6 is the fact that some t-dependent constants of motion for a k-contact Lie system can be described via another Lie system (6.3). This illustrates one of the reasons why k-contact geometry is interesting for k-contact Lie systems. On the other hand, these examples also showed that it is natural to associate every k-contact $\boldsymbol{\eta}$-Hamiltonian function with a vector field taking values in the Reeb distribution.

Remark 6.7. Note that (6.4) and the conditions in Proposition 6.6 are the key to many new methods. Assume that I_θ is t-independent, the Lie algebra $\langle[\mathfrak{W},\mathfrak{W}],\theta\rangle$ of a k-contact Lie–Hamilton algebra $\mathfrak{W}$ consists of constants and the $R_{\boldsymbol{h}_\alpha}\langle \boldsymbol{h}_\beta,\theta\rangle$ are constants (as it will happen in some examples of this work). Under the given conditions, one has, in view of (6.4), that $b(t)=\frac{dI_\theta}{dt}$ for a t-dependent function $b(t)$. Hence, $\int^t b(t')\mathrm{d}t' - I_\theta$ is a constant of motion for $(M,\boldsymbol{\eta},X)$.

Recall that a *generalized master symmetry of order* m for a system of differential equations associated with a vector field X on a manifold M is a vector field Z on M such that $\mathrm{ad}_X^m Z=0$, where $\mathrm{ad}_X^m=(m-\text{times})\mathrm{ad}_X\circ\ldots\circ\mathrm{ad}_X$ (see [27]). Generalized master symmetries allow one to obtain functions, the *generators of constants of motion of order* m, whose derivative of order m in terms of the time is zero, but the derivative of order $m-1$ is not. The same idea, which appears in symplectic mechanics, can be applied to t-independent projectable k-contact Lie systems. For instance, assume that $\mathfrak{W}_\theta$ is nilpotent of order m consisting of first integrals of the Reeb vector fields of a k-contact Lie system $(M,\boldsymbol{\eta},X)$. Consider also the η-Lie Hamilton algebra $\mathfrak{W}$ for V^X. In such a case,

$$\frac{\mathrm{d}^m I_\theta}{\mathrm{d}t^m}=(-1)^m\overbrace{\{h_\theta,\{h_\theta\ldots,\{h_\theta,I_\theta\}\ldots\}}^{m\text{-times}}=0,\quad I_\theta\in\mathfrak{W}^\theta,$$

and I_θ is a t-dependent function of order, at most $m-1$, on solutions. Meanwhile, X_{I_θ} becomes a master symmetry of order m of X.

7. *k*-Contact Lie Systems and Other Geometric Structures

It is worth noting that a Lie system may potentially be Hamiltonian relative to different types of geometric structures. In some cases, some of them can be more appropriate than others. The following proposition shows how projectable k-contact Lie systems induce some k-symplectic Lie systems on other spaces.

Proposition 7.1. *If $(M,\boldsymbol{\eta},X)$ is a projectable k-contact Lie system, the space of integral curves of the Reeb distribution $\mathcal{D}^R:=\ker\mathrm{d}\boldsymbol{\eta}$, let us say $M/\mathcal{D}^R$, is a manifold, and $\mathfrak{p}_R: M\to M/\mathcal{D}^R$ is the canonical projection that becomes a submersion, then $(M/\mathcal{D}^R,\boldsymbol{\omega},\mathfrak{p}_{R*}X)$, where $\mathfrak{p}_R^*\boldsymbol{\omega}=\mathrm{d}\boldsymbol{\eta}$, is a k-symplectic Lie system relative to the k-symplectic form $\boldsymbol{\omega}$ on $M/\mathcal{D}^R$, namely the vector fields $\mathfrak{p}_{R*}X_t$, with $t\in\mathbb{R}$, are $\boldsymbol{\omega}$-Hamiltonian relative to the k-symplectic form $\boldsymbol{\omega}$.*

Proof. Since $(M,\boldsymbol{\eta},X)$ is projectable, the Reeb vector fields $R_1,\ldots,R_k$ commute with the $\boldsymbol{\eta}$-Hamiltonian vector fields of V^X. Therefore, all the elements of V^X are projectable onto $M/\mathcal{D}^R$ and their projections span a finite-dimensional Lie algebra of vector fields on $M/\mathcal{D}^R$. Moreover, the $\boldsymbol{\eta}$-Hamiltonian k-functions of the elements of V^X are first integrals of the Reeb vector fields of $\boldsymbol{\eta}$. Hence, they are also projectable. Moreover, $\mathscr{L}_{R_\alpha}\mathrm{d}\boldsymbol{\eta} = 0$ and $\iota_{R_\alpha}\mathrm{d}\boldsymbol{\eta} = 0$ for $\alpha = 1, \ldots, k$. Hence, $\mathrm{d}\boldsymbol{\eta}$ can be projected onto $M/\mathcal{D}^R$. In other words, there exists a unique two-form, $\boldsymbol{\omega}$, on $M/\mathcal{D}^R$ such that $\mathfrak{p}_R^*\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\eta}$. Note that $\boldsymbol{\omega}$ is closed. Moreover, if $\iota_{Y_{[x]}}\boldsymbol{\omega}_{[x]} = 0$ for a tangent vector $Y_{[x]} \in T_{[x]}(M/\mathcal{D}^R)$ where $[x]$ is the leave of $\mathcal{D}^R$ passing through a point $x \in M$, then there exists a tangent vector $\widetilde{Y}_x \in T_xM$ such that $T_x\mathfrak{p}_R(\widetilde{Y}_x) = Y_{[x]}$. Hence, $T_x\mathfrak{p}_R^T\iota_{Y_{[x]}}\boldsymbol{\omega}_{[x]} = \iota_{\widetilde{Y}_x}(\mathrm{d}\boldsymbol{\eta})_x = 0$ and $\widetilde{Y}_x \in \ker(\mathrm{d}\boldsymbol{\eta})_x$. Moreover, $\mathfrak{p}_{R*x}\widetilde{Y}_x = 0$ and $\boldsymbol{\omega}$ is non-degenerate. Since $\boldsymbol{\omega}$ is closed too, it becomes a k-symplectic form and the vector fields of $\mathfrak{p}_{R*}V^X$ span a finite-dimensional Lie algebra of Hamiltonian vector fields relative to $\boldsymbol{\omega}$. In fact, given a basis of $\boldsymbol{\eta}$-Hamiltonian k-functions $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_r$ for a basis $X_1, \ldots, X_R$ of vector fields of V^X, one may write that $\boldsymbol{h}_\alpha = \mathfrak{p}_R^*\boldsymbol{g}_\alpha$ for $\alpha = 1, \ldots, k$ and some k-functions $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r \in \mathscr{C}^\infty(M/\mathcal{D}^R, \mathbb{R}^k)$. Moreover,

$$\mathfrak{p}_R^*(\iota_{\mathfrak{p}_{R*}X_\alpha}\boldsymbol{\omega}) = \iota_{X_\alpha}\mathrm{d}\boldsymbol{\eta} = \mathrm{d}\boldsymbol{h}_\alpha - \sum_{\mu,\beta=1}^{k}(R_\beta h_\alpha^\mu)\eta^\beta \otimes e_\mu = \mathrm{d}\mathfrak{p}_R^*\boldsymbol{g}_\alpha = \mathfrak{p}_R^*(\mathrm{d}\boldsymbol{g}_\alpha).$$

Hence, $\mathfrak{p}_{R*}X_\alpha$ is $\boldsymbol{\omega}$-Hamiltonian relative to $\boldsymbol{\omega}$. Therefore, the time-dependent vector field $\mathfrak{p}_{R*}X$, namely the t-parametric family of vector fields $(\mathfrak{p}_{R*}X)_t = \mathfrak{p}_{R*}X_t$ for every $t \in \mathbb{R}$, becomes a k-symplectic Lie system relative to $\boldsymbol{\omega}$. □

Let us provide a proposition that allows one to study k-contact Lie systems via presymplectic Lie systems. Notwithstanding, this implies that one has to study a problem on a higher-dimensional manifold with different properties than the initial one, which may make the problem harder to study in some cases [22], but easier in others. It is worth noting that Proposition 7.2 is a particular case of [23, Proposition 7.14] focusing on a particular case of relevance for us.

Proposition 7.2. *Every k-contact Lie system $(M, \boldsymbol{\eta}, X_h)$ gives rise to a presymplectic Lie system $(\mathbb{R}^k \times M, \boldsymbol{\omega} = \mathrm{d}(\sum_{\alpha=1}^k z_\alpha\widehat{\eta}^\alpha), \sum_{\alpha,\beta=1}^k z_\alpha(R_\beta h^\alpha)\partial/\partial z^\beta + X_{\boldsymbol{h}})$, where $\hat{\eta}^\alpha$ is the lift to $\mathbb{R}^k \times M$, via the natural projection from $\mathbb{R}^k \times M$ onto M, of η^α for $\alpha = 1, \ldots, k$.*

As it happens for the similar proposition in the contact case, Proposition 7.2 may be inappropriate to study k-contact Hamiltonian systems on M via Hamiltonian systems on presymplectic manifolds $\mathbb{R}^k \times M$ since the dynamics of a k-contact Hamiltonian vector field on M may significantly differ from the presymplectic Hamiltonian system on $\mathbb{R}^k \times M$ used to investigate it. For instance, a k-contact Hamiltonian vector field X on M may have stable points, while $\sum_{\alpha,\beta=1}^k z_\alpha(R_\beta h^\alpha)\partial/\partial z^\beta + X_{\boldsymbol{h}}$, which is its associated Hamiltonian vector field on $\mathbb{R}^k \times M$, has not. This has relevance in certain theories, like the energy–momentum method [49]. Additionally, enlarging the manifold where the dynamics of a system is considered poses a problem that should have more advantages than drawbacks.

Example 7.3. Note that Example 5.4 satisfies the conditions of Proposition 7.1. Then, the Reeb vector fields have constants of motion given by the common first integrals of the vector fields

$$x_1\frac{\partial}{\partial x_1}+v_1\frac{\partial}{\partial v_1},\qquad x_2\frac{\partial}{\partial x_2}+v_2\frac{\partial}{\partial v_2}.$$

We can define the first integrals $f_1 = v_1/x_1$ and $f_2 = v_2/x_2$ of the previous vector fields. Hence, the integral leaves of the distribution spanned by Y_4, Y_2 can be described by means of the surfaces with f_1, f_2 equal to constants. Hence, f_1, f_2 are coordinates in the space of leaves. In those coordinates, the $\boldsymbol{\eta}_{is}$-Hamiltonian vector fields $X_1, \ldots, X_4$ read

$$\begin{aligned}
X_1 &= f_1x_1\frac{\partial}{\partial x_1}+f_2x_2\frac{\partial}{\partial x_2}-f_1^2\frac{\partial}{\partial f_1}-f_2^2\frac{\partial}{\partial f_2},\\
X_2 &= \frac{1}{2}x_1\frac{\partial}{\partial x_1}+\frac{1}{2}x_2\frac{\partial}{\partial x_2}-f_1\frac{\partial}{\partial f_1}-f_2\frac{\partial}{\partial f_2},\\
X_3 &= -\frac{\partial}{\partial f_1}-\frac{\partial}{\partial f_2},\qquad X_4 = x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2},
\end{aligned}$$

and the differential of two-contact form becomes

$$\mathrm{d}\boldsymbol{\eta}_{is} = 2\Upsilon^1\wedge\Upsilon^3\otimes e_1 = \left(\frac{2}{(f_2-f_1)^2}\mathrm{d}f_1\wedge\mathrm{d}f_2\right)\otimes e_1.$$

We can define a mapping $\mathfrak{p}_R\colon (f_1, f_2, x_1, x_2) \in \mathbb{R}^4 \mapsto (f_1, f_2) \in \mathbb{R}^2$ that projects the initial space onto the space of leaves, which, by Proposition 7.1, is the two-symplectic manifold with two-symplectic form $\boldsymbol{\omega}$ defined by $\boldsymbol{p}_R^*\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\eta}_{is}$. Moreover, $\boldsymbol{p}_{R_*}X_i$, where $i = 1, \ldots, 4$, are given by

$$\begin{aligned}
\boldsymbol{p}_{R_*}X_1 &= -f_1^2\frac{\partial}{\partial f_1}-f_2^2\frac{\partial}{\partial f_2},\qquad \boldsymbol{p}_{R_*}X_2 = -f_1\frac{\partial}{\partial f_1}-f_2\frac{\partial}{\partial f_2},\\
\boldsymbol{p}_{R_*}X_3 &= -\frac{\partial}{\partial f_1}-\frac{\partial}{\partial f_2},
\end{aligned}$$

and $\boldsymbol{p}_{R_*}X_4 = 0$, are Hamiltonian vector fields relative to the two-symplectic form $\mathrm{d}\boldsymbol{\eta}_{is}$, with Hamiltonian k-functions on $\mathbb{R}^2$ of the form

$$\begin{aligned}
\boldsymbol{h}_1' &= -\frac{2f_1f_2}{f_1-f_2}e_1,\qquad \boldsymbol{h}_2' = \left(1+\frac{2f_1}{f_2-f_1}\right)e_1,\\
\boldsymbol{h}_3' &= -\frac{2}{f_1-f_2}e_1,\qquad \boldsymbol{h}_4' = -e_2.
\end{aligned}$$

Note that $\boldsymbol{p}_R^*\boldsymbol{h}_i' = \boldsymbol{h}_i$ for $i = 1, \ldots, 4$.

It is interesting to analyze how to recover the solution to the initial projectable $\boldsymbol{\eta}$-contact Lie system from the information of the projected k-symplectic one. Some particular results in this direction have recently appeared in the literature [8]. We expect to investigate this topic further in the future.

8. Methods to Construct k-Contact Lie Systems

Let us develop a method to turn a Lie system X into a k-contact Lie system by finding a compatible k-contact form, i.e. one turning V^X into $\boldsymbol{\eta}$-Hamiltonian vector fields for a certain k-contact form $\boldsymbol{\eta}$. This will be more or less explicitly used in further sections to determine potential applications of k-contact Lie systems. Our method for deriving $\boldsymbol{\eta}$ relies on distributions, a strategy that generally simplifies the process compared to directly calculating $\boldsymbol{\eta}$.

Recall that a *locally automorphic Lie system* is a triple (M, X, V), where X is a Lie system on a manifold M so that V is a Lie algebra of dimension $\dim M$ whose vector fields span the tangent space to M. It is known [24, 33] that locally automorphic Lie systems are locally diffeomorphic to a Lie system on a Lie group G of the form

$$X^R = \sum_{\alpha=1}^{r} b_\alpha(t) X^R_\alpha, \tag{8.1}$$

where $X^R_1, \ldots, X^R_r$ is a basis of right-invariant vector fields on G spanning a Lie algebra isomorphic to V and $b_1(t), \ldots, b_r(t)$ are t-dependent functions. This structure is extremely useful, as one can treat locally automorphic Lie systems on general manifolds as Lie systems of a very specific type on Lie groups. Moreover, there are many applications of locally automorphic Lie systems [9, 13, 24]. For instance, there are many locally automorphic Lie systems related to control systems (cf. [55]) and we will study examples of this type in the following sections.

Consider now the Maurer–Cartan equations on G, which satisfy that

$$\mathrm{d}\eta^\alpha_L + \frac{1}{2}\sum_{\beta,\gamma=1}^{r} c_{\beta\gamma}{}^{\alpha}\eta^\beta_L \wedge \eta^\gamma_L = 0, \quad \alpha = 1, \ldots, r, \tag{8.2}$$

where $c_{\beta\gamma}{}^{\alpha}$, with $\alpha, \beta, \gamma = 1, \ldots, r$, are the constants of structure of a basis of left-invariant vector fields on G given by $X^L_1, \ldots, X^L_r$, while $\eta^1_L, \ldots, \eta^r_L$ is its dual basis, which consists of left-invariant one-forms on G. The above expression (8.2) follows immediately by evaluating $\mathrm{d}\eta^\alpha_L$ on pairs of left-invariant vector fields chosen among $X^L_1, \ldots, X^L_r$. Then, the differentials of left-invariant one-forms can be determined by the Lie algebra structure of V.

In particular, it is simple to find Lie algebras admitting k particular different indexes $\alpha_1, \ldots, \alpha_k \in \{1, \ldots, r\}$ so that $c_{\alpha_j\beta}{}^{\alpha_i} = 0$ for every β and $i, j = 1, \ldots, k$. In view of (8.2), this implies that the elements of $\langle X^L_{\alpha_1}, \ldots, X^L_{\alpha_k}\rangle$ take values in $\bigcap_{i=1}^k \ker \mathrm{d}\eta^{\alpha_i}_L$. If additionally $\bigcap_{i=1}^k \ker \mathrm{d}\eta^{\alpha_i}_L = \langle X^L_{\alpha_1}, \ldots, X^L_{\alpha_k}\rangle$, then $\boldsymbol{\eta} = \sum_{i=1}^k \eta^{\alpha_i}_L \otimes e_{\alpha_i}$ is a k-contact form that is invariant relative to the vector fields $X^R_1, \ldots, X^R_r$. Recall that Lie groups of this type are related to the so-called *k-contact Lie groups*

[23], namely Lie groups with a left-invariant k-contact form.[d] In practice, it is relatively simple to determine Lie groups of this type, which will be shown in latter examples. Moreover, one-contact Lie groups were analyzed in [22]. On the opposite, Lie groups not admitting a left-invariant contact form were discussed in [36] in the context of hydrodynamic equations.

Right-invariant vector fields leave invariant $\ker \boldsymbol{\eta}$, which can be spanned by left-invariant vector fields, which implies that the vector fields of the Vessiot–Guldberg Lie algebra $\langle X_1^R, \ldots, X_r^R \rangle$ consist of $\boldsymbol{\eta}$-Hamiltonian vector fields relative to $\boldsymbol{\eta}$. With no loss of generality (see [23] for details), the basis of left-invariant vector fields of a k-contact Lie group can be chosen so that the Reeb vector fields are given by $X_1^L, \ldots, X_k^L$, which commute with $X_1^R, \ldots, X_r^R$. Then, the Lie systems of the form

$$X^R = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R$$

are projectable k-contact Lie systems regardless of the functions $b_1(t), \ldots, b_r(t)$. Indeed, it follows that the $\boldsymbol{\eta}$-Hamiltonian k-functions of $X_1^R, \ldots, X_r^R$ are first integrals of the Reeb vector fields, which are left-invariant, and the Lie system can be projected onto $G/\ker \mathrm{d}\boldsymbol{\eta}$, where $\ker \mathrm{d}\boldsymbol{\eta}$ is the distribution given by the Reeb vector fields. More specifically, it turns out that the k-contact Lie system is projectable onto a k-symplectic one [25].

There is another manner to obtain, in general non-projectable, k-contact Lie systems from locally automorphic ones. As before, we analyze a locally automorphic Lie system via its associated Lie system (8.1) on a Lie group. Consider that the Lie algebra of the Lie group is such that there exists a linear subspace spanned by left-invariant vector fields, $X_1^L, \ldots, X_{r-k}^L$, that is maximally non-integrable, i.e. it admits no element leaving invariant the subspace via the commutator of vector fields, which is relatively straightforward to determine due to the Lie algebra structure of left-invariant vector fields. Consider also that there exists a supplementary abelian Lie subalgebra assumed to have, with no loss of generality, a basis of the form $X_{r-k+1}^L, \ldots, X_r^L$. Then, let us analyze the maximally non-integrable distribution spanned by

$$\mathcal{D} = \langle X_1^L, \ldots, X_{r-k}^L \rangle$$

and the commutative Lie algebra

$$\langle X_{r-k+1}^R, \ldots, X_r^R \rangle.$$

Note that the right-invariant vector fields are chosen so that $X_\alpha^R(e) = X_\alpha^L(e)$ for $\alpha = 1, \ldots, r$, which makes them to have opposite commutation relations than left-invariant vector fields. The latter right-invariant vector fields $X_{r-k+1}^R, \ldots, X_r^R$ leave $\mathcal{D}$ invariant and commute among themselves. This gives rise to a k-contact

[d]Note that the approach that follows could be developed similarly with a basis of left-invariant vector fields and a right-invariant k-contact form.

distribution $\mathcal{D}$ and the vector fields $X^R_{r-k+1},\dots,X^R_r$ leave it invariant. Hence, $X^R_1,\dots,X^R_r$ become k-contact vector fields and one obtains a k-contact Lie system related to (8.1). In this approach, the k-contact form must be obtained as in the proof of Theorem 4.15. Namely, one has to derive the k one-forms $\eta^{r-k+1},\dots,\eta^r$ that annihilate $\langle X^L_1,\dots,X^L_{r-k}\rangle$ and are duals to the vector fields $X^R_{r-k+1},\dots,X^R_r$. It is worth noting that this approach works, at least locally at a point where $X^L_1,\dots,X^L_{r-k},X^R_{r-k+1},\dots,X^R_r$ are linearly independent, which can be always found locally at almost every point. The obtained one-forms are the components of a k-contact form $\boldsymbol{\eta}=\sum_{\alpha=r-k+1}^r\eta^\alpha_L\otimes e_\alpha$ turning the vector fields $X^R_1,\dots,X^R_r$ into $\boldsymbol{\eta}$-Hamiltonian vector fields. Note that the k-contact form $\boldsymbol{\eta}$ described above is not, in general, a left-invariant one-form, while X^R does not need to be projectable.

There are several manners to obtain a maximally non-integrable $\mathcal{D}$. For instance, if $[X^L_1,X^L_2]\wedge X^L_1\wedge X^L_2$ is not vanishing, then $\mathcal{D}$ is maximally non-integrable. This is a relatively straightforward case to identify, as it suffices to locate two left-invariant vector fields X and Y, such that their commutator $[X,Y]$ does not take values within the distribution spanned by X and Y at any point. Note that if $[X,Y]$ does not take values in the distribution spanned by X,Y at any point, then it is impossible that $X\wedge Y=0$ on an open set as, if this would happen,

$$\begin{aligned}0&=\mathcal{L}_X(X\wedge Y)=X\wedge\mathcal{L}_XY=\mathcal{L}_Y(X\wedge Y)\\&=\mathcal{L}_YX\wedge Y\quad\Rightarrow\quad[X,Y]\wedge X\wedge Y=0,\end{aligned}$$

which is a contradiction. Moreover, assume that $E=\langle X^L_1,X^L_2,X^L_3\rangle\subset\langle X^L_1,\dots,X^L_r\rangle$ is such that there is no element of $Y\in E$ such that $[Y,E]$ is not included in E. Then, E is also maximally non-integrable. These ideas to obtain maximally non-integrable distributions will be used in our applications in Sec. 10.

9. Diagonal Prolongation of k-Contact Lie Systems

It is possible to describe superposition rules of Lie systems geometrically. To do so, it is convenient to introduce the so-called *diagonal prolongation* of a t-dependent vector field and its basic properties [14]. Before that, let us introduce the projections

$$\mathfrak{pr}\colon M^{\ell+1}\ni(x_{(0)},\dots,x_{(\ell)})\longmapsto(x_{(1)},\dots,x_{(\ell)})\in M^\ell,\tag{9.1}$$

$$\mathfrak{pr}_0^\alpha\colon M^{\ell+1}\ni(x_{(0)},\dots,x_{(\ell)})\longmapsto x_{(\alpha)}\in M,\tag{9.2}$$

for $\alpha=0,\dots,\ell$. Recall that the group $S^{\ell+1}$ of permutations $x_{(\alpha)}\leftrightarrow x_{(\beta)}$, with $0\le\alpha\ne\beta\le\ell$, acts on $M^{\ell+1}$.

Definition 9.1. Given a t-dependent vector field on M locally of the form

$$X(t,x_{(0)})=\sum_{i=1}^n X^i(t,x_{(0)})\frac{\partial}{\partial x^i_{(0)}},\tag{9.3}$$

its *diagonal prolongation* to $M^{\ell+1}$ is the t-dependent vector field on $M^{\ell+1}$ given by

$$X^{[\ell+1]}(t, x_{(0)}, \ldots, x_{(\ell)}) = \sum_{a=0}^{\ell} \sum_{i=1}^{n} X^i(t, x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}. \tag{9.4}$$

This definition can be rewritten in the following manner.

Definition 9.2. Given a t-dependent vector field X on M, its *diagonal prolongation* to $M^{\ell+1}$ is the unique t-dependent vector field $X^{[\ell+1]}$ on $M^{\ell+1}$ such that:

(i) The t-dependent vector field $X^{[\ell+1]}$ is invariant under the action of the symmetry group $S^{\ell+1}$ on $M^{\ell+1}$;
(ii) The vector fields $X_t^{[\ell+1]}$ are projectable under the projections $\mathfrak{pr}_0^\alpha$ given by (9.2) and $\mathfrak{pr}_{0*}^\alpha X_t^{[\ell+1]} = X_t$ for every $t \in \mathbb{R}$.

Several lemmas, covered in [9, 14], describe properties of the diagonal projection necessary to formulate the following proposition (see [9, pp. 18–21] for details).

Proposition 9.3. *For every family of linearly independent (over* $\mathbb{R}$*) vector fields* $X_1, \ldots, X_r \in \mathfrak{X}(M)$*, there exists an integer* ℓ *with* $\ell \dim M \geq r$ *such that their prolongations to* M^ℓ *are linearly independent at a generic point.*

In the contact setting, the diagonal prolongation of a contact Lie system is not a contact Lie system because the manifold of the diagonal prolongation is not always odd. Nevertheless, k-contact distributions allow for a diagonal prolongation procedure as follows.

Proposition 9.4. *The diagonal prolongation to* $N^{\ell+1}$ *of a* k*-contact Lie system on* N *is a* $k(\ell+1)$*-contact Lie system on* $N^{\ell+1}$*.*

Proof. Let $X_1, \ldots, X_r$ span the smallest Lie algebra V^X of the k-contact Lie system $(N, \boldsymbol{\eta}_N, X)$. There exists the $k(\ell+1)$-contact form

$$\boldsymbol{\eta}_{N^{\ell+1}} = \sum_{\alpha=0}^{\ell} \boldsymbol{\eta}_N^\alpha = \sum_{\alpha=0}^{\ell} \sum_{\beta=1}^{k} \eta_\beta^\alpha \otimes e_{\alpha k+\beta},$$

where $\boldsymbol{\eta}_N^\alpha = \mathfrak{pr}_0^{\alpha *} \boldsymbol{\eta}_N$. Indeed,

$$\begin{aligned} \ker \boldsymbol{\eta}_{N^{\ell+1}} &= \ker \boldsymbol{\eta}_N^0 \cap \ker \boldsymbol{\eta}_N^1 \cap \cdots \cap \ker \boldsymbol{\eta}_N^\ell, \\ \ker \mathrm{d}\boldsymbol{\eta}_{N^{\ell+1}} &= \ker \mathrm{d}\boldsymbol{\eta}_N^0 \cap \ker \mathbf{d}\eta_N^1 \cap \cdots \cap \ker \mathrm{d}\boldsymbol{\eta}_N^\ell \end{aligned}$$

and $\ker \boldsymbol{\eta}_{N^{\ell+1}} \cap \ker \mathrm{d}\boldsymbol{\eta}_{N^{\ell+1}} = 0$. Moreover, $\ker \boldsymbol{\eta}_{N^{\ell+1}}$ has corank $(\ell+1)k$ and $\ker \mathrm{d}\boldsymbol{\eta}_{N^{\ell+1}}$ has rank $k(\ell+1)$. Hence, we have obtained a $k(\ell+1)$-contact form.

By contracting each $X_i^{[\ell+1]} = \sum_{\alpha=0}^{\ell} X_i^{(\alpha)}$, where $X_i^{(\alpha)} = X_i(x_{(\alpha)})$ and $i = 1,\ldots,r$, with $\boldsymbol{\eta}_{N^{\ell+1}}$ and its differential

$$\iota_{X_i^{[\ell]}}\boldsymbol{\eta}_{N^\ell} = \sum_{\alpha,\beta=0}^{\ell} \iota_{X_i^{(\alpha)}}\boldsymbol{\eta}_N^\beta = \sum_{\alpha=0}^{\ell} \iota_{X_i^{(\alpha)}}\boldsymbol{\eta}_N^\alpha = -\sum_{\alpha=0}^{\ell} \boldsymbol{h}_\alpha^i, \quad i = 1,\ldots,r,$$

$$\iota_{X_i^{[\ell]}}\mathrm{d}\boldsymbol{\eta}_{N^{\ell+1}} = \sum_{\alpha=0}^{\ell} \iota_{X_i^{(\alpha)}}\mathrm{d}\boldsymbol{\eta}_N^\alpha = \sum_{\alpha=0}^{\ell} \mathrm{d}\boldsymbol{h}_\alpha^i - \sum_{\mu=0}^{\ell}\sum_{\beta=1}^{k} R_\beta^\alpha \boldsymbol{h}_\alpha^i \eta_\beta^\alpha, \quad i = 1,\ldots,r,$$

where R_β^α, for $\beta = 1,\ldots,k$, are Reeb vector fields on $(\alpha+1)$th copy of N in $N^{\ell+1}$. Thus, we obtain that each prolongation is an $\boldsymbol{\eta}_{N^{\ell+1}}$-Hamiltonian vector field with the $\boldsymbol{\eta}_{N^{\ell+1}}$-Hamiltonian $k(\ell+1)$-function $\boldsymbol{h}^i = \sum_{\alpha=0}^{\ell} \boldsymbol{h}_\alpha^i$ for $i = 1,\ldots,r$. □

The reason why the procedure for the diagonal prolongations of contact Lie systems relied on passing to the use of Jacobi setting was because this has a good diagonal prolongation related to Jacobi manifolds. This is no longer necessary in the above, more natural, realm.

There is another relevant fact about the fact that the diagonal prolongations to $N_{\ell+1}$ of $\boldsymbol{\eta}$-Hamiltonian vector fields are $\boldsymbol{\eta}_{N^{\ell+1}}$-Hamiltonian. Consider a k-contact Lie system $(M,\boldsymbol{\eta},X)$ and an $\boldsymbol{\eta}$-Hamiltonian k-function $\boldsymbol{C}$. If $\{h^\theta, C^\theta\}_\theta = 0$, then the $\boldsymbol{\eta}_{N^{\ell+1}}$-Hamiltonian functions of the diagonal prolongations to $N^{\ell+1}$ of $X_{\boldsymbol{h}}$ and $X_{\boldsymbol{C}}$ do also commute, which can be used to obtain constants of motion of $X_{\boldsymbol{h}}^{[\ell+1]}$. This may have applications to obtain superposition rules for $X_{\boldsymbol{h}}$, for instance.

10. Applications of *k*-Contact Lie Systems

This section provides several applications of k-contact Lie systems and illustrates the results obtained in previous sections.

10.1. *A control Lie system*

Let us endow a Lie system related to control systems with a k-contact form. Let us consider the system of differential equations on $\mathbb{R}^5$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{\alpha=1}^{5} b_\alpha(t) X_\alpha, \tag{10.1}$$

where $b_1(t),\ldots,b_5(t)$ are arbitrary t-dependent functions and

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_1^2\frac{\partial}{\partial x_4} + 2x_1x_2\frac{\partial}{\partial x_5},$$

$$X_3 = \frac{\partial}{\partial x_3} + 2x_1\frac{\partial}{\partial x_4} + 2x_2\frac{\partial}{\partial x_5}, \quad X_4 = \frac{\partial}{\partial x_4}, \quad X_5 = \frac{\partial}{\partial x_5}.$$

The above vector fields span a nilpotent Lie algebra V_c of vector fields whose non-vanishing commutation relations read

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = 2X_4, \quad [X_2, X_3] = 2X_5.$$

This makes (10.1) into a Lie system related to a t-dependent vector field $\sum_{\alpha=1}^{5} b_\alpha(t)X_\alpha$ as proved in [55]. The initial motivation to study (10.1) comes from the fact that it covers as a particular case the system of differential equations on $\mathbb{R}^5$ of the form

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= b_1(t), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = b_2(t), \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= b_2(t)x_1, \quad \frac{\mathrm{d}x_4}{\mathrm{d}t} = b_2(t)x_1^2, \\ \frac{\mathrm{d}x_5}{\mathrm{d}t} &= 2b_2(t)x_1x_2, \end{aligned} \tag{10.2}$$

where $b_1(t)$ and $b_2(t)$ are arbitrary t-dependent functions whose interest is due to their relation to certain control problems [52, 55]. Let us define $X_c = b_1(t)X_1 + b_2(t)X_2$.

Since $X_1 \wedge \cdots \wedge X_5 \neq 0$ and $\dim V_c = 5$, one obtains a locally automorphic Lie system $(\mathbb{R}^5, X_c, V_c)$. Then, it can be proved that the Lie algebra of Lie symmetries of the vector fields of V_c is isomorphic to it [33]. Indeed, consider the Lie algebra of Lie symmetries of V_c given by vector fields

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + 2x_3\frac{\partial}{\partial x_4} + x_2^2\frac{\partial}{\partial x_5}, \quad Y_2 = \frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_5}, \\ &Y_3 = \frac{\partial}{\partial x_3}, \quad Y_4 = \frac{\partial}{\partial x_4}, \quad Y_5 = \frac{\partial}{\partial x_5}. \end{aligned}$$

More exactly, $[Y_i, X_j] = 0$ for $i, j = 1, \ldots, 5$. Moreover, $Y_1, \ldots, Y_5$ span a Lie algebra with opposite structure constants to those of $X_1, \ldots, X_5$. The $Y_1, \ldots, Y_5$ admit the corresponding dual one-forms given by

$$\begin{aligned} &\Upsilon^1 = \mathrm{d}x_1, \quad \Upsilon^2 = \mathrm{d}x_2, \quad \Upsilon^3 = -x_2\mathrm{d}x_1 + \mathrm{d}x_3, \\ &\Upsilon^4 = -2x_3\mathrm{d}x_1 + \mathrm{d}x_4, \quad \Upsilon^5 = -x_2^2\mathrm{d}x_1 - 2x_3\mathrm{d}x_2 + \mathrm{d}x_5. \end{aligned}$$

The existence of these dual forms follows by the condition that $Y_1 \wedge \cdots \wedge Y_5$ is non-vanishing on a manifold of dimension five. Additionally, $\Upsilon^1, \ldots, \Upsilon^5$ are invariant relative to the vector fields $X_1, \ldots, X_5$, namely $\mathscr{L}_{X_i}\Upsilon^j = 0$ for $i, j = 1, \ldots, 5$. The differentials of $\Upsilon^1, \ldots, \Upsilon^5$ depend on the commutation relations of the vector fields $X_1, \ldots, X_5$, namely

$$\mathrm{d}\Upsilon^\gamma = \frac{1}{2}\sum_{\alpha,\beta=1}^{5} c_{\alpha\beta}{}^\gamma \Upsilon^\alpha \wedge \Upsilon^\beta, \quad \alpha = 1, \ldots, 5,$$

where $c_{\alpha\beta}{}^\gamma$ are the structure constants of our problem, namely $[X_\alpha, X_\beta] = \sum_{\gamma=1}^{5} c_{\alpha\beta}{}^\gamma X_\gamma$ for $\alpha, \beta = 1, \ldots, 5$. In particular, for every differential one-form $\Upsilon \in \langle \Upsilon^1, \ldots, \Upsilon^5 \rangle$, its differential is both closed and invariant relative to the vector fields $X_1,\dots,X_5$. Consequently, these vector fields become $\mathrm{d}\Upsilon$-Hamiltonian relative to such presymplectic forms Υ.

One has the following relations:

$$\mathrm{d}\Upsilon^1 = 0, \quad \mathrm{d}\Upsilon^2 = 0, \qquad \mathrm{d}\Upsilon^3 = \Upsilon^1\wedge\Upsilon^2,$$
$$\mathrm{d}\Upsilon^4 = 2\Upsilon^1\wedge\Upsilon^3, \qquad \mathrm{d}\Upsilon^5 = 2\Upsilon^2\wedge\Upsilon^3.$$

Let us prove that $\boldsymbol{\eta}_c = \Upsilon^4\otimes e_1 + \Upsilon^5\otimes e_2$ gives rise to a two-contact form. Indeed,

$$\ker\boldsymbol{\eta}_c = \langle Y_1, Y_2, Y_3\rangle = \left\langle \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + 2x_3\frac{\partial}{\partial x_4} + x_2^2\frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_3}\right\rangle.$$

Furthermore, $\ker\mathrm{d}\boldsymbol{\eta}_c = \ker\mathrm{d}\Upsilon^4\cap\ker\mathrm{d}\Upsilon^5$ is a distribution spanned by

$$\ker\mathrm{d}\boldsymbol{\eta}_c = \langle Y_4, Y_5\rangle = \left\langle\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\right\rangle.$$

Clearly, $\ker\mathrm{d}\boldsymbol{\eta}_c\cap\ker\boldsymbol{\eta}_c = \{0\}$ and $\boldsymbol{\eta}_c$ is a two-contact form. Moreover, $R_1 = Y_4, R_2 = Y_5$ are the associated Reeb vector fields. Let us now find the corresponding $\boldsymbol{\eta}_c$-Hamiltonian two-functions $\boldsymbol{h}_1,\dots,\boldsymbol{h}_5$. By verifying the conditions

$$\iota_{X_i}\mathrm{d}\Upsilon^\mu = \mathrm{d}h_i^\mu - \sum_{\alpha=4}^{5}(R_{\alpha-3}h_i^\mu)\Upsilon^\alpha, \quad \iota_{X_i}\Upsilon^\mu = -h_i^\mu, \quad \mu = 4,5, \quad i = 1,\dots,5, \tag{10.3}$$

for $X_1,\dots,X_5$, we get that

$$\boldsymbol{h}_1 = 2x_3e_1 + x_2^2e_2, \quad \boldsymbol{h}_2 = -x_1^2e_1 + (2x_3 - 2x_1x_2)e_2,$$
$$\boldsymbol{h}_3 = -2x_1e_1 - 2x_2e_2, \quad \boldsymbol{h}_4 = -e_1, \quad \boldsymbol{h}_5 = -e_2.$$

In particular, $X_4 = Y_4$ and $X_5 = Y_5$, therefore, X_4 and X_5 have constant Hamiltonian two-functions, as they correspond to two-contact Reeb vector fields. It is worth noting that all the above $\boldsymbol{\eta}_c$-Hamiltonian two-functions are first integrals of the Reeb vector fields, thus (10.1) becomes a two-contact projectable Lie system relative to $\boldsymbol{\eta}_c$. The system (10.1) gives rise to a two-contact Lie–Hamiltonian system $(\mathbb{R}^5, \boldsymbol{\eta}_c, \boldsymbol{h} = \sum_{\alpha=1}^5 b_\alpha(t)\boldsymbol{h}_\alpha)$ with a two-contact Lie–Hamilton algebra $\mathfrak{W}_c = \langle\boldsymbol{h}_1,\dots,\boldsymbol{h}_5\rangle$ relative to $\boldsymbol{\eta}_c$. It is worth noting that X_c is invariant relative to the Reeb vector fields of $\boldsymbol{\eta}_c$. Moreover, $\boldsymbol{h}$ is also invariant relative to the Reeb vector fields of $\boldsymbol{\eta}_c$. In particular, system (10.2) becomes a projectable two-contact Lie system relative to $\boldsymbol{\eta}_c$ and $(\mathbb{R}^5, \boldsymbol{\eta}_c, \boldsymbol{h}_c = b_1(t)\boldsymbol{h}_1 + b_2(t)\boldsymbol{h}_2)$ is a two-contact Lie–Hamiltonian system for $(\mathbb{R}^5, \boldsymbol{\eta}_c, X_c)$ with a two-contact Lie–Hamilton algebra $\mathfrak{W}_c = \langle\boldsymbol{h}_1,\dots,\boldsymbol{h}_5\rangle$.

It is worth emphasizing that whether a given k-contact Lie system is projectable relative to a compatible k-contact form. To illustrate this, let us show that X_c is not projectable relative to another compatible three-contact form constructed via the methods given in Sec. 8. Note that the distribution spanned by $\langle Y_1, Y_2\rangle$ is maximally non-integrable and invariant relative to $X_1,\dots,X_5$. Moreover, it admits commuting Lie symmetries, namely X_3, X_4, X_5, which turns $\langle Y_1, Y_2\rangle$ into a three-contact distribution.

Let us consider the dual forms to Y_1, Y_2, X_3, X_4, X_5, namely

$$\begin{aligned}(\eta')^1 &= \mathrm{d}x_1, \quad (\eta')^2 = \mathrm{d}x_2, \quad (\eta')^3 = -x_2\mathrm{d}x_1 + \mathrm{d}x_3,\\ (\eta')^4 &= 2\,(x_1x_2 - x_3)\,\mathrm{d}x_1 - 2x_1\mathrm{d}x_3 + \mathrm{d}x_4,\\ (\eta')^5 &= x_2^2\mathrm{d}x_1 - 2x_3\mathrm{d}x_2 - 2x_2\mathrm{d}x_3 + \mathrm{d}x_5,\end{aligned}$$

whose non-zero differentials read

$$\mathrm{d}\,(\eta')^3 = \mathrm{d}x_1\wedge\mathrm{d}x_2, \quad \mathrm{d}\,(\eta')^4 = -2x_1\mathrm{d}x_1\wedge\mathrm{d}x_2, \quad \mathrm{d}\,(\eta')^5 = -2x_2\mathrm{d}x_1\wedge\mathrm{d}x_2.$$

One can construct the three-contact form

$$\boldsymbol{\eta}'_c = \sum_{\alpha=3}^{5} (\eta')^\alpha \otimes e_\alpha,$$

satisfying that $\ker\boldsymbol{\eta}'_c = \langle Y_1, Y_2\rangle$ and $\ker\mathrm{d}\boldsymbol{\eta}'_c = \langle X_3, X_4, X_5\rangle$. As X_3, X_4, X_5 span $\ker\mathrm{d}\boldsymbol{\eta}'_c$ and are dual to $\boldsymbol{\eta}'_c$, they are Reeb vector fields.

Note that $X_1 \dots, X_5$ are $\boldsymbol{\eta}'_c$-Hamiltonian vector fields with $\boldsymbol{\eta}'_c$-Hamiltonian three-functions given by

$$\boldsymbol{h}'_1 = x_2e_3 - 2\,(x_1x_2 - x_3)\,e_4 - x_2^2e_5, \quad \boldsymbol{h}'_2 = -x_1e_3 + x_1^2e_4 + 2x_3e_5.$$

$$\boldsymbol{h}'_\mu = -e_\mu, \quad \mu = 3, 4, 5.$$

Clearly not all $\boldsymbol{\eta}'_c$-Hamiltonian three-functions are first integrals of all Reeb vector fields, e.g. $X_3\boldsymbol{h}'_2 = -2e_5$. One can verify that $\boldsymbol{\eta}_c$-Hamiltonian three-functions close a Lie algebra under the Lie bracket given by (4.4) with opposite constant as their respective $\boldsymbol{\eta}_c$-Hamiltonian vector fields. Indeed, the non-vanishing commutation relations read

$$\{\boldsymbol{h}'_1, \boldsymbol{h}'_2\} = -\boldsymbol{h}'_3, \quad \{\boldsymbol{h}'_1, \boldsymbol{h}'_3\} = -2\boldsymbol{h}'_4, \quad \{\boldsymbol{h}'_2, \boldsymbol{h}'_3\} = -2\boldsymbol{h}'_5.$$

Note that $\boldsymbol{h}'_4$ and $\boldsymbol{h}'_5$ are dissipated functions for the three-contact Lie system $(\mathbb{R}^5, \boldsymbol{\eta}'_c, X_c)$. In fact, their associated $\boldsymbol{\eta}'_c$-Hamiltonian vector fields are Lie symmetries of the corresponding three-contact Lie system.

Note that $[V_c, [V_c, [V_c, V_c]]] = 0$ and (10.2) is projectable relative to the Reeb vector fields of $\boldsymbol{\eta}_c$. Using Remark 6.7, one obtains that if $b_1(t), b_2(t)$ are assumed to be constants, then

$$\frac{\mathrm{d}^3\boldsymbol{I}}{\mathrm{d}t^3} = 0, \quad \forall \boldsymbol{I}\in\mathfrak{W}_c,$$

which gives lots of t-dependent constants of motion of such systems. Hence, every $\boldsymbol{I}$ is related to a generator of constants of motion of order, at most, three. In particular, the form of the $\boldsymbol{I}\in\mathfrak{M}_c$ implies that solutions to (10.2) are such that their coordinates $x_1(t), x_2(t), x_3(t)$ are polynomial functions on t up to order two. Moreover, one also gets that

$$\frac{\mathrm{d}^3(x_1x_2)}{\mathrm{d}t^3} = 0,$$

which gives additional restrictions on the form of $x_1(t), x_2(t)$ for particular solutions.

10.2. *The complex Schwarz equation*

The Schwarz derivative plays a significant role in studying the linearization in t-dependent systems, projective systems, the study of special functions, etc. [37, 39, 44]. It is particularly related to the t-dependent complex differential equation given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = a, \quad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{3}{2}\frac{a^2}{v} + 2b(t)v, \quad z, v, a \in \mathbb{C}, \quad t \in \mathbb{R}, \tag{10.4}$$

for a certain complex t-dependent function $b(t)$. System (10.4) can be understood as a complex analogue of the Lie system on $\mathcal{O} = \{(z, v, a) \in T^2\mathbb{R} : v \neq 0\}$ studied in [25]. It represents the complex equation $\{z, t\}_{\mathrm{sc}} = 2b(t)$, where $\{\cdot, \cdot\}_{\mathrm{sc}}$ denotes the so-called *Schwarz derivative*.

$$\{z, t\}_{\mathrm{sc}} = \frac{\mathrm{d}^3 z}{\mathrm{d}t^3}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{-1} - \frac{3}{2}\left(\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\right)^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{-2} = 2b(t).$$

In fact, (10.4) more accurately describes the standard Schwarz derivative compared to the real version in [25]. Note that (10.4) is a differential equation on $\mathcal{O}_{\mathrm{sc}} = \{(z, v, a) \in T^2\mathbb{C} : v \neq 0\}$.

Although (10.4) can be considered as a complex Lie system, we will not develop the due theory here (which requires defining complexifications of tangent bundles and other related notions). Moreover, it can be proved that the complex approach simplifies only a part of the theory, making other computations more complicated.

In real coordinates,

$$v_1 = \mathfrak{Re}(z), \quad v_2 = \mathfrak{Im}(z), \quad v_3 = \mathfrak{Re}(v), \quad v_4 = \mathfrak{Im}(v),$$
$$v_5 = \mathfrak{Re}(a), \quad v_6 = \mathfrak{Im}(a),$$

system (10.4) is associated with the t-dependent vector field

$$X_{\mathrm{sc}} = X_1 + 2b_R(t)X_2 + 2b_I(t)X_3,$$

where $b_R(t) = \mathfrak{Re}(b(t))$, $b_I(t) = \mathfrak{Im}(b(t))$, and

$$X_1 = v_3\frac{\partial}{\partial v_1} + v_4\frac{\partial}{\partial v_2} + v_5\frac{\partial}{\partial v_3} + v_6\frac{\partial}{\partial v_4} + \frac{3}{2}\frac{2v_4v_5v_6 + (v_5^2 - v_6^2)v_3}{v_3^2 + v_4^2}\frac{\partial}{\partial v_5}$$
$$+ \frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2 - v_6^2)}{v_3^2 + v_4^2}\frac{\partial}{\partial v_6},$$

$$X_2 = v_3\frac{\partial}{\partial v_5} + v_4\frac{\partial}{\partial v_6}, \quad X_3 = -v_4\frac{\partial}{\partial v_5} + v_3\frac{\partial}{\partial v_6},$$

$$X_4 = -v_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_4} - 2v_5\frac{\partial}{\partial v_5} - 2v_6\frac{\partial}{\partial v_6},$$

$$X_5 = v_4\frac{\partial}{\partial v_3} - v_3\frac{\partial}{\partial v_4} + 2v_6\frac{\partial}{\partial v_5} - 2v_5\frac{\partial}{\partial v_6},$$

$$X_6 = -v_4\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_2} - v_6\frac{\partial}{\partial v_3} + v_5\frac{\partial}{\partial v_4} - \frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2 - v_6^2)}{(v_3^2+v_4^2)}\frac{\partial}{\partial v_5}$$
$$+\frac{3}{2}\frac{2v_4v_5v_6 + v_3(v_5^2 - v_6^2)}{(v_3^2+v_4^2)}\frac{\partial}{\partial v_6}.$$

These vector fields, defined for $v_3^2 + v_4^2 \neq 0$, satisfy the following commutation relations:

$$\begin{array}{lllll} [X_1, X_2] = X_4, & [X_1, X_3] = X_5, & [X_1, X_4] = X_1, & [X_1, X_5] = X_6, & [X_1, X_6] = 0, \\ & [X_2, X_3] = 0, & [X_2, X_4] = -X_2, & [X_2, X_5] = -X_3, & [X_2, X_6] = -X_5, \\ & & [X_3, X_4] = -X_3, & [X_3, X_5] = X_2, & [X_3, X_6] = X_4, \\ & & & [X_4, X_5] = 0, & [X_4, X_6] = -X_6, \\ & & & & [X_5, X_6] = X_1, \end{array}$$

Then, $X_1, \ldots, X_k$ span a Lie algebra V_{sc} that is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C} \otimes \mathfrak{sl}_2$ as a real vector space. Indeed, $\langle X_1, X_2, X_4\rangle \simeq \mathfrak{sl}_2(\mathbb{R}) \simeq \langle X_3, X_4, X_6\rangle$. Additionally, $\mathbb{C} \otimes \mathfrak{sl}_2(\mathbb{R})$ decomposes as $\langle X_1, X_4, X_2\rangle \oplus \langle X_6, X_5, X_3\rangle$. Then, $V_{\mathrm{sc}} = E_{-1} \oplus E_0 \oplus E_1$, where $E_{-1} = \langle X_6, X_1\rangle$, $E_0 = \langle X_4, X_5\rangle$, and $E_1 = \langle X_3, X_2\rangle$, with $[E_i, E_j] = E_{i+j}$, where the sum $i+j$ is taken relative to the additive group $\{-1, 0, 1\}$ and the direct sum is relative to subspaces of V_{sc}. Relevantly, $X_1 \wedge \cdots \wedge X_6 \neq 0$ on every point of $\mathcal{O}_{\mathrm{sc}}$. This allows us to map, at least locally, these vector fields diffeomorphically into the right-invariant vector fields of a basis of a Lie group with Lie algebra isomorphic to V_{sc}. This relation can be used to obtain Lie symmetries of these vector fields or differential forms that are invariant relative to $X_1, \ldots, X_6$.

Meanwhile, the Lie algebra of symmetries of the system (10.4) reads

$$2Y_1 = (v_1^2 - v_2^2)\frac{\partial}{\partial v_1} + 2v_1v_2\frac{\partial}{\partial v_2} + (2v_1v_3 - 2v_2v_4)\frac{\partial}{\partial v_3} + 2(v_3v_2 + v_1v_4)\frac{\partial}{\partial v_4}$$
$$+ 2(v_3^2 + v_1v_5 - v_4^2 - v_2v_6)\frac{\partial}{\partial v_5} + 2(v_3^2 + v_1v_5 - v_4^2 - v_2v_6)\frac{\partial}{\partial v_6},$$

$$Y_2 = \frac{\partial}{\partial v_1}, \quad Y_3 = \frac{\partial}{\partial v_2},$$

$$Y_4 = -v_1\frac{\partial}{\partial v_1} - v_2\frac{\partial}{\partial v_2} - v_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_4} - v_5\frac{\partial}{\partial v_5} - v_6\frac{\partial}{\partial v_6},$$

$$Y_5 = v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2} + v_4\frac{\partial}{\partial v_3} - v_3\frac{\partial}{\partial v_4} + v_6\frac{\partial}{\partial v_5} - v_5\frac{\partial}{\partial v_6},$$

$$2Y_6 = -2v_1v_2\frac{\partial}{\partial v_1} + (v_1^2 - v_2^2)\frac{\partial}{\partial v_2} - 2(v_2v_3 + v_1v_4)\frac{\partial}{\partial v_3} + (2v_1v_3 - 2v_2v_4)\frac{\partial}{\partial v_4}$$
$$-2(2v_3v_4 + v_2v_5 + v_1v_6)\frac{\partial}{\partial v_5} + 2(v_3^2 - v_4^2 + v_1v_5 - v_2v_6)\frac{\partial}{\partial v_6}.$$

The commutation relations are

$$\begin{array}{lllll}
[Y_1,Y_2]=Y_4, & [Y_1,Y_3]=Y_5, & [Y_1,Y_4]=Y_1, & [Y_1,Y_5]=Y_6, & [Y_1,Y_6]=0,\\
 & [Y_2,Y_3]=0, & [Y_2,Y_4]=-Y_2, & [Y_2,Y_5]=-Y_3, & [Y_2,Y_6]=-Y_5,\\
 & & [Y_3,Y_4]=-Y_3, & [Y_3,Y_5]=Y_2, & [Y_3,Y_6]=Y_4,\\
 & & & [Y_4,Y_5]=0, & [Y_4,Y_6]=-Y_6,\\
 & & & & [Y_5,Y_6]=Y_1.
\end{array}$$

Note that $Y_1,\ldots,Y_6$ admit identical structure constants as $X_1,\ldots,X_6$. One can choose one-forms $\Upsilon^1,\ldots,\Upsilon^6$ to be the dual to $Y_1,\ldots,Y_6$. These differential forms are locally diffeomorphic to a basis of right-invariant one-forms on a Lie group. The existence of these dual forms is ensured by the condition $Y_1\wedge\cdots\wedge Y_6 \neq 0$ and the dimension of the manifold $\mathbb{R}^6$. These dual forms remain invariant concerning the Lie derivatives of the vector fields $X_1\ldots,X_6$, i.e. $\mathscr{L}_{X_i}\Upsilon^j = 0$ for $i,j = 1,\ldots,6$.

Moreover, the differential forms $\mathrm{d}\Upsilon^1,\ldots,\mathrm{d}\Upsilon^6$, or their linear combinations, are closed differential forms that are invariant relative to the Lie derivatives along $X_1,\ldots,X_6$. Moreover (see Lemma A.1), one has

$$\mathrm{d}\Upsilon^i = -\frac{1}{2}\sum_{j,k=1}^{6} c_{jk}{}^i\Upsilon^j\wedge\Upsilon^k, \quad i=1,\ldots,6.$$

In particular,

$$\begin{array}{ll}
\mathrm{d}\Upsilon^1 = -\Upsilon^5\wedge\Upsilon^6 - \Upsilon^1\wedge\Upsilon^4, & \mathrm{d}\Upsilon^2 = -\Upsilon^3\wedge\Upsilon^5 - \Upsilon^4\wedge\Upsilon^2,\\
\mathrm{d}\Upsilon^3 = -\Upsilon^4\wedge\Upsilon^3 - \Upsilon^5\wedge\Upsilon^2, & \mathrm{d}\Upsilon^4 = -\Upsilon^1\wedge\Upsilon^2 - \Upsilon^3\wedge\Upsilon^6,\\
\mathrm{d}\Upsilon^5 = -\Upsilon^1\wedge\Upsilon^3 - \Upsilon^6\wedge\Upsilon^2, & \mathrm{d}\Upsilon^6 = -\Upsilon^1\wedge\Upsilon^5 - \Upsilon^6\wedge\Upsilon^4.
\end{array}$$

Table 1. Elements of $\boldsymbol{\eta}_{\rm sc}$-Hamiltonian functions $\boldsymbol{h}_1,\dots,\boldsymbol{h}_6$ of the vector fields $X_1,\dots,X_6$, with $\gamma=\frac{-1}{v_3^2+v_4^2}$, for the complex Schwarz equation.

$h_1^4=\frac{\gamma^3}{2}([v_2v_4(v_4^2-3v_3^2)-v_1(v_3^3-3v_3v_4^2)]v_6^2$ $-2v_6[v_5(v_1v_4(v_4^2-3v_3^2)+v_2(v_3^3-3v_3v_4^2))$ $+v_4(v_3^2+v_4^2)^2]+v_5[v_5(v_1(v_3^3-3v_3v_4^2)$ $-v_2v_4(v_4^2-3v_3^2))-2v_3(v_3^2+v_4^2)^2])$	$h_1^5=\frac{\gamma^3}{2}([v_1v_4(v_4^2-3v_3^2)+v_2(v_3^3-3v_3v_4^2)]v_6^2$ $-2v_6[v_5(v_2v_4(v_4^2-3v_3^2)-v_1(v_3^3-3v_3v_4^2))$ $+v_3(v_3^2+v_4^2)^2]+v_5[2v_4(v_3^2+v_4^2)^2$ $+(v_1v_4(v_4^2-3v_3^2)+v_2(v_3^3-3v_3v_4^2))v_5])$
$h_2^4=\gamma(v_1v_3+v_2v_4)$	$h_2^5=\gamma(v_2v_3-v_1v_4)$
$h_3^4=\gamma(v_1v_4-v_2v_3)$	$h_3^5=\gamma(v_1v_3+v_2v_4)$
$h_4^4=-\gamma^2(v_3^4+(2v_4^2-v_1v_5+v_2v_6)v_3^2$ $-2v_4(v_2v_5+v_1v_6)v_3$ $+v_4^2(v_4^2+v_1v_5-v_2v_6))$	$h_4^5=-\gamma^2(v_1(-v_6v_3^2+2v_4v_5v_3+v_4^2v_6)$ $+v_2(-2v_4v_6v_3+v_4^2v_5-v_3^2v_5))$
$h_5^4=-\gamma^2(v_2((v_3^2-v_4^2)v_5+2v_3v_4v_6)$ $+v_1((v_3^2-v_4^2)v_6-2v_3v_4v_5))$	$h_5^5=-\gamma^2(v_3^4+(2v_4^2-v_1v_5+v_2v_6)v_3^2$ $-2v_4(v_2v_5+v_1v_6)v_3$ $+v_4^2(v_4^2+v_1v_5-v_2v_6))$
$h_6^4=\frac{\gamma^3}{2}(v_6^2(v_1v_4(v_4^2-3v_3^2)+v_2(v_3^3-3v_3v_4^2))$ $+2v_6[v_5(v_2v_4(v_4^2-3v_3^2)-v_1(v_3^3-3v_3v_4^2))$ $+v_3(v_3^2+v_4^2)^2]+v_5[-v_5(v_1v_4(v_4^2-3v_3^2)$ $+v_2(v_3^3-3v_3v_4^2))-2v_4(v_3^2+v_4^2)^2])$	$h_6^5=\frac{\gamma^3}{2}((v_2v_4(v_4^2-3v_3^2)-v_1(v_3^3-3v_3v_4^2))v_6^2$ $-2v_6[v_5(v_1v_4(v_4^2-3v_3^2)+v_2(v_3^3-3v_3v_4^2))$ $+v_4(v_3^2+v_4^2)^2]+v_5[v_5(v_1(v_3^3-3v_3v_4^2)$ $-v_2v_4(v_4^2-3v_3^2))-2v_3(v_3^2+v_4^2)^2])$

Hence,

$$\sum_{\alpha=1}^{6}\mathrm{d}\Upsilon^\alpha\otimes e_\alpha=-\frac{1}{2}\sum_{\alpha,\beta,\gamma=1}^{6}c_{\beta\gamma}{}^{\alpha}\Upsilon^\beta\wedge\Upsilon^\gamma\otimes e_\alpha=:-\frac{1}{2}\sum_{\mu,\nu=1}^{6}[\Upsilon^\mu\otimes e_\mu,\Upsilon^\nu\otimes e_\nu].$$

Then, for $\boldsymbol{\Upsilon}=\sum_{\alpha=1}^{6}\Upsilon^\alpha\otimes e_\alpha$, one has

$$\mathrm{d}\boldsymbol{\Upsilon}=-\frac{1}{2}[\boldsymbol{\Upsilon},\boldsymbol{\Upsilon}]\Rightarrow\mathrm{d}\boldsymbol{\Upsilon}+\frac{1}{2}[\boldsymbol{\Upsilon},\boldsymbol{\Upsilon}]=0,$$

where $\{e_1,\dots,e_6\}$ is a basis of the Lie algebra $T_e\mathrm{SL}_2(\mathbb{C})$ with the commutation relations of the vector fields $X_1,\dots,X_6$.

Let us prove that $\Upsilon^4,\Upsilon^5,\mathrm{d}\Upsilon^4,\mathrm{d}\Upsilon^5$ give rise to a two-contact form $\boldsymbol{\eta}_{\rm sc}=\Upsilon^4\otimes e_4+\Upsilon^5\otimes e_5$. Indeed, note that Y^4,Y^5 commute between themselves, take values in $\ker\mathrm{d}\Upsilon^4\cap\ker\mathrm{d}\Upsilon^5$, and span the Reeb distribution. Moreover, the space $\langle Y^1,Y^2,Y^3,Y^6\rangle$ spans the associated two-contact distribution. It follows that Y^4,Y^5 are the Reeb vector fields of our two-contact distribution. By verifying conditions (10.3) for $X_1,\dots,X_6$, we obtain their $\boldsymbol{\eta}_{\rm sc}$-Hamiltonian two-functions, whose components are given by Table 1. Hence, $(\mathcal{O}_{\rm sc},\boldsymbol{\eta}_{\rm sc},X_{\rm sc})$ becomes a two-contact Lie system with a two-contact Lie–Hamiltonian system $(\mathcal{O}_{\rm sc},\boldsymbol{\eta}_{\rm sc},\boldsymbol{h}_{\rm sc}=\boldsymbol{h}_1+2b_R(t)\boldsymbol{h}_2+2b_I(t)\boldsymbol{h}_3)$.

10.3. *Generalization to degree three of the Brockett system*

Let us study now two examples of Lie systems, whose motivation is due to its interest in control theory, Wei–Norman equations, and the theory of Lie systems [50, 51, 55]. Both of them are control systems on $\mathbb{R}^8$.

Let $\{x_1, \ldots, x_8\}$ be linear coordinates in $\mathbb{R}^8$. Consider the system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = b_1(t), \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = b_2(t), \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = b_2(t)x_1, \qquad \frac{\mathrm{d}x_4}{\mathrm{d}t} = b_1(t)x_3,$$
$$\frac{\mathrm{d}x_5}{\mathrm{d}t} = b_2(t)x_3, \qquad \frac{\mathrm{d}x_6}{\mathrm{d}t} = b_1(t)x_4, \qquad \frac{\mathrm{d}x_7}{\mathrm{d}t} = b_2(t)x_4, \qquad \frac{\mathrm{d}x_8}{\mathrm{d}t} = b_2(t)x_5, \tag{10.5}$$

where $b_1(t)$ and $b_2(t)$ are the so-called control functions. From our point of view, they are just arbitrary t-dependent functions whose behavior can lead to interesting features.

The solutions of (10.5) are the integral curves of t-dependent vector field $X_{3b} = b_1(t)\,X_1 + b_2(t)\,X_2$, with

$$X_1 = \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_6},$$
$$X_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_5} + x_4\frac{\partial}{\partial x_7} + x_5\frac{\partial}{\partial x_8}.$$

Taking the successive Lie brackets of X_1, X_2, we obtain the smallest Lie algebra of vector fields containing X_1, X_2. In fact,

$$X_3 = [X_1,\, X_2] = \frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4} + x_3\frac{\partial}{\partial x_7}, \qquad X_4 = [X_1,\, X_3] = -2\frac{\partial}{\partial x_4} + x_1\frac{\partial}{\partial x_6},$$
$$X_5 = [X_2,\, X_3] = [X_1, X_6] = -\frac{\partial}{\partial x_5} + 2x_1\frac{\partial}{\partial x_7}, \qquad X_6 = [X_1,\, X_4] = 3\frac{\partial}{\partial x_6},$$
$$X_7 = [X_1,\, X_5] = 2\frac{\partial}{\partial x_7}, \qquad X_8 = [X_2,\, X_5] = \frac{\partial}{\partial x_8},$$

along with X_1, X_2 define an eight-dimensional Lie algebra of vector fields $V_{3b} = \langle X_1,\, \ldots,\, X_8\rangle$. It is worth noting that $X_1 \wedge \cdots \wedge X_8 \neq 0$ and these vector fields span $T\mathbb{R}^8$. Hence, X_{3b} gives rise to a locally automorphic Lie system $(\mathbb{R}^8, X_{3b}, V_{3b})$. Moreover, the fact that the vector fields $X_1, \ldots, X_8$ span the tangent space means that the system is *controllable*, which has special relevance in control theory [40, 56].

The non-zero commutation relations between the elements of the given basis of V read

$$[X_1,\, X_2] = X_3, \quad [X_1,\, X_3] = X_4, \quad [X_1,\, X_4] = X_6, \quad [X_1,\, X_5] = X_7,$$
$$[X_2,\, X_3] = X_5, \quad [X_2,\, X_4] = X_7, \quad [X_2,\, X_5] = X_8.$$

This implies that V_{3b} is a nilpotent Lie algebra.

As the Brockett system (10.5) gives rise to a locally automorphic Lie system $(\mathbb{R}^8, X_{3b}, V_{3b})$, it can be considered to be locally diffeomorphic to the so-called automorphic Lie system on a Lie group [33]. This means that one can consider the

Lie algebra of Lie symmetries of such vector fields, which will span a Lie algebra isomorphic to the previous one [33]. In fact, one can consider a basis $Y_1, \ldots, Y_8$ of Lie symmetries of $X_1, \ldots, X_8$ on $\mathbb{R}^8$ taking the same values as $X_1, \ldots, X_8$ at $0 \in \mathbb{R}^8$. Hence, the structure constants for $Y_1, \ldots, Y_8$ are opposite to the ones for $X_1, \ldots, X_8$. To illustrate our theory, the Lie symmetries of the vector fields $X_1, \ldots, X_8$ are spanned by the basis given by

$$Y_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + (x_1 x_2 - x_3) \frac{\partial}{\partial x_4} + \frac{x_2^2}{2} \frac{\partial}{\partial x_5} + \frac{x_1(x_1 x_2 - x_3) - x_4}{2} \frac{\partial}{\partial x_6} + (x_3 x_2 - 2x_5) \frac{\partial}{\partial x_7} + \frac{x_2^3}{6} \frac{\partial}{\partial x_8},$$

$$Y_2 = \frac{\partial}{\partial x_2}, \quad Y_3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \frac{x_1^2}{2} \frac{\partial}{\partial x_6} + x_3 \frac{\partial}{\partial x_7} + \frac{x_2^2}{2} \frac{\partial}{\partial x_8},$$

$$Y_4 = -2 \frac{\partial}{\partial x_4} - 2x_1 \frac{\partial}{\partial x_6} - 2x_2 \frac{\partial}{\partial x_7}, \quad Y_5 = -\frac{\partial}{\partial x_5} - x_2 \frac{\partial}{\partial x_8}, \quad Y_6 = 3 \frac{\partial}{\partial x_6},$$

$$Y_7 = 2 \frac{\partial}{\partial x_7}, \quad Y_8 = \frac{\partial}{\partial x_8}.$$

Previous vector fields take the same values as $X_1, \ldots, X_8$ at $0 \in \mathbb{R}^8$, namely $X_\alpha(0) = Y_\alpha(0)$ for $\alpha = 1, \ldots, 8$. Moreover, the non-vanishing commutation relations for $Y_1, \ldots, Y_8$ read

$$[Y_1, Y_2] = -Y_3, \quad [Y_1, Y_3] = -Y_4, \quad [Y_1, Y_4] = -Y_6, \quad [Y_1, Y_5] = -Y_7,$$
$$[Y_2, Y_3] = -Y_5, \quad [Y_2, Y_4] = -Y_7, \quad [Y_2, Y_5] = -Y_8.$$

Let us construct a six-contact distribution that is invariant relative to the action by Lie brackets of the vector fields $X_1, \ldots, X_8$. Note that any distribution spanned by $Y_1, \ldots, Y_8$ is invariant relative to the action of the vector fields $X_1, \ldots, X_8$. In particular, consider the distribution spanned by the vector fields Y_1, Y_2. Since $[Y_1, Y_2] = -Y_3$ and $Y_1 \wedge Y_2 \wedge Y_3$ is never vanishing, the distribution is maximally non-integrable. Consider the basis of vector fields on $\mathbb{R}^8$ given by

$$Y_1, Y_2, X_3, \ldots, X_8.$$

On the one hand, Y_1, Y_2 span a maximally non-integrable distribution, while $X_3, \ldots, X_8$ are Lie symmetries of $\langle Y_1, Y_2 \rangle$ spanning an abelian Lie algebra and a distribution that is supplementary to the one spanned by Y_1, Y_2. This gives rise to a six-contact distribution on $\mathbb{R}^8$. Moreover, $X_1, \ldots, X_8$ leave invariant the distribution spanned by Y_1, Y_2, which turns them into six-contact Hamiltonian vector fields.

The vector fields $X_3, \ldots, X_8$ are the Reeb vector fields of the six-contact form $\boldsymbol{\eta}_{3b}$ to be described next. Nevertheless, the $\boldsymbol{\eta}_{3b}$-Hamiltonian k-functions of X_1, X_2 will not be in general first integrals of the Reeb vector fields of $\boldsymbol{\eta}_{3b}$. In particular, the vector fields $X_3, \ldots, X_8$ do not commute with X_1, X_2, which implies that the $\boldsymbol{\eta}_{3b}$-Hamiltonian six-functions of X_1, X_2 are not first integrals of the Reeb distribution, which is spanned by $X_3, \ldots, X_8$.

It is worth noting that the potential applications of this method are quite large as the scheme used is quite general: it only depends on a subspace, which is not a Lie algebra, but is invariant relative to a supplementary commutative Lie subalgebra.

Then, the dual forms to $Y_1, Y_2, X_3, \ldots, X_8$ take the form

$$\begin{aligned}
\eta^1 &= \mathrm{d}x_1, \quad \eta^2 = \mathrm{d}x_2, \quad \eta^3 = \mathrm{d}x_3 - x_2\mathrm{d}x_1,\\
\eta^4 &= \left(x_1x_2 - \frac{1}{2}x_3\right)\mathrm{d}x_1 - \frac{1}{2}x_1\mathrm{d}x_3 - \frac{1}{2}\mathrm{d}x_4, \quad \eta^5 = \frac{1}{2}x_2^2\mathrm{d}x_1 - \mathrm{d}x_5,\\
\eta^6 &= \frac{1}{6}\left(2x_3x_1 - 3x_2x_1^2 + x_4\right)\mathrm{d}x_1 + \frac{1}{6}x_1^2\mathrm{d}x_3 + \frac{1}{6}x_1\mathrm{d}x_4 + \frac{1}{3}\mathrm{d}x_6,\\
\eta^7 &= \left(x_5 - \frac{1}{2}x_1x_2^2\right)\mathrm{d}x_1 - \frac{1}{2}x_3\mathrm{d}x_3 + x_1\mathrm{d}x_5 + \frac{1}{2}\mathrm{d}x_7,\\
\eta^8 &= -\frac{1}{6}x_2^3\mathrm{d}x_1 + \mathrm{d}x_8.
\end{aligned}$$

It is immediate that $\eta^3, \ldots, \eta^8$ give rise to a six-contact form

$$\boldsymbol{\eta}_{3b} = \sum_{\alpha=3}^{8} \eta^\alpha \otimes e_\alpha.$$

In fact, $\ker \boldsymbol{\eta}_{3b} = \langle Y_1, Y_2\rangle$ and the $\boldsymbol{\eta}_{3b}$-Hamiltonian six-functions for $X_3, \ldots, X_8$ are easily given by

$$\boldsymbol{h}_\alpha = -e_\alpha, \quad \alpha = 3, \ldots, 8,$$

because $X_3, \ldots, X_8$ are the Reeb vector fields for $\boldsymbol{\eta}_{3b}$. Meanwhile,

$$\begin{aligned}
\boldsymbol{h}_1 &= x_2e_3 - (x_1x_2 - x_3)e_4 - \frac{1}{2}x_2^2e_5 - \frac{1}{2}(x_1(x_3 - x_1x_2) + x_4)e_6\\
&\quad + (x_1x_2^2/2 - x_5)e_7 + \frac{1}{6}x_2^3e_8,\\
\boldsymbol{h}_2 &= -x_1e_3 + \frac{1}{2}x_1^2e_4 + x_3e_5 - \frac{x_1^3}{6}e_6 - \frac{1}{2}(x_1x_3 + x_4)e_7 - x_5e_8.
\end{aligned}$$

Note that these $\boldsymbol{\eta}_{3b}$-Hamiltonian six-functions are not first integrals of all the Reeb vector fields, e.g. $X_5\boldsymbol{h}_2 = e_8$.

To verify that all our results are correct in a new way, let us write $\mathrm{d}\boldsymbol{\eta}_{3b}$, which reads

$$\mathrm{d}\boldsymbol{\eta}_{3b} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(e_3 - x_1e_4 - x_2e_5 + \frac{1}{2}x_1^2e_6 + x_2x_1e_7 + \frac{1}{2}x_2^2e_8\right),$$

which means that $X_3, \ldots, X_8$ take values in $\ker \mathrm{d}\boldsymbol{\eta}_{3b}$ and are "dual" to $\boldsymbol{\eta}_{3b}$. Finally, $(\mathbb{R}^8, \boldsymbol{\eta}_{3b}, \boldsymbol{h}_{3b} = b_1(t)\boldsymbol{h}_1 + b_2(t)\boldsymbol{h}_2)$ is a six-contact Lie–Hamiltonian system for X_{3b}. Moreover, $(\mathbb{R}^8, \boldsymbol{\eta}_{3b}, X_{3b})$ is a six-contact Lie system.

10.4. *A new control system*

Another example of a Lie system, given in [55, Sec. 7.2.3.2], is a control system in $\mathbb{R}^8$ given by

$$\begin{aligned} &\frac{\mathrm{d}x_1}{\mathrm{d}t} = b_1(t), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = b_2(t), \quad \frac{\mathrm{d}x_3}{\mathrm{d}t} = b_2(t)x_1 - b_1(t)x_2, \\ &\frac{\mathrm{d}x_4}{\mathrm{d}t} = b_2(t)x_1^2, \quad \frac{\mathrm{d}x_5}{\mathrm{d}t} = b_1(t)x_2^2, \quad \frac{\mathrm{d}x_6}{\mathrm{d}t} = b_2(t)x_1^3, \\ &\frac{\mathrm{d}x_7}{\mathrm{d}t} = b_1(t)x_2^3, \quad \frac{\mathrm{d}x_8}{\mathrm{d}t} = b_1(t)x_1^2x_2 + b_2(t)x_1x_2^2, \end{aligned} \tag{10.6}$$

where $b_1(t), b_2(t)$ are arbitrary t-dependent functions. The above system is related to a generalization to third degree of a Brockett system [6].

System (10.6) is associated with the t-dependent vector field $X_{g3b} = b_1(t)X_1 + b_2(t)X_2$ on $\mathbb{R}^8$, with

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} + x_2^2\frac{\partial}{\partial x_5} + x_2^3\frac{\partial}{\partial x_7} + x_1^2x_2\frac{\partial}{\partial x_8}, \\ X_2 &= \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_1^2\frac{\partial}{\partial x_4} + x_1^3\frac{\partial}{\partial x_6} + x_1x_2^2\frac{\partial}{\partial x_8}. \end{aligned}$$

By considering the vector fields[e]

$$X_3 = [X_1, X_2] = 2\frac{\partial}{\partial x_3} + 2x_1\frac{\partial}{\partial x_4} - 2x_2\frac{\partial}{\partial x_5} + 3x_1^2\frac{\partial}{\partial x_6} + \left(x_2^2 - x_1^2\right)\frac{\partial}{\partial x_8},$$

$$X_4 = [X_1, X_3] = 2\frac{\partial}{\partial x_4} + 6x_1\frac{\partial}{\partial x_6} - 2x_1\frac{\partial}{\partial x_8}, \quad X_5 = [X_2.X_3] = -2\frac{\partial}{\partial x_5} + 2x_2\frac{\partial}{\partial x_8},$$

$$X_6 = [X_1, X_4] = 6\frac{\partial}{\partial x_6} - 2\frac{\partial}{\partial x_8}, \quad X_7 = [X_2, X_5] = 2\frac{\partial}{\partial x_8},$$

we obtain a nilpotent Lie algebra $V_{g3b} = \langle X_1, \ldots, X_7\rangle$ whose structure is defined by the non-vanishing commutation relations

$$\begin{aligned} &[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_1, X_4] = X_6, \\ &[X_2, X_3] = X_5, \quad [X_2, X_5] = X_7. \end{aligned}$$

It should be noted that there exists another vector field, $X_8 = \frac{\partial}{\partial x_7}$, which commutes with $X_1, \ldots, X_7$ so that $\{X_1, \ldots, X_8\}$ span the tangent space at each point of $\mathbb{R}^8$. This gives rise to a locally automorphic Lie system $(\mathbb{R}^8, X_{g3b}, V_{g3b})$. Since every automorphic Lie system is locally diffeomorphic to an automorphic Lie system and V_{g3b} is mapped into the Lie algebra of right-invariant vector fields on a Lie group with Lie algebra isomorphic to V_{g3b}, there exists a Lie algebra of Lie symmetries of the vector fields of V_{g3b} on $\mathbb{R}^8$ isomorphic to V_{g3b}, which are locally diffeomorphic

[e]Note that there exists a small mistake in the form of the vector field X_5 in [55, p. 185].

to left-invariant vector fields on a Lie group G with Lie algebra isomorphic to V_{g3b} (see [33] for details). Indeed, the Lie symmetries of $X_1, \ldots, X_8$ are spanned by

$$Y_1 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + (x_1x_2 + x_3)\frac{\partial}{\partial x_4} + 3x_4\frac{\partial}{\partial x_6} + \left(x_1^2x_2 + \frac{1}{3}x_2^3 - x_4\right)\frac{\partial}{\partial x_8},$$
$$Y_2 = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + (x_1x_2 - x_3)\frac{\partial}{\partial x_5} + \left(\frac{1}{3}x_1^3 + x_1x_2^2 - x_5\right)\frac{\partial}{\partial x_8}, \quad Y_3 = 2\frac{\partial}{\partial x_3},$$
$$Y_4 = 2\frac{\partial}{\partial x_4}, \quad Y_5 = -2\frac{\partial}{\partial x_5}, \quad Y_6 = 6\frac{\partial}{\partial x_6} - 2\frac{\partial}{\partial x_8}, \quad Y_7 = 2\frac{\partial}{\partial x_8}, \quad Y_8 = \frac{\partial}{\partial x_7}.$$

Since $Y_1, \ldots, Y_8$ are linearly independent, there exist dual forms $\theta^1, \ldots, \theta^8$. Therefore, we can consider the distribution $\mathcal{D} = \langle Y_1, Y_2, Y_3\rangle$ as the kernel of five-contact form $\boldsymbol{\theta} = \sum_{i=4}^8 \theta^i \otimes e_i$ and a vector bundle mapping in Definition 4.13 is given by $\rho(v_\alpha, v_\beta) = \boldsymbol{\theta}([Z_\alpha, Z_\beta])$ for $Z_\alpha, Z_\beta \in \mathcal{D}$. As the degeneracy of ρ is locally equivalent to existence of a non-zero vector field $Z \in \mathcal{D}$ such that $[Z, W] \in \mathcal{D}$ for all $W \in \mathcal{D}$, one can check that if $Z = c_1Y_1 + c_2Y_2 + c_3Y_3$, then

$$\mathcal{D} \ni [Z, Y_1] = [c_1Y_1 + c_2Y_2 + c_3Y_3, Y_1] = c_3[Y_3, Y_1] + c_2[Y_2, Y_1] \Rightarrow c_3 = 0,$$
$$\mathcal{D} \ni [Z, Y_3] = [c_1Y_1 + c_2Y_2, Y_3] = c_1[Y_1, Y_3] + c_2[Y_2, Y_3] \Rightarrow c_1 = c_2 = 0,$$

thus ρ is non-degenerate and $\mathcal{D}$ is maximally non-integrable. Additionally, it admits five commuting Lie symmetries $X_4, \ldots, X_8$, hence it is a five-contact distribution invariant with respect to $X_1, \ldots, X_8$, which are $\boldsymbol{\eta}_{gb3}$-Hamiltonian with regard to five-contact form $\boldsymbol{\eta}_{gb3}$ to be described next.

Consider the dual forms to $Y_1, \ldots, Y_3, X_4, \ldots, X_8$, namely

$$\eta^1 = \mathrm{d}x_1, \quad \eta^2 = \mathrm{d}x_2, \quad \eta^3 = -\frac{1}{2}x_2\mathrm{d}x_1 + \frac{1}{2}x_1\mathrm{d}x_2 + \frac{1}{2}\mathrm{d}x_3,$$
$$\eta^4 = -\frac{1}{2}(x_1x_2 + x_3)\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_4, \quad \eta^5 = \frac{1}{2}(x_1x_2 - x_3)\,\mathrm{d}x_2 - \frac{1}{2}\mathrm{d}x_5,$$
$$\eta^6 = \frac{1}{2}\left(x_1\left(x_1x_2 + x_3\right) - x_4\right)\mathrm{d}x_1 - \frac{1}{2}x_1\mathrm{d}x_4 + \frac{1}{6}\mathrm{d}x_6,$$
$$\eta^7 = -\frac{1}{6}x_2\left(3x_1^2 + x_2^2\right)\mathrm{d}x_1 + \frac{1}{6}\left(-x_1^3 - 6x_1x_2^2 + 3\left(x_2x_3 + x_5\right)\right)\mathrm{d}x_2$$
$$+\frac{1}{2}x_2\mathrm{d}x_5 + \frac{1}{6}\mathrm{d}x_6 + \frac{1}{2}\mathrm{d}x_8,$$
$$\eta^8 = \frac{3}{2}(x_1x_2 - x_3)\mathrm{d}x_2 - \frac{3}{2}\mathrm{d}x_5 + \mathrm{d}x_7.$$

Dual forms associated with $X_4,\ldots,X_8$ give rise to a five-contact form $\boldsymbol{\eta}_{g3b} = \sum_{\alpha=4}^{8}\eta^\alpha\otimes e_\alpha$, and the $\boldsymbol{\eta}_{g3b}$-Hamiltonian functions of $X_1,\ldots,X_8$ read

$$
\begin{aligned}
\boldsymbol{h}_1 &= \frac{1}{2}(x_1x_2+x_3)e_4+\frac{1}{2}x_2^2e_5+\frac{1}{2}(-x_1(x_1x_2+x_3)+x_4)e_6\\
&\quad -\frac{1}{3}x_2^3e_7+\frac{1}{2}(3-2x_2)x_2^2e_8,\\
\boldsymbol{h}_2 &= -\frac{1}{2}x_1^2e_4+\frac{1}{2}(-x_1x_2+x_3)e_5+\frac{1}{3}x_1^3e_6+\frac{1}{2}\left(x_1x_2^2-x_2x_3-x_5\right)e_7\\
&\quad +\frac{3}{2}(-x_1x_2+x_3)e_8,\\
\boldsymbol{h}_3 &= -x_1e_4-x_2e_5+\frac{1}{2}x_1^2e_6+\frac{1}{2}x_2^2e_7,\quad \boldsymbol{h}_\alpha=-e_\alpha,\quad \alpha=4,\ldots,8.
\end{aligned}
$$

By taking Lie brackets of $\boldsymbol{h}_1,\ldots,\boldsymbol{h}_8$, one obtains that their non-vanishing relations read

$$
\begin{aligned}
&\{\boldsymbol{h}_1,\boldsymbol{h}_2\}=-\boldsymbol{h}_3,\quad \{\boldsymbol{h}_1,\boldsymbol{h}_3\}=-\boldsymbol{h}_4,\quad \{\boldsymbol{h}_1,\boldsymbol{h}_4\}=-\boldsymbol{h}_6,\\
&\{\boldsymbol{h}_2,\boldsymbol{h}_3\}=-\boldsymbol{h}_5,\quad \{\boldsymbol{h}_2,\boldsymbol{h}_5\}=-\boldsymbol{h}_7.
\end{aligned}
$$

Then, X_{g3b} admits a five-contact Lie–Hamiltonian system $(\mathbb{R}^8,\boldsymbol{\eta}_{g3b},\boldsymbol{h}_{g3b} = b_1(t)\boldsymbol{h}_1+b_2(t)\boldsymbol{h}_2)$. In fact, $(\mathbb{R}^8,\boldsymbol{\eta}_{g3b},X_{g3b})$ is a five-contact Lie system.

10.5. *Front-wheel driven kinematic car*

Let us consider another example of control system, namely

$$
\frac{\mathrm{d}x}{\mathrm{d}t}=c_1(t),\quad \frac{\mathrm{d}y}{\mathrm{d}t}=c_1(t)\tan\theta,\quad \frac{\mathrm{d}\phi}{\mathrm{d}t}=c_2(t),\quad \frac{\mathrm{d}\theta}{\mathrm{d}t}=c_1(t)\frac{\tan\phi}{\cos\theta},\tag{10.7}
$$

defined on the configuration manifold $M=\mathbb{R}^2\times I^2$, where $I=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, with coordinates $\{x,y,\theta,\phi\}$, respectively, and arbitrary t-dependent functions $c_1(t)$ and $c_2(t)$. System (10.7) is of interest as it describes a simple model of a car with front and rear wheels [55]. Its associated t-dependent vector field is $X=c_1(t)Z_1+c_2(t)Z_2$, where

$$
Z_1=\frac{\partial}{\partial x}+\tan\theta\frac{\partial}{\partial y}+\frac{\tan\phi}{\cos\theta}\frac{\partial}{\partial\phi},\quad Z_2=\frac{\partial}{\partial\phi}.
$$

By taking their commutators

$$
Z_3=[Z_1,Z_2]=-\frac{1}{\cos\theta\cos^2\phi}\frac{\partial}{\partial\theta},\quad Z_4=[Z_1,Z_3]=\frac{1}{\cos^3\theta\cos^2\phi}\frac{\partial}{\partial y},
$$

one sees that Z_1,Z_2,Z_3,Z_4 span TM. These vector fields do not close any finite-dimensional Lie algebra, indeed, the successive Lie brackets of Y_2 with Y_4, namely $[Y_2,Y_4],[Y_2,[Y_2,Y_4]],\ldots,$ span an infinite-dimensional Lie algebra of vector fields. When $c_1(t)$ and $c_2(t)$ are not linearly dependent, one then has that (10.7) is not a Lie system.

Nonetheless, (10.7) can be transformed into a Lie system related to a nilpotent Vessiot–Guldberg Lie algebra by using the following transformation (see [15] for details):

$$c_1(t) = b_1(t), \quad c_2(t) = -3\sin^2\phi \frac{\tan\theta}{\cos\theta} b_1(t) + \cos^3\theta\cos^2\phi\, b_2(t).$$

Indeed, the successive change of coordinates

$$x_1 = x, \quad x_2 = \frac{\tan\phi}{\cos^3\theta}, \quad x_3 = \tan\theta, \quad x_4 = y$$

transforms (10.7) into the control system on $\mathbb{R}^4$ given by

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = b_1(t), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = b_2(t), \quad \frac{\mathrm{d}x_3}{\mathrm{d}t} = b_1(t)x_2, \quad \frac{\mathrm{d}x_4}{\mathrm{d}t} = b_1(t)x_3. \tag{10.8}$$

System (10.8) describes the integral curves of the t-dependent vector field $X_{fw} = b_1(t)X_1 + b_2(t)X_2$, with

$$X_1 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}, \quad X_2 = \frac{\partial}{\partial x_2}.$$

Along with the following vector fields

$$X_3 = [X_1, X_2] = -\frac{\partial}{\partial x_3}, \quad X_4 = [X_1, X_3] = \frac{\partial}{\partial x_4},$$

the set $\{X_1, X_2, X_3, X_4\}$ generates a nilpotent Lie algebra V_{fw} with non-vanishing commuting relations

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4.$$

Because $\langle X_1, X_2, X_3, X_4\rangle = T\mathbb{R}^4$, one has that $(\mathbb{R}^4, X_{fw}, V_{fw})$ is a locally automorphic Lie system. Therefore, there exist, at least locally, Lie symmetries of $X_1, \ldots, X_4$ spanning Lie algebra isomorphic to V_{fw} [33], namely

$$Y_1 = \frac{\partial}{\partial x_1}, \quad Y_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + \frac{1}{2}x_1^2\frac{\partial}{\partial x_4}, \quad Y_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}, \quad Y_4 = \frac{\partial}{\partial x_4}.$$

Since $[Y_1, Y_2] = Y_3$ and $Y_1 \wedge Y_2 \wedge Y_3$ is non-vanishing, the distribution $\mathcal{D} = \langle Y_1, Y_2\rangle$ is maximally non-integrable. Moreover, $\mathcal{D}$ is a two-contact distribution, since it admits two commuting Lie symmetries X_3 and X_4 such that Y_1, Y_2, X_3, X_4 span $T\mathbb{R}^4$. One can construct its associated two-contact form $\boldsymbol{\eta}_{fw}$ by considering dual forms to Y_1, Y_2, X_3, X_4, namely

$$\eta^1 = \mathrm{d}x_1, \quad \eta^2 = \mathrm{d}x_2, \quad \eta^3 = x_1\mathrm{d}x_2 - \mathrm{d}x_3, \quad \eta^4 = -\frac{1}{2}x_1^2\mathrm{d}x_2 + \mathrm{d}x_4$$

which are duals to Y_1, Y_2, X_3, X_4. Then,

$$\boldsymbol{\eta}_{fw} = \eta^3 \otimes e_3 + \eta^4 \otimes e_4$$

is a two-contact form with $\ker\boldsymbol{\eta}_{fw} = \mathcal{D}$. Indeed, both Y_1 and Y_2 span $\ker\boldsymbol{\eta}_{fw}$, $\ker\mathrm{d}\boldsymbol{\eta}_{fw} = \langle X_3, X_4\rangle$ and $\ker\boldsymbol{\eta}_{fw} \cap \ker\mathrm{d}\boldsymbol{\eta}_{fw} = 0$. Because $X_1, \ldots, X_4$ leave invariant $\mathcal{D}$, they are $\boldsymbol{\eta}_{fw}$-Hamiltonian vector fields with $\boldsymbol{\eta}_{fw}$-Hamiltonian two-functions

$$\boldsymbol{h}_1 = x_2 e_3 - x_3 e_4, \qquad \boldsymbol{h}_2 = -x_1 e_3 + \frac{1}{2} x_1^2 e_4, \quad \boldsymbol{h}_\alpha = -e_\alpha, \qquad \alpha = 3, 4,$$

which non-vanishing Lie brackets are

$$\{\boldsymbol{h}_1, \boldsymbol{h}_2\} = -\boldsymbol{h}_3, \qquad \{\boldsymbol{h}_1, \boldsymbol{h}_3\} = -\boldsymbol{h}_4.$$

Note that $(\mathbb{R}^4, \boldsymbol{\eta}_{fw}, X_{fw})$ is a four-contact Lie system and $(\mathbb{R}^4, \boldsymbol{\eta}_{fw}, \boldsymbol{h}_{fw} = b_1(t)\boldsymbol{h}_1 + b_2(t)\boldsymbol{h}_2)$ is an associated $\boldsymbol{\eta}_{fw}$-Lie Hamiltonian system.

Using Remark 6.7, one has that

$$\frac{\mathrm{d}\langle \boldsymbol{h}_1, e^3\rangle}{\mathrm{d}t} = -X_{\boldsymbol{h}}\langle \boldsymbol{h}_1, e^3\rangle = -\langle\{b_1(t)\boldsymbol{h}_1 + b_2(t)\boldsymbol{h}_2, \boldsymbol{h}_1\}, e^3\rangle - R_{\boldsymbol{h}_1}\langle \boldsymbol{h}_{fw}, e^3\rangle = b_2(t)$$

and $\int^t b_2(t')\mathrm{d}t' - \langle \boldsymbol{h}_1, e^3\rangle$ is a t-dependent constant of motion for (10.8).

11. PDE Lie Systems with a Compatible k-Contact Manifold

Let us briefly introduce PDE Lie systems [9, 55] and show that a Lie algebra of $\boldsymbol{\eta}$-Hamiltonian vector fields relative to a co-oriented k-contact manifold $(M, \boldsymbol{\eta})$ allows us to construct PDE Lie systems with a compatible k-contact manifold, the so-called *k-contact PDE Lie systems*. These PDE Lie systems can be understood as a certain type of Hamilton–De Donder–Weyl equations in the k-contact realm [58]. Moreover, methods developed for k-contact Lie systems can be easily generalized to k-contact PDE Lie systems.

A *t-dependent k-vector field* on M with $t \in \mathbb{R}^k$ is a mapping $\boldsymbol{X}\colon \mathbb{R}^k \times M \to \bigoplus^k TM$ such that $\tau^k\colon \bigoplus^k TM \to M$ satisfies $\tau^k \circ \boldsymbol{X} = \pi$, where $\pi\colon \mathbb{R}^k \times M \to M$ is the natural projection onto M. Every t-dependent k-vector field on M with $t \in \mathbb{R}^k$ amounts to a series of t-parametrized k-vector fields $\boldsymbol{X}_t : M \to \bigoplus^k TM$, which in turn implies that $\boldsymbol{X}$ can be considered as a family $\boldsymbol{X} = (X_1, \ldots, X_k)$ of t-dependent vector fields on M with $(X_\alpha)_t = \tau^\alpha \circ \boldsymbol{X}_t$ for $\alpha = 1, \ldots, k$ and every $t \in \mathbb{R}^k$.

Given a t-dependent k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ on M, its *associated system* is the system of first-order partial differential equations

$$\frac{\partial \gamma}{\partial t^\alpha} = X_\alpha(t, \gamma), \quad \alpha = 1, \ldots, k, \quad \gamma \in M, \quad t = (t^1, \ldots, t^k) \in \mathbb{R}^k. \qquad (11.1)$$

Each particular solution $\gamma : \mathbb{R}^k \to M$ with $\gamma(0) = x_0$ is called an *integral section* of $\mathbf{X}$ with initial condition x_0 at $t = 0$. Every t-dependent k-vector field has an associated t-dependent system of partial differential equations in normal form, and every t-dependent system of first-order partial differential equations in normal form uniquely determines the integral sections of some t-dependent k-vector field. We can then use $\mathbf{X}$ to refer both to a t-dependent k-vector field and the t-dependent system of partial differential equations determining its integral sections. It is worth stressing that not every t-dependent k-vector field $\boldsymbol{X}$ is such that its associated system (11.1) is integrable.

As for ordinary differential equations, an integrable system of first-order partial differential equations on M of the form (11.1) is said to admit a *superposition rule* [14] if there exists a t-independent map $\Phi\colon M^\ell \times M \to M$ of the form

$$x = \Phi(x_{(1)}, \ldots, x_{(\ell)}; \lambda),$$

such that every generic solution of (11.1) can be written as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(\ell)}(t); \lambda),$$

where $x_{(1)}(t), \ldots, x_{(\ell)}(t)$ is any generic family of particular solutions of system (3.1) and λ is an element of M.

Theorem 11.1 (PDE Lie Theorem). *A t-dependent system of PDEs $\boldsymbol{X}$, with $t \in \mathbb{R}^k$, admits a superposition rule if and only if $\mathbf{X} = (X_1, \ldots, X_k)$ is integrable and it can be cast into the form*

$$X_\alpha(t,x) = \sum_{\mu=1}^{r} b_\alpha^\mu(t) Y_\mu(x), \quad \alpha = 1, \ldots, k,$$

where $Y_1, \ldots, Y_r$ is a family of vector fields on M spanning an r-dimensional Lie algebra of vector fields V on M and $b_\alpha^\mu(t)$, with $\alpha = 1, \ldots, k$ and $\mu = 1, \ldots, r$, as arbitrary t-dependent functions.

As in the case of Lie systems, the Lie algebra spanned by $Y_1, \ldots, Y_r$ is called a Vessiot–Guldberg Lie algebra of the PDE Lie system.

Let us explain how a PDE Lie system on a Lie group G allows us to solve a PDE Lie system on a manifold M whose Vessiot–Guldberg Lie algebra is given by the fundamental vector fields of a Lie group action $\Phi\colon G \times M \to M$ [55].

Theorem 11.2. *Given a PDE Lie system on a Lie group G given by*

$$\frac{\partial g}{\partial t^\mu} = \sum_{\alpha=1}^{r} b_\mu^\alpha(t,g) X_\alpha^R(g), \quad g \in G, \quad \mu = 1, \ldots, k, \tag{11.2}$$

and a Lie group action $\Phi\colon G \times M \to M$, the solution of the PDE Lie system on M given by

$$\frac{\partial y}{\partial t^\mu} = \sum_{\alpha=1}^{r} b_\mu^\alpha(t,y) X_\alpha(y), \quad y \in M, \quad \mu = 1, \ldots, k,$$

where $X_\alpha(y) = \Phi_{y*} X_\alpha^R(e)$ for the right-invariant vector fields $X_\alpha^R \in \mathfrak{g} \cong T_e G$ with $\alpha = 1, \ldots, r$, as

$$y(t) = \Phi(g(t), y(0))$$

for any $y(0) \in M$ and assuming that $g(t)$ is the particular solution to (11.2) with $g(0) = e$.

Let us provide several known and new examples of PDE Lie systems.

Example 11.3. Let us consider an example coming from a reduction of Wess–Zumino–Witten–Novikov equations [12, 28]. Let us define a system of PDEs induced by a Lie algebra $\mathfrak{g}$ of the form

$$\frac{\partial \psi}{\partial t^i} = \sum_{\beta=1}^{r} f_i^\beta(t) L_\psi M_\beta, \quad \frac{\partial \psi}{\partial t^{-i}} = \sum_{\beta=1}^{r} f_{-i}^\beta(t) R_\psi M_\beta, \quad \psi \in G, \quad i = 1, \ldots, n, \tag{11.3}$$

where $\{M_1, \ldots, M_r\}$ is a basis of $\mathfrak{g}$, while $i = 1, \ldots, n$, $t = (t^{-n}, \ldots, t^{-1}, t^1, \ldots, t^n) \in \mathbb{R}^{2n}$ and $f_{-n}^\beta, \ldots, f_{-1}^\beta, f_1^\beta, \ldots, f_n^\beta \in \mathscr{C}^\infty(\mathbb{R}^{2n})$ are arbitrary functions for every $\beta = 1, \ldots, r$. Recall that L_ψ and R_ψ are the left- and right- multiplications in G by $\psi \in G$. Let us choose bases $\{Y_1^L, \ldots, Y_r^L\}$ and $\{Y_1^R, \ldots, Y_r^R\}$ of left- and right-invariant vector fields on G, respectively, so that system (11.3) can be then rewritten as

$$\frac{\partial \psi}{\partial t^i} = \sum_{\beta=1}^{r} f_i^\beta(t) Y_\beta^L(\psi), \quad \frac{\partial \psi}{\partial t^{-i}} = \sum_{\beta=1}^{r} f_{-i}^\beta(t) Y_\beta^R(\psi), \quad \psi \in G, \quad i = 1, \ldots, n,$$

which is the associated system of the t-dependent $2n$-vector field $\mathbf{X} = (X_{-n}, \ldots, X_{-1}, X_1, \ldots, X_n)$ of the form

$$X_{-i} = \sum_{\beta=1}^{r} f_{-i}^\beta(t) Y_\beta^L, \quad X_i = \sum_{\beta=1}^{r} f_i^\beta(t) Y_\beta^R, \quad i = 1, \ldots, n.$$

The vector fields $Y_1^L, \ldots, Y_r^L, Y_1^R, \ldots, Y_r^R$ span a finite-dimensional Lie algebra, as every left-invariant vector field commutes with every right-invariant vector field, and together span a Lie algebra isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$, provided there is no vector field that is left- and right-invariant simultaneously.

The integrability condition for (11.3) reads

$$\left[\frac{\partial}{\partial t^i} + X_i \, , \, \frac{\partial}{\partial t^j} + X_j\right] = 0, \quad i, j = -n, \ldots, -1, 1, \ldots, n.$$

One can see that if $f_j^\beta = f_j^\beta(t^1, \ldots, t^n)$, $f_{-j}^\beta = f_{-j}^\beta(t^{-1}, \ldots, t^{-n})$, for $j = 1, \ldots, n$ and $\beta = 1, \ldots, r$, and systems of PDEs given by the left- and right-hand sides in (11.3) are integrable, then our WZNW system is a PDE Lie system.

Example 11.4. Let us analyze a system of PDEs associated with Floquet theory and geometric phases for periodic Lie systems [29], which is here shown to be a PDE Lie system for the first time. This allows one to extend Floquet theory, which is applied to t-dependent periodic linear systems, to general nonlinear t-dependent periodic Lie systems. More in detail, the co-adjoint action of the Lie group G on the dual $\mathfrak{g}^*$ to its Lie algebra $\mathfrak{g}$ induces a system of PDEs of the form

$$\frac{\partial \theta}{\partial t} = \mathrm{ad}^*_{p_1(t,s)} \theta, \quad \frac{\partial \theta}{\partial s} = \mathrm{ad}^*_{p_2(t,s)} \theta, \quad \theta \in \mathfrak{g}^*, \quad (t, s) \in \mathbb{R}^2, \tag{11.4}$$

for certain functions $p_1, p_2 \colon \mathbb{R}^2 \to \mathfrak{g}$. In particular, additional assumptions on the periodicity of p_1, p_2 can be assumed in Floquet theory. Anyway, we will skip hereafter these conditions, which could be implemented if desired. Let us choose a basis

$\{\xi_1,\ldots,\xi_r\}$ of $\mathfrak{g}$, which induces a family of fundamental vector fields of the coadjoint action $X^{\mathfrak{g}^*}_{\xi_\alpha}(\theta) = -\mathrm{ad}^*_{\xi_\alpha}\theta$ for $\alpha = 1,\ldots,r$ and $\theta \in \mathfrak{g}^*$. Then, (11.4) reads

$$\frac{\partial\theta}{\partial t} = -\sum_{\alpha=1}^{r} b_1^\alpha(t,s)X^{\mathfrak{g}^*}_{\xi_\alpha}(\theta), \quad \frac{\partial\theta}{\partial s} = -\sum_{\alpha=1}^{r} b_2^\alpha(t,s)X^{\mathfrak{g}^*}_{\xi_\alpha}(\theta), \quad \theta\in\mathfrak{g}^*.$$

The solutions of (11.4) are the integral sections of the (t,s)-dependent two-vector field on $\mathfrak{g}^*$ given by

$$\boldsymbol{X} = \left(-\sum_{\alpha=1}^{r} b_1^\alpha(t,s)X^{\mathfrak{g}^*}_{\xi_\alpha}(\theta), -\sum_{\alpha=1}^{r} b_2^\alpha(t,s)X^{\mathfrak{g}^*}_{\xi_\alpha}(\theta)\right),$$

which makes (11.4) a PDE Lie system provided the integrability condition

$$\left[\frac{\partial}{\partial t} - \sum_{\alpha=1}^{r} b_1^\alpha(t,s)X^{\mathfrak{g}^*}_{\xi_\alpha}(\theta), \frac{\partial}{\partial s} - \sum_{\alpha=1}^{r} b_2^\alpha(t,s)X^{\mathfrak{g}^*}_{\xi_\alpha}(\theta)\right] = 0$$

is satisfied.

Example 11.5. Let us consider the so-called $\mathfrak{g}$-*structure*, as defined in [26], which is related to completely integrable distributions on M with simply transitive symmetry algebras and Cartan connections. A $\mathfrak{g}$-structure is a differential one-form, $\boldsymbol{\Upsilon}$, on an m-dimensional manifold M taking values in a Lie algebra $\mathfrak{g}$ so that

$$\mathrm{d}\boldsymbol{\Upsilon}(X_1,X_2) = -\big[\boldsymbol{\Upsilon}(X_1),\boldsymbol{\Upsilon}(X_2)\big] = -\frac{1}{2}[\boldsymbol{\Upsilon},\boldsymbol{\Upsilon}](X_1,X_2) \tag{11.5}$$

for all vector fields X_1, X_2 on M. In particular, Maurer–Cartan forms are $\mathfrak{g}$-structures [41]. Assume that $\mathfrak{g}$ is the Lie algebra of the Lie group G, and G acts effectively on an n-dimensional manifold N. A *first integral* of $\boldsymbol{\Upsilon}$ is a function $f\colon M\to G$ such that $R\circ Tf = \boldsymbol{\Upsilon}$, where $R\colon v_g\in TG\mapsto R_{g^{-1}*}v_g\in\mathfrak{g}$. By fixing a point $b\in N$ and introducing a Lie algebra homomorphism $\alpha\colon\mathfrak{g}\to\mathfrak{X}(N)$, we can find a local integral $f\colon M\to G$ of $\boldsymbol{\Upsilon}$ by means of the mapping $F\colon q\in M\mapsto f(q)b\in N$, with $b\in N$ and $f(q)b$ the action of $f(q)$ on b, satisfying

$$(F_q)_* = \alpha_{F(q)}\circ\Upsilon_q, \tag{11.6}$$

where $\alpha_q\colon\mathfrak{g}\to T_qN$ is given by $\alpha_q(X) = \alpha(X)_q$. In local coordinates, (11.6) reads

$$\frac{\partial F(q)}{\partial q^i} = \sum_{\beta=1}^{\dim\mathfrak{g}} \alpha_\beta(F(q))\boldsymbol{\Upsilon}_i^\beta(q), \quad i=1,\ldots,m, \tag{11.7}$$

for $M\ni q=(q^1,\ldots,q^m)$. It follows that (11.6) is a system of PDEs associated with the M-dependent m-vector field $\boldsymbol{X} = (X_1,\ldots,X_m)$ on N of the form

$$X_i = \sum_{j=1}^{\dim N}\sum_{\beta=1}^{\dim\mathfrak{g}} \alpha^j_\beta(y)\Upsilon_i^\beta(q)\frac{\partial}{\partial y^j}, \quad i=1,\ldots,m, \quad y\in N.$$

Moreover, one can verify that $\boldsymbol{X}$ is integrable (see Proposition A.2). This makes (11.6) a PDE Lie system, which admits a superposition rule described in [26].

It is worth noting that all locally automorphic Lie systems admit a related $\mathfrak{g}$-structure. Above example shows how the concept given in [26] is really important in the theory of Lie systems and their generalizations, which seems to passed unadvertised so far.

Example 11.6. The next example of PDE Lie system that we are going to study is related to Lax pairs for systems of PDEs [30]. In this case, a system of PDEs is written as the compatibility condition for the integrability of a system of PDEs of the form

$$\frac{\partial g}{\partial t} = A(t,s)g, \quad \frac{\partial g}{\partial s} = B(t,s)g, \quad g \in G, \tag{11.8}$$

where $A(t,x)$ and $B(t,s)$ are functions taking values in the matrix algebra $\mathfrak{g}$ of a matrix group G. Moreover, the integrability condition for (11.8), also called the *zero curvature condition*, can be written as

$$\frac{\partial A}{\partial s} - \frac{\partial B}{\partial t} + [A,B] = 0. \tag{11.9}$$

The coefficients of the matrix functions A, B depend on a function $u\colon \mathbb{R}^2 \to M$ that is a solution of the initial system of PDEs if and only if (11.8) is integrable. In other words, the above system of PDEs (11.9), when considered as a system of PDEs on the variable u, retrieves the system of PDEs under study. Meanwhile, (11.8) can be used to study the properties of (11.9), e.g. constants of motion, Bäcklund transformations, and other properties [21]. In particular, (11.8) can be understood as a linear spectral problem for studying immersed submanifolds related to integrable systems [21].

The adjoint Lie group action of G on $\mathfrak{g}$ relates (11.8) with systems of PDEs of the form

$$\frac{\partial X}{\partial s} = [A, X], \quad \frac{\partial X}{\partial t} = [B, X],$$

where X takes values in the matrix Lie algebra of G. In general, any system of the form

$$\frac{\partial F}{\partial t} = \sum_{\alpha=1}^{r} A_1^\alpha(t,s)X_\alpha, \quad \frac{\partial F}{\partial s} = \sum_{\alpha=1}^{r} A_2^\alpha(t,s)X_\alpha,$$

where $X_1, \ldots, X_r$ are fundamental vector fields of an action of G on a manifold M, which is a PDE Lie system related to (11.8).

Example 11.7. Let us again analyze a system of PDEs associated with Floquet theory [29], which is here shown to be a PDE Lie system for the first time. Additionally, we will show how we relate it to k-contact geometry. Let G be an r-dimensional Lie group and let σ be a smooth map $\mathbb{R}^2 \ni (s,t) \mapsto \sigma(s,t) \in G$. The tangent vectors to the image of σ are spanned by

$$\sigma_{s^*}(s,t) \equiv \frac{\partial \sigma(s,t)}{\partial s} = \sum_{\alpha=1}^{r} f_s^\alpha(s,t) R_{\sigma(s,t)^*} v_\alpha,$$

$$\sigma_{t^*}(s,t) \equiv \frac{\partial \sigma(s,t)}{\partial t} = \sum_{\alpha=1}^{r} f_t^\alpha(s,t) R_{\sigma(s,t)^*} v_\alpha, \tag{11.10}$$

where $\{v_1, \ldots, v_r\}$ is a basis of the Lie algebra $\mathfrak{g} \cong T_e G$. By choosing a base of right-invariant vector fields $X_1^R, \ldots, X_r^R$ on G, system (11.10) can be written as

$$\frac{\partial \sigma(s,t)}{\partial s} = \sum_{\alpha=1}^{r} f_s^\alpha(s,t) X_\alpha^R(\sigma(s,t)),$$

$$\frac{\partial \sigma(s,t)}{\partial t} = \sum_{\alpha=1}^{r} f_t^\alpha(s,t) X_\alpha^R(\sigma(s,t)).$$

The associated $\mathbb{R}^2$-dependent two-vector field $\boldsymbol{X}$ on G is given by

$$\boldsymbol{X} = \left(\sum_{\alpha=1}^{r} f_s^\alpha(s,t) X_\alpha^R(\sigma), \sum_{\alpha=1}^{r} f_t^\alpha(s,t) X_\alpha^R(\sigma) \right)$$

and the integrability condition reads

$$\left[\frac{\partial}{\partial s} + \sum_{\alpha=1}^{r} f_s^\alpha(s,t) X_\alpha^R(\sigma) \ , \ \frac{\partial}{\partial t} + \sum_{\alpha=1}^{r} f_t^\alpha(s,t) X_\alpha^R(\sigma) \right] = 0.$$

In certain cases, one can transform the right-invariant vector fields $(X_1)_{(s,t)} = \sum_{\alpha=1}^{r} f_s^\alpha(s,t) X_\alpha^R$, $(X_2)_{(s,t)} = \sum_{\alpha=1}^{r} f_t^\alpha(s,t) X_\alpha^R$ into $\boldsymbol{\eta}$-Hamiltonian vector fields related to a k-contact form $\boldsymbol{\eta}$, making (11.10) a k-contact PDE Lie system. It is enough to use the properties of the Lie algebra of a Lie group G to construct a left-invariant k-contact form as done in [23] for Lie algebras of certain special unitary Lie groups.

The above example suggests us the following definition.

Definition 11.8. A *k-contact PDE Lie system* is a triple $(M, \boldsymbol{\eta}, \boldsymbol{X} : \mathbb{R}^k \times M \to \bigoplus^k TM)$, where $\boldsymbol{\eta}$ is a k-contact form on M and $\boldsymbol{X}$ is a PDE Lie system on M with $t \in \mathbb{R}^k$, whose smallest Lie algebra, V^X, consists of $\boldsymbol{\eta}$-Hamiltonian vector fields.

Let us prove that a finite-dimensional Lie algebra of Hamiltonian vector fields relative to a k-contact form gives rise to a PDE Lie system which can be understood as some Hamilton–De Donder–Weyl equations.

Theorem 11.9. *If $X_1, \ldots, X_k$ are $\boldsymbol{\eta}$-Hamiltonian vector fields relative to a co-oriented $\boldsymbol{\eta}$-contact manifold $(M, \boldsymbol{\eta})$, then $(X_1, \ldots, X_k)$ is an $\boldsymbol{\eta}$-Hamiltonian k-vector field.*

Proof. Since $X_1, \ldots, X_k$ are $\boldsymbol{\eta}$-Hamiltonian vector fields relative to $(M, \boldsymbol{\eta})$, they possess a series of $\boldsymbol{\eta}$-contact k-functions $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_k$ such that

$$\iota_{X_\alpha} \mathrm{d}\boldsymbol{\eta} = \mathrm{d}\boldsymbol{h}_\alpha - \mathfrak{R}_{\boldsymbol{\eta}} \boldsymbol{h}_\alpha, \quad \iota_{X_\alpha} \boldsymbol{\eta} = -\boldsymbol{h}_\alpha, \quad \alpha = 1, \ldots, k.$$

If $\{e^1, \ldots, e^k\}$ is the dual basis to $\{e_1, \ldots, e_k\}$ in $\mathbb{R}^k$, one has that

$$\langle \iota_{X_\alpha} \mathrm{d}\boldsymbol{\eta}, e^\alpha \rangle = \langle \mathrm{d}\boldsymbol{h}_\alpha, e^\alpha \rangle - \langle \mathfrak{R}_{\boldsymbol{\eta}} \boldsymbol{h}_\alpha, e^\alpha \rangle, \quad \langle \iota_{X_\alpha} \boldsymbol{\eta}, e^\alpha \rangle = -\langle \boldsymbol{h}_\alpha, e^\alpha \rangle, \quad \alpha = 1, \ldots, k.$$

Summing over $\alpha = 1, \ldots, k$, one gets

$$\sum_{\alpha=1}^{k} \iota_{X_\alpha} \mathrm{d}\eta^\alpha = \mathrm{d} \sum_{\alpha=1}^{k} \langle \boldsymbol{h}_\alpha, e^\alpha \rangle - \mathfrak{R}_{\boldsymbol{\eta}} \sum_{\alpha=1}^{k} \langle \boldsymbol{h}_\alpha, e^\alpha \rangle, \quad \iota_{X_\alpha} \eta^\alpha = -\sum_{\alpha=1}^{k} \langle \boldsymbol{h}_\alpha, e^\alpha \rangle.$$

In other words, $(X_1, \ldots, X_k)$ is an $\boldsymbol{\eta}$-Hamiltonian k-vector field. □

Example 11.10. Let us analyze Floquet theory for Lie systems related to a Lie algebra isomorphic to $\mathfrak{gl}_2$. Lie systems of this type appear in numerous applications [2, 4, 29]. For instance, it appears in the Floquet theory and geometric phases for oscillators with t-dependent periodic frequency, mass and drift [24]. Consider the system of PDEs of the form

$$\frac{\partial A}{\partial t} = A f_1(t, s), \qquad \frac{\partial A}{\partial s} = A f_2(t, s), \quad A \in \mathrm{GL}_2, \tag{11.11}$$

where $f_1, f_2 \colon \mathbb{R}^2 \to \mathfrak{gl}_2$ are functions such that the above system of PDEs is integrable, while GL_2 is the general linear group of invertible 2×2 matrices. In other words, the above system of PDEs can be written as

$$\frac{\partial A}{\partial t} = X_1^L(t, s, A), \quad \frac{\partial A}{\partial s} = X_2^L(t, s, A),$$

where $X_i^L(t, s, A)$, with $i = 1, 2$, are $\mathbb{R}^2$-dependent vector fields on GL_2. Then, (11.11) is a PDE Lie system associated with the (s, t)-dependent two-vector field $\boldsymbol{X} = (X_1^L, X_2^L)$ on GL_2. Moreover,

$$\left[\frac{\partial}{\partial t} + X_1^L \ , \ \frac{\partial}{\partial s} + X_2^L \right] = 0.$$

Moreover, one can consider a basis of right-invariant one-forms $\eta_1^R, \ldots, \eta_4^R$ dual at the neutral element e in GL_2 to a basis of $\mathfrak{gl}_2$ such that

$$[v_1, v_2] = v_1, \quad [v_1, v_3] = 2v_2, \quad [v_2, v_3] = v_3,$$

and $[v_4, v_i] = 0$ for $i = 1, \ldots, 4$. Then, $\boldsymbol{\eta}^{\mathrm{GL}_2} = \eta_2^R \otimes e_1 + \eta_4^R \otimes e_2$ is a two-contact form since $\mathrm{d}\eta_2^R = 2\eta_1^R \wedge \eta_3^R$ and $\mathrm{d}\eta_4^R = 0$. By recalling Example 5.4, one gets that $X_1^L(t, s, A), X_2^L(t, s, A)$ are $\mathbb{R}^2$-dependent $\boldsymbol{\eta}^{\mathrm{GL}_2}$-Hamiltonian vector fields with $\mathbb{R}^2$-dependent $\boldsymbol{\eta}^{\mathrm{GL}_2}$-Hamiltonian two-functions $\boldsymbol{h}_1, \boldsymbol{h}_2$. Therefore, Theorem 11.9 ensures that the $\boldsymbol{X}$ associated with (11.11) is an $\mathbb{R}^2$-dependent $\boldsymbol{\eta}^{\mathrm{GL}_2}$-Hamiltonian two-vector field with the $\mathbb{R}^2$-dependent Hamiltonian function $h = \langle \boldsymbol{h}_1, e^1 \rangle + \langle \boldsymbol{h}_2, e^2 \rangle$. Moreover, $(\mathrm{GL}_2, \boldsymbol{\eta}^{\mathrm{GL}_2}, \boldsymbol{X} : \mathbb{R}^2 \times \mathrm{GL}_2 \to \bigoplus^2 T\mathrm{GL}_2)$ is a two-contact PDE Lie system.

The whole theory of k-contact Lie systems can be quite straightforwardly generalized to k-contact PDE Lie systems. For instance, every k-contact PDE Lie system $\boldsymbol{X}$ admits a Vessiot–Guldberg Lie algebra V of Hamiltonian vector fields relative to a k-contact form $\boldsymbol{\eta}$. In turn, V is related to a Lie algebra $\mathfrak{W}$ of $\boldsymbol{\eta}$-Hamiltonian k-functions. Moreover, every k-contact PDE Lie system $\boldsymbol{X}$ has k-components $X_1, \ldots, X_k$ and every X_α of them admits an $\mathbb{R}^k$-dependent $\boldsymbol{\eta}$-Hamiltonian k-function $\boldsymbol{g}_\alpha$ taking values in $\mathfrak{W}$. Then, the analogue of (6.4) for a k-contact PDE Lie systems $\boldsymbol{X}$ and a general $\boldsymbol{\eta}$-Hamiltonian k-function $\boldsymbol{I}$ is

$$\frac{\partial \boldsymbol{I}}{\partial t^\beta}(t) = \frac{\partial \boldsymbol{I}}{\partial t^\beta}(t) + (X_\beta)_t \boldsymbol{I}_t = \frac{\partial \boldsymbol{I}}{\partial t^\beta}(t) + \{\boldsymbol{I}_t, \boldsymbol{g}_\beta\}, \quad \beta = 1, \ldots, k.$$

A whole theory of master symmetries and constants of motion for k-contact PDE Lie systems can be developed on the basis of the above formula and the methods for k-contact Lie systems.

12. Conclusions and Outlook

By using some new ideas based on the notions in [23], this paper introduces and establishes the theoretical and practical significance of k-contact Lie systems. Many examples of k-contact Lie systems have been presented, and k-contact geometry has been employed to study some of their features like t-dependent constants of motion, generators of constants of motion of order s, and master symmetries. Applications demonstrate the approach versatility of our ideas. The work also inspects PDE Lie systems and their potential compatibility with k-contact forms.

It is quite certain that a coalgebra method is available for k-contact Lie systems of projectable type. The method could be also extended to k-contact PDE Lie systems admitting a projectable condition. Our future aim will be to analyze its properties and main applications in the future. Note also that the study of master symmetries and generators of constants of motion of order s has been just started, and there are many further possibilities that can be analyzed. In particular, we aim to apply these ideas more in detail to PDE Lie systems with a compatible k-contact form. Moreover, it is still convenient to characterize k-contact Lie groups on Lie groups of dimension larger than three to extend the work [22], which characterizes k-contact Lie systems on Lie groups of dimension three.

Appendix A

This appendix provides some technical issues that can be also found in [23] and are here given to make the work more self-contained. Nevertheless, they may be skipped in a first lecture and used just in case of necessity.

Proposition A.1. *The map* (4.5) *is well defined.*

Proof. The map does not depend on the vector fields X, X' chosen to extend v, v'. Moreover, any $\boldsymbol{\zeta} \in \Omega^1(U, \mathbb{R}^k)$ such that $\ker \boldsymbol{\zeta} = \mathcal{D}|_U$ gives rise to a trivialization

$TU/\mathcal{D}|_U \simeq U \times \mathbb{R}^k$ and ρ can be described as

$$\begin{aligned}\rho(v, v') = \boldsymbol{\zeta}_x([X, X']_x) &= X_x \iota_{X'} \boldsymbol{\zeta} - X'_x \iota_X \boldsymbol{\zeta} - \mathrm{d}\boldsymbol{\zeta}_x(X_x, X'_x) \\ &= -\mathrm{d}\boldsymbol{\zeta}_x(v, v'), \quad \forall v, v' \in \mathcal{D}_x. \end{aligned} \tag{A.1}$$

□

Proof of Proposition 4.5. If X is an $\boldsymbol{\eta}$-Hamiltonian vector field, then $[X, \mathcal{D}] \subset \mathcal{D}$. This implies that $\mathscr{L}_X \eta^\alpha = \sum_{\beta=1}^k f_\beta^\alpha \eta^\beta$ for certain functions $f_\beta^\alpha \in \mathscr{C}^\infty(M)$ and $\alpha, \beta = 1, \ldots, k$. Then, defining $h^\alpha = -\iota_X \eta^\alpha$ for $\alpha = 1, \ldots, k$, one has

$$(\mathrm{d}\iota_X + \iota_X \mathrm{d})\eta^\alpha = \sum_{\beta=1}^k f_\beta^\alpha \eta^\beta,$$

$$\alpha = 1, \ldots, k \quad \Rightarrow \quad \iota_X \mathrm{d}\eta^\alpha = \mathrm{d}h^\alpha + \sum_{\beta=1}^k f_\beta^\alpha \eta^\beta, \quad \alpha = 1, \ldots, k.$$

Contracting with R_γ, one obtains

$$0 = \iota_{R_\gamma} \iota_X \mathrm{d}\eta^\alpha = R_\gamma h^\alpha + f_\gamma^\alpha, \quad \alpha, \gamma = 1, \ldots, k, \tag{A.2}$$

which implies that $f_\gamma^\alpha = -R_\gamma h^\alpha$, for $\alpha, \gamma = 1, \ldots, k$, and thus

$$\iota_X \mathrm{d}\eta^\alpha = \mathrm{d}h^\alpha - \sum_{\beta=1}^k \eta^\beta (R_\beta h^\alpha), \quad \alpha = 1, \ldots, k.$$

The converse is immediate.

Proof of Proposition 4.14. Let us proceed by reducing to absurd. If $\mathrm{d}\boldsymbol{\eta}$ is degenerate when restricted to $\ker \boldsymbol{\eta}$ at a certain $x \in M$, then there exists a non-zero tangent vector $v \in \ker \boldsymbol{\eta}_x$ such that $\mathrm{d}\boldsymbol{\eta}_x(v, w) = 0$ for every tangent vector $w \in \ker \boldsymbol{\eta}_x$. One gets that $\mathrm{d}\boldsymbol{\eta}_x(v, R_x) = 0$ for each Reeb vector field R at x. Since $\boldsymbol{\eta}$ has k Reeb vector fields spanning $\ker \mathrm{d}\boldsymbol{\eta}$, these vector fields, along with a basis of $\ker \boldsymbol{\eta}$, give rise at x to a basis of $T_x M$. It follows that $0 \neq v \in \ker \mathrm{d}\boldsymbol{\eta}_x \cap \ker \boldsymbol{\eta}_x$ and $\boldsymbol{\eta}$ is not a k-contact form. This is a contradiction and thus $\mathrm{d}\boldsymbol{\eta}$ must be non-degenerate when restricted to $\ker \boldsymbol{\eta}$. In view of (A.1), it follows that $\ker \boldsymbol{\eta}$ is maximally non-integrable.

Proof of Proposition 4.15. The direct part is a consequence of Proposition 4.14 and the Reeb vector fields of $\boldsymbol{\eta}$.

Let us prove the converse part. Given the vector fields $S_1, \ldots, S_k$ on U, consider the differential one-forms $\eta^1, \ldots, \eta^k$ vanishing on $\mathcal{D}$ and dual to $S_1, \ldots, S_k$ on U. Such differential one-forms are unique and exist due to decomposition (4.6). Then, they give rise to $\boldsymbol{\eta} = \sum_{\alpha=1}^k \eta^\alpha \otimes e_\alpha \in \Omega^1(U, \mathbb{R}^k)$ and $\eta^1 \wedge \cdots \wedge \eta^k \neq 0$ on U. For a basis $X_{k+1}, \ldots, X_m$ of $\ker \boldsymbol{\eta}$ around x, one has that the fact that $[S_i, \mathcal{D}] \subset \mathcal{D}$ and $[S_i, S_j] = 0$ for $i, j = 1, \ldots, k$ implies that

$$\mathrm{d}\boldsymbol{\eta}(S_\beta, S_\gamma) = 0, \quad \mathrm{d}\boldsymbol{\eta}(S_\beta, X_\alpha) = 0, \quad \alpha = k+1, \ldots, m, \quad \beta, \gamma = 1, \ldots, k.$$

Therefore, the rank of $\ker \mathrm{d}\boldsymbol{\eta}$ is at least k, and $S_1, \ldots, S_k$ span a supplementary distribution to $\mathcal{D}$ around $x \in M$. Moreover, there exists no tangent vector $v'_{x'} \in \ker \mathrm{d}\boldsymbol{\eta}$ such that $v_{x'} \wedge S_1(x') \wedge \cdots \wedge S_k(x') \neq 0$ for $x' \in U$, as otherwise there would be an element $0 \neq v_{x'} \in \ker \boldsymbol{\eta}_{x'} \cap \ker \mathrm{d}\boldsymbol{\eta}_{x'}$. Such an element would belong to the kernel of $\mathrm{d}\boldsymbol{\eta}_{x'}$ restricted to $\ker \boldsymbol{\eta}_{x'}$ and, since $\mathcal{D}$ is maximally non-integrable, one has that $v_{x'} = 0$. This is a contradiction, $\ker \mathrm{d}\boldsymbol{\eta}$ has rank k, while $\operatorname{rk} \mathcal{D} > 0$ and $\mathcal{D}$ has corank k, and $\ker \mathrm{d}\boldsymbol{\eta} \cap \ker \boldsymbol{\eta} = 0$. Hence, $\boldsymbol{\eta}$ is a k-contact form and $\mathcal{D}$ is a k-contact distribution.

Lemma A.1. *Given an n-dimensional Lie algebra of vector fields $\langle Y_1, \ldots, Y_n \rangle$ on an n-dimensional manifold M whose elements span TM, and let $\Upsilon_1, \ldots, \Upsilon_n$ be a dual basis to $Y_1, \ldots, Y_n$, one has*

$$\mathrm{d}\Upsilon^i = -\frac{1}{2} \sum_{j,k=1}^{n} c_{jk}{}^i \Upsilon^j \wedge \Upsilon^k, \quad i = 1, \ldots, n,$$

for $[Y_i, Y_j] = \sum_{k=1}^n c_{ij}{}^k Y_k$ and $1 \leq i < j \leq n$.

Proof of Proposition 4.15. By using the formula for the differential of a one-form, it follows that

$$\begin{aligned} \mathrm{d}\Upsilon^i(Y_j, Y_k) &= Y_j \Upsilon^i(Y_k) - Y_k \Upsilon^i(Y_j) - \Upsilon^i([Y_j, Y_k]) = -\Upsilon^i([Y_j, Y_k]) \\ &= -\Upsilon^i\left(\sum_{l=1}^{6} c_{jk}{}^l Y_l\right) = -c_{jk}{}^i \end{aligned}$$

for every $i, j, k = 1, \ldots, n$. Since $Y_1, \ldots, Y_n$ form a basis of vector fields on M, this finishes the proof.

Lemma A.2. *The system of PDEs* (11.7) *is integrable.*

Proof of Proposition 4.15. The result follows from the fact that the vector fields $\partial/\partial q^i + X_i$, with $i = 1, \ldots, m$, on $M \times N$ commute among themselves, as shown in the following calculation:

$$\begin{aligned} &\left[\frac{\partial}{\partial q^i} + \sum_{\beta=1}^{\dim \mathfrak{g}} \alpha_\beta(y) \Upsilon_i^\beta(q), \frac{\partial}{\partial q^j} + \sum_{\gamma=1}^{\dim \mathfrak{g}} \alpha_\gamma(y) \Upsilon_j^\gamma(q)\right] \\ &= \sum_{\gamma,\beta=1}^{\dim \mathfrak{g}} \Upsilon_i^\beta(q) \Upsilon_j^\gamma(q) [\alpha_\beta, \alpha_\gamma](y) + \sum_{\beta=1}^{\dim \mathfrak{g}} \alpha_\beta(y) \left(\frac{\partial \Upsilon_j^\beta(q)}{\partial q^i} - \frac{\partial \Upsilon_i^\beta(q)}{\partial q^j}\right) \\ &= \sum_{\gamma=1}^{\dim \mathfrak{g}} \left(\frac{\partial \Upsilon_j^\gamma}{\partial q_i} - \frac{\partial \Upsilon_i^\gamma}{\partial q_j} + \sum_{\beta,\epsilon=1}^{\dim \mathfrak{g}} \Upsilon_i^\beta \Upsilon_j^\epsilon c_{\beta\epsilon}{}^\gamma\right)(q) \alpha_\gamma(y) = 2\left(\mathrm{d}\boldsymbol{\Upsilon} + \frac{1}{2}[\boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}]\right) = 0, \end{aligned}$$

where the last equality follows from (11.5).

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