


## A survey on geometric frameworks for action-dependent classical field theories and their relationship

Jordi Gaset Rifa 

*Department of Mathematics, CUNEF Universidad  
Calle Almansa 101, Madrid 28040, Spain  
jordi.gaset@cunef.edu*

Xavier Rivas 

*Department of Computer Engineering and Mathematics  
Universitat Rovira i Virgili  
Avinguda Països Catalans 26, Tarragona 43007, Spain  
xavier.rivas@urv.cat*

Narciso Román-Roy \*

*Department of Mathematics  
Universitat Politècnica de Catalunya  
C. de Jordi Girona 31, Barcelona 08034, Spain  
narciso.roman@upc.edu*

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This work presents a comprehensive overview of three recently developed geometric frameworks for the study of classical action-dependent field theories. Specifically, we introduce the three underlying geometric structures, namely  $k$ -contact,  $k$ -cocontact and multicontact, and we use them to develop the Lagrangian and Hamiltonian formalisms for such theories. Finally, we analyze the relationships among these three structures in the case of trivial bundles, and we compare them with alternative definitions of multicontact structures found in the literature.

*Keywords:* Classical field theories; action-dependent theories; Lagrangian and Hamiltonian formalisms; multicontact;  $k$ -contact;  $k$ -cocontact structures.

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\*Corresponding author.

## 1. Introduction

In the second half of the 20th century, *symplectic geometry* proved to be a highly effective geometric framework for formalizing analytical mechanics. This success is well documented in numerous classical treatises on the subject (see, for instance, [1, 3, 31, 56, 64, 68], and the references therein).

More recently, particularly during the early 21st century, there has been growing interest in the use of *contact geometry* [7, 46, 58] to describe a specific class of mechanical systems: those exhibiting dissipation or, equivalently, non-conservative behavior (see [11, 18] for a motivated introduction). Beyond this, contact geometry has found broader applications in modeling various physical theories, such as thermodynamics, quantum mechanics, electric circuits and control theory [12, 19, 58]. This renewed interest has led to a significant body of literature. For example, contact Hamiltonian systems have been studied in [13, 14, 26, 51], while their Lagrangian counterparts are addressed in [19, 25, 41]. Other relevant developments include non-autonomous systems [21, 44, 73], quantization [19] and variational formulations dating back to the original work of Herglotz [54, 55]. (This list of references is by no means exhaustive.)

Just as mechanical systems can exhibit non-conservative dynamics, so too can classical field theories. These are generally known in physics as *action-dependent theories*, which are extensions of standard models in which the corresponding Lagrangian and Hamiltonian functions incorporate additional variables related to the action. This results in extra terms in the dynamical or field equations, which can be interpreted as encoding dissipative effects (although the applications of these theories extend well beyond dissipation).

In the conservative case, several geometric frameworks generalizing symplectic geometry have been developed to describe classical field theories. Among them are the so-called *k-(co)symplectic* and *polysymplectic* formalisms, introduced in [4–6] and later expanded and applied to describe Lagrangian and Hamiltonian field theories (see [32, 48, 57] and references therein). However, the most general framework is *multisymplectic geometry* [59, 60], for which there exists extensive literature. We refer to [50, 74, 75] as general sources for its application to field theories and, in particular, [2, 39, 76] for the Lagrangian formalism, [17, 38, 49, 53] for the Hamiltonian setting and [28] for the singular case.

In the context of action-dependent field theories, recent efforts have been made to construct geometric frameworks analogous to the *k*-symplectic, *k*-cosymplectic and multisymplectic formalisms. In particular, in [33, 40, 42, 52, 71], the so-called *k*-contact and *k*-cocontact structures are introduced as natural extensions of contact geometry, built upon the *k*-symplectic and *k*-cosymplectic foundations. Additionally, the fusion of contact and multisymplectic frameworks has recently led to the definition of the *multicontact* structure, proposed in [22–24, 72]. Other less general approaches to similar geometric frameworks appear in [9, 37, 67, 78]; in particular, we highlight the different version of *multicontact manifolds* presented in [79].

As in the case of contact mechanics, the field equations for action-dependent field theories can be derived from a variational principle [47, 62], and for a precise and general variational formulation in this multicontact framework, see [22, 43, 45]. Note that the notion of  $k$ -contact manifold is unrelated to the concept of  $k$ -contact manifold, which is a generalization of *Sasakian manifolds*; these are, in turn, the odd-dimensional counterparts of *Kähler manifolds* (see [8, 10, 77] for more details).

The aim of this paper is two-fold. First, we review the geometric formulations previously introduced for first-order action-dependent field theories, namely the  $k$ -contact,  $k$ -cocontact and multicontact frameworks, as developed in [22–24, 40, 42, 71]. In this review, we restrict our attention to the case of regular theories, that is, those defined by regular Lagrangian functions. For a detailed analysis of singular cases, we refer the reader to the aforementioned references. Second, we establish a correspondence between these geometric structures in the particular setting where the phase bundles associated with the field theories are assumed to be trivial. In this regard, our aim is not to give a detailed description of these formulations, but rather to outline their main features. Readers interested in a detailed development of physically relevant examples may consult [23, 70, 71].

The paper is organized as follows. Sections 2–4 provide a review of the  $k$ -contact,  $k$ -cocontact and multicontact formulations, respectively. In each case, we first introduce the underlying geometric structure and then develop the corresponding Lagrangian and Hamiltonian formalisms for action-dependent field theories. Sections 5 and 6 contain the main original contributions of this work. In particular, Sec. 5 is devoted to studying the relationship between the geometric frameworks mentioned above under the assumption of trivial phase bundles and, in Sec. 6, our notion of a multicontact structure is compared to that given in [79].

All manifolds are real, second-countable and class  $\mathcal{C}^\infty$ , and the mappings are assumed to be smooth. Sum over crossed repeated indices is understood.

The following notation will be used throughout, adhering to standard conventions:

- $\mathcal{C}^\infty(\mathcal{M})$ : Smooth functions in a manifold  $\mathcal{M}$ .
- $\Omega^k(\mathcal{M})$ : Module of differential forms of degree  $k$  in a manifold  $\mathcal{M}$ .
- $\mathfrak{X}(\mathcal{M})$ : Module of vector fields in a manifold  $\mathcal{M}$ .
- $\mathfrak{X}^k(\mathcal{M})$ : Module of  $k$ -multivector fields in a manifold  $\mathcal{M}$ .
- $\iota(X)\Omega$  or  $\iota_X\Omega$ : Inner contraction of a vector field  $X \in \mathfrak{X}(\mathcal{M})$  and a  $k$ -form  $\Omega \in \Omega^k(\mathcal{M})$ .
- $\mathcal{L}(X)$  or  $\mathcal{L}_X$ : Lie derivative by a vector field  $X \in \mathfrak{X}(\mathcal{M})$ .
- $d$ : Exterior differential of differential forms.

## 2. $k$ -Contact Field Theories

In this section, we present the most simple geometric description of action-dependent field theories, using a new framework that is an evolution of the

$k$ -symplectic formulation of classical field theories and contact mechanics. These kinds of formulation are specific for field theories that have the peculiarity that the Lagrangian or Hamiltonian functions describing them are independent of the space-time coordinates (or those analogous to these).

**2.1.  $k$ -contact structures and  $k$ -contact Hamiltonian systems**

The  $k$ -contact (and  $k$ -precontact) manifolds  $(\mathcal{M}, \eta^\alpha)$  and Hamiltonian systems defined on them were introduced in [40], where you can find more details on their definitions and properties. See also [33] for a detailed study of the geometry of  $k$ -contact manifolds.

2.1.1.  $k$ -Contact structures

Given an  $N$ -dimensional manifold  $\mathcal{M}$ , recall that, for every non-vanishing, differential 1-form  $\eta \in \Omega^1(\mathcal{M})$ , its annihilator is a distribution of corank 1, denoted  $\langle \eta \rangle^\circ \subset T\mathcal{M}$ , which can be described as the kernel of the vector bundle morphism  $\widehat{\eta}: T\mathcal{M} \rightarrow \mathcal{M} \times \mathbb{R}$  defined by  $\eta$ . Furthermore,  $\eta$  generates a regular codistribution of rank 1, denoted by  $\langle \eta \rangle \subset T^*\mathcal{M}$ .

The following definition generalizes the concept of *contact structure* (which is recovered as a particular case, when  $k = 1$ ):

**Definition 2.1.** Given  $k$  differential 1-forms  $\eta^1, \dots, \eta^k \in \Omega^1(\mathcal{M})$ , consider the following associated distributions and codistributions:

$$\begin{aligned} \mathcal{C}^C &= \langle \eta^1, \dots, \eta^k \rangle \subset T^*\mathcal{M}, \\ \mathcal{D}^C &= (\mathcal{C}^C)^\circ = \ker \widehat{\eta}^1 \cap \dots \cap \ker \widehat{\eta}^k \subset T\mathcal{M}, \\ \mathcal{D}^R &= \ker \widehat{d\eta}^1 \cap \dots \cap \ker \widehat{d\eta}^k \subset T\mathcal{M}, \\ \mathcal{C}^R &= (\mathcal{D}^R)^\circ \subset T^*\mathcal{M}. \end{aligned}$$

The family  $\{\eta^\alpha\}$  is said to be a  **$k$ -contact structure** on  $\mathcal{M}$  if:

- (i)  $\mathcal{D}^C \subset T\mathcal{M}$  is a regular distribution of corank  $k$ ; or, equivalently,  $\eta^1 \wedge \dots \wedge \eta^k \neq 0$ , at every point.
- (ii)  $\mathcal{D}^R \subset T\mathcal{M}$  is a regular distribution of rank  $k$ .
- (iii)  $\mathcal{D}^C \cap \mathcal{D}^R = \{0\}$  or, what is equivalent,  $\bigcap_{\alpha=1}^k (\ker \widehat{\eta}^\alpha \cap \ker \widehat{d\eta}^\alpha) = \{0\}$ .

A  $k$ -contact manifold is a manifold  $\mathcal{M}$  endowed with a  $k$ -contact structure and is denoted  $(\mathcal{M}, \eta^\alpha)$ ,  $1 \leq \alpha \leq k$ . We call  $\mathcal{C}^C$  the *contact codistribution*,  $\mathcal{D}^C$  the *contact distribution*,  $\mathcal{D}^R$  the *Reeb distribution*, and  $\mathcal{C}^R$  the *Reeb codistribution* of the  $k$ -contact structure.

**Remark 2.2.** If conditions (i) and (ii) hold, then (iii) is equivalent to

$$(iii') \quad T\mathcal{M} = \mathcal{D}^C \oplus \mathcal{D}^R.$$

**Theorem 2.3.** Let  $(\mathcal{M}, \eta^\alpha)$  be a  $k$ -contact manifold.

- (1) The Reeb distribution  $\mathcal{D}^R$  is involutive and therefore integrable.
- (2) There exist  $k$  vector fields  $R_\alpha \in \mathfrak{X}(\mathcal{M})$ , called Reeb vector fields, which are uniquely defined by the relations

$$\iota_{R_\beta} \eta^\alpha = \delta_{\beta}^\alpha, \quad \iota_{R_\beta} d\eta^\alpha = 0. \tag{2.1}$$

- (3) The Reeb vector fields commute,  $[R_\alpha, R_\beta] = 0$ , and they generate the Reeb distribution  $\mathcal{D}^R$ .

**Proposition 2.4.** Let  $(\mathcal{M}, \eta^\alpha)$  be a  $k$ -contact manifold. Around every point of  $\mathcal{M}$ , there is a local chart of coordinates  $(U; z^I; s^\alpha)$ ,  $U \subset \mathcal{M}$ , such that

$$R_\alpha|_U = \frac{\partial}{\partial s^\alpha}, \quad \eta^\alpha|_U = ds^\alpha - f_I^\alpha(z^J) dz^I,$$

which are called adapted coordinates (to the  $k$ -contact structure).

The existence of canonical coordinates is only assured for a particular kind of  $k$ -contact manifolds.

**Theorem 2.5 (Darboux theorem for  $k$ -contact manifolds).** Let  $(\mathcal{M}, \eta^\alpha)$  be a  $k$ -contact manifold of dimension  $n + kn + k$  such that there exists an integrable subdistribution  $\mathcal{V}$  of  $\mathcal{D}^C$  with  $\text{rank } \mathcal{V} = nk$ . Then, around every point of  $\mathcal{M}$ , there exists a local chart of coordinates  $(U; y^a, p_a^\alpha, s^\alpha)$ ,  $1 \leq \alpha \leq k$ ,  $1 \leq a \leq n$ , such that

$$\eta^\alpha|_U = ds^\alpha - p_a^\alpha dy^a, \quad \mathcal{D}^R|_U = \left\langle R_\alpha = \frac{\partial}{\partial s^\alpha} \right\rangle, \quad \mathcal{V}|_U = \left\langle \frac{\partial}{\partial p_a^\alpha} \right\rangle.$$

They are called the canonical or Darboux coordinates of the  $k$ -contact manifold.

The following example constitutes the canonical model for these kinds of  $k$ -contact manifolds.

**Example 2.6.** Given  $k \geq 1$ , let  $Q$  be an  $n$ -dimensional differentiable manifold; consider the vector bundle  $\oplus^k T^*Q := T^*Q \oplus \dots \oplus^k T^*Q$ , which is called the  $k$ -cotangent bundle or bundle of  $k^1$ -momenta of  $Q$ . Then, the manifold  $\mathcal{M} = (\oplus^k T^*Q) \times \mathbb{R}^k$  has a canonical  $k$ -contact structure defined by the 1-forms

$$\eta^\alpha = ds^\alpha - \theta^\alpha,$$

where  $s^\alpha$  is the  $\alpha$ th cartesian coordinate of  $\mathbb{R}^k$ , and  $\theta^\alpha$  is the pull-back of the canonical 1-form of  $T^*Q$  to  $(\oplus^k T^*Q) \times \mathbb{R}^k$  by the corresponding projection  $(\oplus^k T^*Q) \times \mathbb{R}^k \rightarrow T^*Q$ . Using coordinates  $y^a$  on  $Q$  and natural coordinates  $(y^a, p_a^\alpha)$  on each  $T^*Q$ , their local expressions are

$$\eta^\alpha = ds^\alpha - p_a^\alpha dy^a, \tag{2.2}$$

and the Reeb vector fields are

$$R_\alpha = \frac{\partial}{\partial s^\alpha}.$$

**Remark 2.7.** Note that, if  $(\mathcal{M}, \eta^\alpha)$  admits a polarization  $\mathcal{V}$ , then  $\dim \mathcal{M} = n + nk + k$  for some  $n, k \in \mathbb{N}$ . One may wonder when  $\dim \mathcal{M}$  can be written in this form. Observe that

$$n + nk + k = (n + 1)(k + 1) - 1,$$

so  $\dim \mathcal{M} + 1$  must be a composite number. In other words, if  $(\mathcal{M}, \eta^\alpha, \mathcal{V})$  is a polarized  $k$ -contact manifold,  $\dim \mathcal{M} + 1$  cannot be a prime number.

In order to apply Darboux’s theorem, the manifold must admit a polarization. The following example illustrates this by providing a 2-contact manifold that does not admit Darboux coordinates.

**Example 2.8.** Let  $\mathcal{M} = \mathbb{R}^6$  with coordinates  $(x, y, p, q, s, t)$ . The differential 1-forms

$$\eta^1 = ds - \frac{1}{2}(ydx - xdy), \quad \eta^2 = dt - pdx - qdy$$

define a 2-contact structure on  $\mathcal{M}$ . Let us check the conditions of the definition. First, the 1-forms are clearly linearly independent. Then,

$$d\eta^1 = dx \wedge dy, \quad d\eta^2 = dx \wedge dp + dy \wedge dq,$$

from which  $\mathcal{D}^R = \langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle$ , which has rank 2. Obviously, none of these two vector fields belong to the kernel of the 1-forms, which is condition (iii) in Definition 2.1. The Reeb vector fields are

$$R_1 = \frac{\partial}{\partial s}, \quad R_2 = \frac{\partial}{\partial t}.$$

Note that the coordinates in this example are not Darboux-like. In fact, in view of Remark 2.7, this two-contact manifold cannot be written as a polarized  $k$ -contact manifold: its dimension, six, is not a composite number minus one.

**Remark 2.9.** In this section, we have defined  $k$ -contact manifolds inspired by the canonical model given in Example 2.6. Assuming the existence of a polarization, this guaranties the existence of well-defined Reeb vector fields,  $k$ -contact forms, Darboux coordinates and so on.

However, jet bundles may be considered as a more general model. Jet bundles are endowed with the so-called *Cartan distribution* [34, 39, 76], which can be locally described as the kernel of a  $k$ -contact structure. This makes it possible to define a  $k$ -contact manifold as a manifold that is locally a jet bundle. This is the approach adopted in [33].

### 2.1.2. $k$ -Contact Hamiltonian systems

First, let  $\oplus^k \text{T}\mathcal{M} := \text{T}\mathcal{M} \oplus \cdots \oplus^k \text{T}\mathcal{M}$  be the so-called  $k$ -tangent bundle or bundle of  $k^1$ -velocities of  $\mathcal{M}$ . It is endowed with natural projections to each direct summand

and to the base manifold:

$$\tau_\alpha: \oplus^k \text{T}\mathcal{M} \rightarrow \text{T}\mathcal{M}, \quad \tau_{\mathcal{M}}^1: \oplus^k \text{T}\mathcal{M} \rightarrow \mathcal{M}.$$

Then, a *k*-vector field on  $\mathcal{M}$  is a section  $\mathbf{X}: \mathcal{M} \rightarrow \oplus^k \text{T}\mathcal{M}$  of the projection  $\tau_{\mathcal{M}}^1$ . It is specified by giving *k* vector fields  $X_1, \dots, X_k \in \mathfrak{X}(\mathcal{M})$ , obtained as  $X_\alpha = \tau_\alpha \circ \mathbf{X}$ . Then, the *k*-vector field is specified as  $\mathbf{X} = (X_1, \dots, X_k)$ . Every *k*-vector field  $\mathbf{X} = (X_1, \dots, X_k)$  induces a decomposable, contravariant, skew-symmetric tensor field,  $X_1 \wedge \dots \wedge X_k$ , which is a section of the bundle  $\Lambda^k \text{T}\mathcal{M} \rightarrow \mathcal{M}$ , and hence this also induces a tangent distribution on  $\mathcal{M}$ . The sections of this bundle are generically called *k*-multivector fields in  $\mathcal{M}$  and, when they are of the form  $X_1 \wedge \dots \wedge X_k$  (at least locally), are called (locally) decomposable *k*-multivector fields (see Appendix A).

Let  $\psi: D \subset \mathbb{R}^k \rightarrow \mathcal{M}$  be an immersion. If  $t = (1, \dots, k)$  denote the canonical coordinates in  $\mathbb{R}^k$ , let  $\psi(x) = (\psi^I(x))$ ,  $1 \leq I \leq N$ . Then, the *first prolongation* of  $\psi$  to  $\oplus^k \text{T}\mathcal{M}$  is the map  $\psi': D \subset \mathbb{R}^k \rightarrow \oplus^k \text{T}\mathcal{M}$  defined by

$$\psi'(x) = \left( \psi^I(x), \text{T}\psi \left( \frac{\partial}{\partial x^1} \Big|_x \right), \dots, \text{T}\psi \left( \frac{\partial}{\partial x^k} \Big|_x \right) \right) \equiv (\psi(x); \psi'_\alpha(x)).$$

We say that  $\psi$  is an *integral map* of a *k*-vector field  $\mathbf{X} = (X_1, \dots, X_k)$  if

$$\psi' = \mathbf{X} \circ \psi, \tag{2.3}$$

or, equivalently, if  $\text{T}\psi \circ \frac{\partial}{\partial x^\alpha} = X_\alpha \circ \psi$ , for every  $\alpha$ . A *k*-vector field  $\mathbf{X}$  is *integrable* if every point of  $Q$  is in the image of an integral map of  $\mathbf{X}$ . In coordinates, if

$$X_\alpha = X_\alpha^a \frac{\partial}{\partial y^a} + X_{\alpha\beta}^a \frac{\partial}{\partial y_\beta^a} + X_\alpha^\beta \frac{\partial}{\partial s^\beta},$$

then  $\psi(x) = (y^a(x), y_\alpha^a(x), s^\alpha(x))$  is an integral map of  $\mathbf{X}$  if and only if it is a solution to the system of partial differential equations,

$$\frac{\partial y^a}{\partial x^\alpha} = X_\alpha^a(\psi), \quad \frac{\partial y_\beta^a}{\partial x^\alpha} = X_{\alpha\beta}^a(\psi(x)), \quad \frac{\partial s^\beta}{\partial x^\alpha} = X_\alpha^\beta(\psi(x)). \tag{2.4}$$

Now, we define the following.

**Definition 2.10.** A *k*-contact Hamiltonian system is a family  $(\mathcal{M}, \eta^\alpha, \mathcal{H})$ , where  $(\mathcal{M}, \eta^\alpha)$  is a *k*-contact manifold, and  $\mathcal{H} \in \mathcal{C}^\infty(\mathcal{M})$  is called a *Hamiltonian function*.

The field equations of a contact Hamiltonian system can be expressed in geometric form in two alternative ways:

**Definition 2.11.** The *k*-contact Hamilton–de Donder–Weyl equations for a map  $\psi: D \subset \mathbb{R}^k \rightarrow \mathcal{M}$  are

$$\begin{cases} \iota_{\psi'_\alpha} d\eta^\alpha = (d\mathcal{H} - (\mathcal{L}_{R_\alpha} \mathcal{H})\eta^\alpha) \circ \psi, \\ \iota_{\psi'_\alpha} \eta^\alpha = -\mathcal{H} \circ \psi. \end{cases} \tag{2.5}$$

The *k*-contact Hamilton–de Donder–Weyl equations for a *k*-vector field  $\mathbf{X} = (X_1, \dots, X_k)$  in  $\mathcal{M}$  are

$$\begin{cases} \iota_{X_\alpha} d\eta^\alpha = d\mathcal{H} - (\mathcal{L}_{R_\alpha} \mathcal{H})\eta^\alpha, \\ \iota_{X_\alpha} \eta^\alpha = -\mathcal{H}. \end{cases} \tag{2.6}$$

Their solutions are called *Hamiltonian k-vector fields*.

Bearing in mind the definition of the integral maps of an integrable *k*-vector field (see Eqs. (2.3) and (2.4)), it is immediate to prove the following.

**Proposition 2.12.** *If  $\mathbf{X}$  is an integrable *k*-vector field in  $\mathcal{M}$ , then every integral map  $\psi: D \subset \mathbb{R}^k \rightarrow \mathcal{M}$  of  $\mathbf{X}$  satisfies the *k*-contact equation (2.5) if and only if  $\mathbf{X}$  is a solution to (2.6).*

If  $(\mathcal{M}, \eta^\alpha, \mathcal{H})$  is a contact Hamiltonian system; using canonical coordinates for the contact structure  $(\mathcal{M}, \eta^\alpha)$ , if  $\psi(x) = (y^a(x), p_a^\alpha(x), s^\alpha(x))$  is a solution to Eqs. (2.5), then  $\psi'_\alpha = (y^a, p_a^\alpha, s^\alpha, \frac{\partial y^a}{\partial \beta}, \frac{\partial p_a^\alpha}{\partial \beta}, \frac{\partial s^\alpha}{\partial \beta})$ , and Eqs. (2.5) read

$$\begin{cases} \frac{\partial y^a}{\partial x^\alpha} = \frac{\partial \mathcal{H}}{\partial p_a^\alpha} \circ \psi, \\ \frac{\partial p_a^\alpha}{\partial x^\alpha} = - \left( \frac{\partial \mathcal{H}}{\partial y^a} + p_a^\alpha \frac{\partial \mathcal{H}}{\partial s^\alpha} \right) \circ \psi, \\ \frac{\partial s^\alpha}{\partial x^\alpha} = \left( p_a^\alpha \frac{\partial \mathcal{H}}{\partial p_a^\alpha} - \mathcal{H} \right) \circ \psi. \end{cases}$$

Furthermore, if  $\mathbf{X} = (X_\alpha)$ , with  $X_\alpha = X_\alpha^\beta \frac{\partial}{\partial s^\beta} + X_\alpha^a \frac{\partial}{\partial y^a} + X_{\alpha a}^\beta \frac{\partial}{\partial p_a^\beta}$ , is a *k*-vector field solution to (2.6), then these equations lead to

$$\begin{cases} X_\alpha^a = \frac{\partial \mathcal{H}}{\partial p_a^\alpha}, \\ X_{\alpha a}^\beta = - \left( \frac{\partial \mathcal{H}}{\partial y^a} + p_a^\alpha \frac{\partial \mathcal{H}}{\partial s^\alpha} \right), \\ X_\alpha^\alpha = p_a^\alpha \frac{\partial \mathcal{H}}{\partial p_a^\alpha} - \mathcal{H}. \end{cases} \tag{2.7}$$

And, from these last equations (2.7) we obtain that (see also [40]).

**Proposition 2.13.** *If  $(\mathcal{M}, \eta^\alpha, \mathcal{H})$  is a contact Hamiltonian system, then there exist solutions to Eqs. (2.6), although they are neither unique nor necessarily integrable.*

**Remark 2.14.** An equivalent way to write Eqs. (2.6) is as follows:

$$\begin{cases} \mathcal{L}_{X_\alpha} \eta^\alpha = -(\mathcal{L}_{R_\alpha} \mathcal{H}) \eta^\alpha, \\ \iota_{X_\alpha} \eta^\alpha = -\mathcal{H}. \end{cases}$$

Another alternative and partially equivalent expression for the Hamilton–De Donder–Weyl equations, without using the Reeb vector fields  $R_\alpha$ , is as follows: consider the 2-forms  $\Omega^\alpha = -\mathcal{H} d\eta^\alpha + d\mathcal{H} \wedge \eta^\alpha$ . On the open set  $\mathcal{O} = \{p \in \mathcal{M} \mid \mathcal{H}(p) \neq 0\}$ , if a  $k$ -vector field  $\mathbf{X} = (X_\alpha)$  satisfies

$$\begin{cases} \iota_{X_\alpha} \Omega^\alpha = 0, \\ \iota_{X_\alpha} \eta^\alpha = -\mathcal{H}, \end{cases}$$

then  $\mathbf{X}$  is a solution of the Hamilton–De Donder–Weyl equations (2.6). Then, the integral maps  $\psi$  of such a  $k$ -vector fields are solutions to

$$\begin{cases} \iota_{\psi'_\alpha} \Omega^\alpha = 0, \\ \iota_{\psi'_\alpha} \eta^\alpha = -\mathcal{H} \circ \psi. \end{cases}$$

This approach, which avoids the need for Reeb vector fields, may be useful when dealing with field theories described by singular Lagrangians, where the Reeb vector fields may fail to exist or to be unique. It is noteworthy that, unlike in mechanics, the field theories arising in modern physics are typically described by singular Lagrangians. See [23, 52] for examples of singular action-dependent field theories.

**Remark 2.15.** Let us briefly consider the case in which the manifold does not admit Darboux coordinates, but only weaker adapted coordinates  $(x^I; s^\alpha)$  (see Proposition 3.5). We have

$$\begin{aligned} R_\alpha &= \frac{\partial}{\partial s^\alpha}, \quad \eta^\alpha = ds^\alpha - f_I^\alpha(x) dx^I, \\ d\eta^\alpha &= \frac{1}{2} \omega_{IJ}^\alpha dx^I \wedge dx^J, \quad \text{with } \omega_{IJ}^\alpha = \frac{\partial f_I^\alpha}{\partial x^J} - \frac{\partial f_J^\alpha}{\partial x^I}. \end{aligned}$$

The map  $\psi$  is expressed as  $\psi(t) = (x^I(t), s^\beta(t))$ , and  $\psi'_\alpha = (x^I, s^\beta; \frac{\partial x^I}{\partial t^\alpha}, \frac{\partial s^\beta}{\partial t^\alpha})$ . Then, Hamilton–De Donder–Weyl equations (2.5) read

$$\begin{cases} \frac{\partial x^J}{\partial t^\alpha} \omega_{JI}^\alpha = \frac{\partial \mathcal{H}}{\partial x^I} + \frac{\partial \mathcal{H}}{\partial s^\alpha} f_I^\alpha, \\ \frac{\partial s^\alpha}{\partial t^\alpha} - f_I^\alpha \frac{\partial x^I}{\partial t^\alpha} = -\mathcal{H}. \end{cases}$$

### 2.2. $k$ -contact Lagrangian formalism

Now we describe the Lagrangian formalism of action-dependent field theories, using  $k$ -contact structures.

2.2.1. Geometry of the phase bundle

Let  $Q$  be an  $n$ -dimensional differentiable manifold, and consider its  $k$ -tangent bundle  $\oplus^k TQ = TQ \oplus \dots \oplus TQ$ . The natural coordinates in  $\oplus^k TQ$  are denoted by  $(y^a, y^a_\alpha)$ , with  $1 \leq i \leq n$  and  $1 \leq \alpha \leq k$ .

The  $k$ -tangent bundle has some canonical structures, which are induced on it by the canonical structures of the tangent bundle  $TQ$ . In particular, first we have the so-called *canonical  $k$ -tangent structure*, which is the set  $(J^1, \dots, J^k)$  of tensor fields of type  $(1, 1)$  in  $\oplus^k TQ$  whose local expressions in natural coordinates are  $J^\alpha = \frac{\partial}{\partial y^a_\alpha} \otimes dy^a$ . Second, we have the *Liouville vector field*  $\Delta \in \mathfrak{X}(\oplus^k TQ)$ , which is the infinitesimal generator of dilations in the fibers of the bundle  $\oplus^k TQ \rightarrow TQ$ ; that is, whose flow  $\psi: \mathbb{R} \times \oplus^k TQ \rightarrow \oplus^k TQ$  is given by  $\psi(t; v_{1q}, \dots, v_{kq}) = (e^t v_{1q}, \dots, e^t v_{kq})$ . In coordinates,  $\Delta = y^a_\alpha \frac{\partial}{\partial y^a_\alpha}$ .

A map  $\varphi: D \subset \mathbb{R}^k \rightarrow \oplus^k TQ$  is said to be *holonomic* if it is the first prolongation of a map  $\phi: D \subset \mathbb{R}^k \rightarrow Q$ . In coordinates, if  $\phi(x) = (\phi^a(x))$ , then  $\phi'(x) = (\phi^a(x), \frac{\partial \phi^a}{\partial x^\alpha}(x))$ . (See [32] for more details on all the above topics.)

Action-dependent Lagrangian field theories are developed in a bundle that is built by enlarging the above  $k$ -tangent bundle to include the dissipation variables. Thus, consider the bundle  $P \equiv \oplus^k TQ \times \mathbb{R}^k$ ; whose natural coordinates are  $(y^a, y^a_\alpha, s^\alpha)$ . We have the canonical projections

$$\begin{aligned} \bar{\tau}_1: P \equiv \oplus^k TQ \times \mathbb{R}^k &\rightarrow \oplus^k TQ, & \bar{\tau}_2: P \equiv \oplus^k TQ \times \mathbb{R}^k &\rightarrow \mathbb{R}^k, \\ s^\alpha: P \equiv \oplus^k TQ \times \mathbb{R}^k &\rightarrow \mathbb{R}, & \bar{\tau}_{Q \times \mathbb{R}^k}: P \equiv \oplus^k TQ \times \mathbb{R}^k &\rightarrow Q \times \mathbb{R}^k. \end{aligned}$$

The manifolds  $P$  and  $Q \times \mathbb{R}^k$  are called the  *$k^1$ -velocity phase space* and the *configuration space* of the  $k$ -contact field theory, respectively.

**Definition 2.16.** Let  $\psi: \mathbb{R}^k \rightarrow Q \times \mathbb{R}^k$  and  $\phi: \mathbb{R}^k \rightarrow Q$  be smooth maps such that  $\psi(x) = (\phi^a(x), s^\alpha(x))$ . The *first prolongation* of  $\psi$  to  $P = \oplus^k TQ \times \mathbb{R}^k$  is the map  $\psi: \mathbb{R}^k \rightarrow \oplus^k TQ \times \mathbb{R}^k$  given by  $\psi = (\phi', s^\alpha)$ ; where  $\phi': \mathbb{R}^k \rightarrow \oplus^k TQ$  is the first prolongation of  $\phi$  to  $\oplus^k TQ$ . The map  $\psi$  is said to be *holonomic* in  $P$ .

In coordinates, the expression of a holonomic map in  $P$  is

$$\psi(x) = \left( \phi^a(x), \frac{\partial \phi^a}{\partial x^\alpha}(x), s^\alpha(x) \right). \tag{2.8}$$

**Definition 2.17.** A  $k$ -vector field  $\Gamma$  in  $P$  is said to be *holonomic* or a *second-order partial differential equation (SOPDE)* if it is integrable and its integral maps are holonomic in  $P$ .

If  $\psi$  is locally given by (2.8), and it is an integral map of a SOPDE  $\Gamma$ , whose vector field components have local expressions as

$$\Gamma_\alpha = \Gamma^a_\alpha \frac{\partial}{\partial y^a} + \Gamma^{\alpha\beta} \frac{\partial}{\partial y^a_\beta} + \Gamma^\beta_\alpha \frac{\partial}{\partial s^\beta}.$$

Then, from (2.3) we see that the components of  $\psi(x)$  are the solution to the system of second-order partial differential equations

$$\frac{\partial \phi^a}{\partial \alpha} = \Gamma_\alpha^a(\psi(x)), \quad \frac{\partial^2 \phi^a}{\partial \alpha \partial \beta} = \Gamma_{\alpha\beta}^a(\psi(x)). \tag{2.9}$$

Therefore, the local expressions of the vector field components of a SOPDE are

$$\Gamma_\alpha = y_\alpha^a \frac{\partial}{\partial y^a} + \Gamma_{\alpha\beta}^a \frac{\partial}{\partial y_\beta^a} + \Gamma_\alpha^\beta \frac{\partial}{\partial s^\beta}, \tag{2.10}$$

and observe that from the second equation of (2.9), we obtain  $\Gamma_{\alpha\beta}^a = \Gamma_{\beta\alpha}^a$ .

**Remark 2.18.** Since  $\oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \oplus^k \text{T}Q$  is a trivial bundle, the canonical structures in  $\oplus^k \text{T}Q$ ; i.e. the canonical  $k$ -tangent structure and the Liouville vector field, can be extended to  $P \equiv \oplus^k \text{T}Q \times \mathbb{R}^k$  in a natural way. They are denoted with the same notation,  $(J^\alpha)$  and  $\Delta$ , and have the same coordinate expressions as above. Then, using these structures, we have the following alternative geometric characterizations for SOPDE  $k$ -vector fields in  $P$ .

Then, a simple calculation in coordinates leads to the following result.

**Proposition 2.19.** *An integrable  $k$ -vector field  $\mathbf{\Gamma} = (\Gamma_\alpha)$  in  $P$  is a SOPDE if and only if  $J^\alpha(\Gamma_\alpha) = \Delta$ .*

**Remark 2.20.** The  $k$ -vector fields that satisfy the above condition,  $J^\alpha(\Gamma_\alpha) = \Delta$ , whose local expression is (2.10), are called *semi-holonomic  $k$ -vector fields*.

### 2.2.2. $k$ -Contact Lagrangian systems

Now, we can state the Lagrangian formalism for action-dependent field theories (see [42]).

**Definition 2.21.** A *Lagrangian function* is a function  $\mathcal{L} \in \mathcal{C}^\infty(P)$ . The *Lagrangian energy* associated with  $\mathcal{L}$  is the function

$$E_{\mathcal{L}} := \Delta(\mathcal{L}) - \mathcal{L} \in \mathcal{C}^\infty(P).$$

The *Cartan forms* associated with the Lagrangian function  $\mathcal{L}$  are

$$\theta_{\mathcal{L}}^\alpha = {}^t(J^\alpha) \circ d\mathcal{L} \in \Omega^1(P), \quad \omega_{\mathcal{L}}^\alpha = -d\theta_{\mathcal{L}}^\alpha \in \Omega^2(P).$$

Finally, we can define the forms

$$\eta_{\mathcal{L}}^\alpha = ds^\alpha - \theta_{\mathcal{L}}^\alpha \in \Omega^1(P), \quad d\eta_{\mathcal{L}}^\alpha = \omega_{\mathcal{L}}^\alpha \in \Omega^2(P).$$

In natural coordinates  $(y^a, y_\alpha^a, s^\alpha)$  of  $P$ , the local expressions of these elements are

$$E_L = y_\alpha^a \frac{\partial \mathcal{L}}{\partial y_\alpha^a} - \mathcal{L}, \quad \eta_{\mathcal{L}}^\alpha = ds^\alpha - \frac{\partial \mathcal{L}}{\partial y_\alpha^a} dy^a. \tag{2.11}$$

**Definition 2.22.** The Legendre map associated with a Lagrangian  $\mathcal{L} \in \mathcal{C}^\infty(P)$  is the fiber derivative of  $\mathcal{L}$ , considered as a function on the vector bundle  $\bar{\tau}_Q \times \mathbb{R}^k : P \rightarrow Q \times \mathbb{R}^k$ ; that is, the map  $\mathcal{FL} : P \equiv \oplus^k TQ \times \mathbb{R}^k \rightarrow P^* \equiv \oplus^k T^*Q \times \mathbb{R}^k$ , given by

$$\mathcal{FL}(v_{1q}, \dots, v_{kq}; s^\alpha) = (\mathcal{FL}_s(v_{1q}, \dots, v_{kq}), s^\alpha); \quad (v_{1q}, \dots, v_{kq}) \in \oplus^k TQ,$$

where  $\mathcal{L}_s$  denotes the restriction of the Lagrangian function to the fibers of the projection  $\bar{\tau}_2 : \oplus^k TQ \times \mathbb{R}^k \rightarrow \mathbb{R}^k$  (i.e. with  $s^\alpha$  “frozen”), and  $\mathcal{FL}_s : \oplus^k TQ \rightarrow \oplus^k T^*Q$  is the corresponding fiber derivative.

The local expression of this map is  $\mathcal{FL}(y^a, y_\alpha^a, s^\alpha) = (y^a, \frac{\partial \mathcal{L}}{\partial y_\alpha^a}, s^\alpha)$ .

**Proposition 2.23.** For a Lagrangian function  $\mathcal{L}$  the following conditions are equivalent:

- (1)  $(P, \eta_\mathcal{L}^\alpha)$  is a  $k$ -contact manifold.
- (2) The Legendre map  $\mathcal{FL}$  is a local diffeomorphism.
- (3) The Hessian matrix  $(\frac{\partial^2 \mathcal{L}}{\partial y_\alpha^a \partial y_\beta^b})$  is nondegenerate everywhere.

**Definition 2.24.** A Lagrangian function  $\mathcal{L}$  is said to be regular if the equivalent conditions in Proposition 2.23 hold. Otherwise,  $\mathcal{L}$  is a singular Lagrangian. In particular,  $\mathcal{L}$  is said to be hyperregular if  $\mathcal{FL}$  is a global diffeomorphism.

**Definition 2.25.** The pair  $(P, \mathcal{L})$  is called a  $k$ -contact Lagrangian system. It defines a  $k$ -contact Hamiltonian system  $(P, \eta_\mathcal{L}^\alpha, E_\mathcal{L})$ .

For a  $k$ -contact Lagrangian system  $(P, \mathcal{L})$ ; i.e. when  $\mathcal{L}$  is regular, the Reeb vector fields  $(R_\mathcal{L})_\alpha \in \mathfrak{X}(P)$  for this system are the unique solution to Eqs. (2.1), which now read as

$$\iota_{(R_\mathcal{L})_\alpha} d\eta_\mathcal{L}^\beta = 0, \quad \iota_{(R_\mathcal{L})_\alpha} \eta_\mathcal{L}^\beta = \delta_\alpha^\beta.$$

In this case, there exists the inverse  $W_{\alpha\beta}^{ij}$  of the Hessian matrix, namely  $W_{\alpha\beta}^{ab} \frac{\partial^2 \mathcal{L}}{\partial y_\beta^b \partial y_\gamma^c} = \delta_c^\alpha \delta_\alpha^\gamma$ , and then we obtain the following:

$$(R_\mathcal{L})_\alpha = \frac{\partial}{\partial s^\alpha} - W_{\gamma\beta}^{ba} \frac{\partial^2 \mathcal{L}}{\partial s^\alpha \partial y_\gamma^b} \frac{\partial}{\partial y_\beta^a}.$$

### 2.2.3. The $k$ -contact Lagrangian equations

The field equations for the Lagrangian formalism of action-dependent field theories can be expressed in the two alternative ways stated in Definition 2.11.

**Definition 2.26.** Let  $(P, \mathcal{L})$  be a  $k$ -contact Lagrangian system.

- (1) The  $k$ -contact Euler–Lagrange equations for holonomic maps  $\psi : \mathbb{R}^k \rightarrow P$  are

$$\begin{cases} \iota_{\psi'_\alpha} d\eta_\mathcal{L}^\alpha = (dE_\mathcal{L} - (\mathcal{L}_{(R_\mathcal{L})_\alpha} E_\mathcal{L} \eta_\mathcal{L}^\alpha) \circ \psi, \\ \iota_{\psi'_\alpha} \eta_\mathcal{L}^\alpha = -E_\mathcal{L} \circ \psi. \end{cases} \tag{2.12}$$

- (2) The  $k$ -contact Lagrangian equations for holonomic  $k$ -vector fields  $\mathbf{X}_{\mathcal{L}} = ((X_{\mathcal{L}})_{\alpha})$  in  $P$  are

$$\begin{cases} \iota_{(X_{\mathcal{L}})_{\alpha}} d\eta_{\mathcal{L}}^{\alpha} = dE_{\mathcal{L}} - (\mathcal{L}_{(R_{\mathcal{L}})_{\alpha}} E_{\mathcal{L}}) \eta_{\mathcal{L}}^{\alpha}, \\ \iota_{(X_{\mathcal{L}})_{\alpha}} \eta_{\mathcal{L}}^{\alpha} = -E_{\mathcal{L}}. \end{cases} \quad (2.13)$$

A  $k$ -vector field which is a solution to these equations is called a *Lagrangian  $k$ -vector field*. These holonomic  $k$ -vector fields are called *Euler–Lagrange  $k$ -vector fields*.

**Proposition 2.27.** *Let  $(P, \mathcal{L})$  be a  $k$ -contact Lagrangian system. If  $\mathbf{X}_{\mathcal{L}}$  is a holonomic  $k$ -vector field (that is, a SOPDE) solution to the Lagrangian equations (2.13), then its integral sections are the solutions to the multicontact Euler–Lagrange field equations for holonomic sections (2.12) associated with  $\mathcal{L}$ .*

*In addition, if the Lagrangian system is regular (that is,  $k$ -contact) then:*

- (1) *The  $k$ -contact Lagrangian field equations for  $k$ -vector fields (2.13) admit solutions on  $P$ . (The solutions are not unique if  $m > 1$ .)*
- (2) *Every  $k$ -vector field  $\mathbf{X}_{\mathcal{L}}$  that is solution to Eqs. (2.13) is semi-holonomic.*

**Proof.** In a natural chart of coordinates of  $P$ , Eqs. (2.12) read

$$\frac{\partial}{\partial \alpha} \left( \frac{\partial \mathcal{L}}{\partial y_{\alpha}^a} \circ \psi \right) = \left( \frac{\partial \mathcal{L}}{\partial y^a} + \frac{\partial \mathcal{L}}{\partial s^{\alpha}} \frac{\partial \mathcal{L}}{\partial y_{\alpha}^a} \right) \circ \psi, \quad \frac{\partial s^{\alpha}}{\partial \alpha} = \mathcal{L} \circ \psi, \quad (2.14)$$

meanwhile, for a  $k$ -vector field  $\mathbf{X}_{\mathcal{L}} = ((X_{\mathcal{L}})_{\alpha})$ , with

$$(X_{\mathcal{L}})_{\alpha} = (X_{\mathcal{L}})_{\alpha}^a \frac{\partial}{\partial y^a} + (X_{\mathcal{L}})_{\alpha\beta}^a \frac{\partial}{\partial y_{\beta}^a} + (X_{\mathcal{L}})_{\alpha}^{\beta} \frac{\partial}{\partial s^{\beta}},$$

the Lagrangian equations (2.13) are

$$0 = \mathcal{L} + \frac{\partial \mathcal{L}}{\partial y_{\alpha}^a} \left( (X_{\mathcal{L}})_{\alpha}^a - y_{\alpha}^a \right) - (X_{\mathcal{L}})_{\alpha}^{\alpha}, \quad (2.15)$$

$$0 = \left( (X_{\mathcal{L}})_{\alpha}^a - y_{\alpha}^a \right) \frac{\partial^2 \mathcal{L}}{\partial y_{\alpha}^a \partial s^{\beta}}, \quad (2.16)$$

$$0 = \left( (X_{\mathcal{L}})_{\alpha}^a - y_{\alpha}^a \right) \frac{\partial^2 \mathcal{L}}{\partial y_{\beta}^b \partial y_{\alpha}^a}, \quad (2.17)$$

$$\begin{aligned} 0 = & \left( (X_{\mathcal{L}})_{\alpha}^a - y_{\alpha}^a \right) \frac{\partial^2 \mathcal{L}}{\partial y^b \partial y_{\alpha}^a} + \frac{\partial \mathcal{L}}{\partial y^b} - \frac{\partial^2 \mathcal{L}}{\partial s^{\beta} \partial y_{\alpha}^b} (X_{\mathcal{L}})_{\alpha}^{\beta} \\ & - \frac{\partial^2 \mathcal{L}}{\partial y^a \partial y_{\alpha}^b} (X_{\mathcal{L}})_{\alpha}^a - \frac{\partial^2 \mathcal{L}}{\partial y_{\beta}^a \partial y_{\alpha}^b} (X_{\mathcal{L}})_{\alpha\beta}^a + \frac{\partial \mathcal{L}}{\partial s^{\alpha}} \frac{\partial \mathcal{L}}{\partial y_{\alpha}^b}. \end{aligned} \quad (2.18)$$

If  $\mathbf{X}_{\mathcal{L}}$  is a SOPDE, then  $y_{\alpha}^a = (X_{\mathcal{L}})_{\alpha}^a$ ; therefore, Eqs. (2.16) hold identically and (2.15) and (2.18) give

$$(X_{\mathcal{L}})_{\alpha}^a = \mathcal{L}, \tag{2.19}$$

$$\frac{\partial \mathcal{L}}{\partial y^b} - \frac{\partial^2 \mathcal{L}}{\partial s^{\beta} \partial y_{\alpha}^b} (X_{\mathcal{L}})_{\alpha}^{\beta} - \frac{\partial^2 \mathcal{L}}{\partial y^a \partial y_{\alpha}^b} y_{\alpha}^a - \frac{\partial^2 \mathcal{L}}{\partial y_{\beta}^a \partial y_{\alpha}^b} (X_{\mathcal{L}})_{\alpha\beta}^a = -\frac{\partial \mathcal{L}}{\partial s^{\alpha}} \frac{\partial \mathcal{L}}{\partial y_{\alpha}^b}. \tag{2.20}$$

Finally, for the holonomic integrable maps of  $\mathbf{X}_{\mathcal{L}}$ , these last equations lead to the Euler–Lagrange equations (2.14) for its integral maps. In addition, the first equation (2.19) relates the variation of the “dissipation coordinates”  $s^{\alpha}$  to the Lagrangian function.

If  $\mathcal{L}$  is a regular Lagrangian, Eqs. (2.17) lead to  $y_{\alpha}^a = (X_{\mathcal{L}})_{\alpha}^a$ , which is the SOPDE condition for  $\mathbf{X}_{\mathcal{L}}$ . Furthermore, Eqs. (2.20) have always solution for coefficients  $(X_{\mathcal{L}})_{\alpha\beta}^b$  (not necessarily unique, unless  $k = 1$ ), since the Hessian matrix  $(\frac{\partial^2 \mathcal{L}}{\partial y_{\alpha}^a \partial y_{\beta}^b})$  is regular everywhere.  $\square$

### 2.3. $k$ -contact Hamiltonian formalism

Next, we use the developments stated in Sec. 2.1.2 to develop the Hamiltonian formalism for action-dependent field theories.

In the  $k$ -contact ambient, action-dependent Hamiltonian field theories are developed in a manifold which is built enlarging the  $k$ -cotangent bundle of a manifold  $Q$ , as in the Lagrangian setting. Thus, we consider the bundle  $P^* \equiv \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$ ; whose natural coordinates are  $(y^a, p_a^{\alpha}, s^{\alpha})$ . We have the canonical projections

$$\begin{aligned} \tilde{\tau}_1: \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k &\rightarrow \oplus^k \mathbb{T}^*Q, & \tilde{\tau}_2: \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k &\rightarrow \mathbb{R}^k, \\ s^{\alpha}: \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k &\rightarrow \mathbb{R}, & \tilde{\tau}_{Q \times \mathbb{R}^k}: \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k &\rightarrow Q \times \mathbb{R}^k. \end{aligned}$$

Regular or  $k$ -contact Hamiltonian field theories take place in the canonical  $k$ -contact manifold  $(\oplus^k \mathbb{T}^*Q \times \mathbb{R}^k, \theta^{\alpha})$ , giving a *Hamiltonian function*  $\mathcal{H} \in \mathcal{C}^{\infty}(\oplus^k \mathbb{T}^*Q \times \mathbb{R}^k)$ .

**Remark 2.28 (The canonical  $k$ -contact Hamiltonian system associated with a  $k$ -contact Lagrangian system).** In particular, if  $(P = \oplus^k \mathbb{T}Q \times \mathbb{R}^k, \mathcal{L})$  is a  $k$ -contact Lagrangian system, we have that  $\mathcal{FL}$  is a local or global diffeomorphism between  $P$  and  $P^*$ , depending on  $\mathcal{L}$  to be a regular or hyper-regular Lagrangian. Then, bearing in mind the coordinate expressions (2.2) and (2.11) of  $\eta^{\alpha}$   $\eta_{\mathcal{L}}^{\alpha}$ , and of the Legendre map, we have that

$$\theta_{\mathcal{L}}^{\alpha} = \mathcal{FL}^* \theta^{\alpha}, \quad \omega_{\mathcal{L}}^{\alpha} = \mathcal{FL}^* \omega^{\alpha},$$

where  $\omega^{\alpha} = -d\theta^{\alpha}$ . Furthermore, there exists (maybe locally) a function  $\mathcal{H} \in \mathcal{C}^{\infty}(P^*)$  such that

$$\mathcal{H} = E_{\mathcal{L}} \circ \mathcal{FL}^{-1}.$$

Then,  $(P^*, \eta^{\alpha}, \mathcal{H})$  is the *canonical  $k$ -contact Hamiltonian system* associated with the  $k$ -contact Lagrangian system  $(P, \mathcal{L})$  and, for it,  $\mathcal{FL}_*(R_{\mathcal{L}})_{\alpha} = R_{\alpha}$ . Therefore, if

$\mathbf{X}_{\mathcal{L}}$  is an Euler–Lagrange  $k$ -vector field associated with  $\mathcal{L}$  in  $P$ , then  $\mathcal{F}\mathcal{L}_*\mathbf{X}_{\mathcal{L}} = \mathbf{X}_{\mathcal{H}}$  is a contact Hamiltonian  $k$ -vector field associated with  $\mathcal{H}$  in  $P^*$ , and conversely.

### 3. $k$ -Cocontact Field Theories

This section reviews the basics of  $k$ -cocontact manifolds and their applications in modeling non-autonomous action-dependent field theories (see [71] for details).

#### 3.1. $k$ -cocontact structures and $k$ -cocontact Hamiltonian systems

First, we summarize the fundamental concepts and properties about  $k$ -cocontact manifolds and  $k$ -cocontact Hamiltonian systems.

##### 3.1.1. $k$ -Cocontact structures

Given an  $N$ -dimensional manifold  $\mathcal{M}$ , let  $\tau^1, \dots, \tau^k \in \Omega^1(\mathcal{M})$  be a family of closed one-forms on  $\mathcal{M}$  and let  $\eta^1, \dots, \eta^k \in \Omega^1(\mathcal{M})$  be a family of one-forms on  $\mathcal{M}$ . A  $k$ -cocontact structure is characterized by a particular relation between certain intersections of the kernels of these one-forms. To facilitate the interpretation of the results, we will use the following compact notation:

- $\mathcal{C}^{\mathcal{C}} = \langle \eta^1, \dots, \eta^k \rangle \subset \mathbb{T}^*\mathcal{M}$ ,
- $\mathcal{D}^{\mathcal{C}} = (\mathcal{C}^{\mathcal{C}})^{\circ} = \ker \widehat{\eta^1} \cap \dots \cap \ker \widehat{\eta^k} \subset \mathbb{T}\mathcal{M}$ ,
- $\mathcal{D}^{\mathcal{R}} = \ker \widehat{d\eta^1} \cap \dots \cap \ker \widehat{d\eta^k} \subset \mathbb{T}\mathcal{M}$ ,
- $\mathcal{C}^{\mathcal{R}} = (\mathcal{D}^{\mathcal{R}})^{\circ} \subset \mathbb{T}^*\mathcal{M}$ ,
- $\mathcal{C}^{\mathcal{S}} = \langle \tau^1, \dots, \tau^k \rangle \subset \mathbb{T}^*\mathcal{M}$ ,
- $\mathcal{D}^{\mathcal{S}} = (\mathcal{C}^{\mathcal{S}})^{\circ} = \ker \widehat{\tau^1} \cap \dots \cap \ker \widehat{\tau^k} \subset \mathbb{T}\mathcal{M}$ .

With these notations, we can define the notion of  $k$ -cocontact structure.

**Definition 3.1.** A  $k$ -cocontact structure on a manifold  $\mathcal{M}$  is a family of  $k$  closed differential one-forms  $\tau^1, \dots, \tau^k \in \Omega^1(\mathcal{M})$  and a family of  $k$  differential one-forms  $\eta^1, \dots, \eta^k \in \Omega^1(\mathcal{M})$  such that, with the preceding notations,

- (1)  $\mathcal{D}^{\mathcal{C}} \subset \mathbb{T}\mathcal{M}$  is a regular distribution of corank  $k$ ,
- (2)  $\mathcal{D}^{\mathcal{S}} \subset \mathbb{T}\mathcal{M}$  is a regular distribution of corank  $k$ ,
- (3)  $\mathcal{D}^{\mathcal{R}} \subset \mathbb{T}\mathcal{M}$  is a regular distribution of rank  $2k$ ,
- (4)  $\mathcal{D}^{\mathcal{C}} \cap \mathcal{D}^{\mathcal{S}}$  is a regular distribution of corank  $2k$ ,  $\mathcal{D}^{\mathcal{C}} \cap \mathcal{D}^{\mathcal{R}}$  is a regular distribution of rank  $k$ , and  $\mathcal{D}^{\mathcal{S}} \cap \mathcal{D}^{\mathcal{R}}$  is a regular distribution of rank  $k$ ,
- (5)  $\mathcal{D}^{\mathcal{C}} \cap \mathcal{D}^{\mathcal{R}} \cap \mathcal{D}^{\mathcal{S}} = \{0\}$ .

We call  $\mathcal{C}^{\mathcal{C}}$  the *contact codistribution*,  $\mathcal{D}^{\mathcal{C}}$  the *contact distribution*,  $\mathcal{D}^{\mathcal{R}}$  the *Reeb distribution*,  $\mathcal{C}^{\mathcal{R}}$  the *Reeb codistribution*,  $\mathcal{C}^{\mathcal{S}}$  the *space-time codistribution* and  $\mathcal{D}^{\mathcal{S}}$  the *space-time distribution*.

A manifold  $\mathcal{M}$  endowed with a  $k$ -cocontact structure  $\tau^1, \dots, \tau^k, \eta^1, \dots, \eta^k \in \Omega^1(\mathcal{M})$  is a  $k$ -cocontact manifold.

Note that condition  $\mathcal{D}^C \cap \mathcal{D}^R \cap \mathcal{D}^S = \{0\}$  implies that

$$T^*\mathcal{M} = \mathcal{C}^C \oplus \mathcal{C}^R \oplus \mathcal{C}^S.$$

**Remark 3.2.** In the particular case  $k = 1$ , a 1-cocontact structure is given by two one-forms  $\tau, \eta$ , with  $d\tau = 0$ . The conditions in Definition 3.1 mean the following: (1)  $\eta \neq 0$  everywhere, (2)  $\tau \neq 0$  everywhere, (4)  $\tau \wedge \eta \neq 0$ , (5)  $\ker \widehat{\tau} \cap \ker \widehat{\eta} \cap \ker \widehat{d\eta} = \{0\}$ , which implies that  $\ker \widehat{d\eta}$  has rank 0, 1 or 2, and (3) implies that  $\ker \widehat{d\eta}$  has rank 2. Thus, a 1-cocontact structure coincides with the cocontact structure introduced in [21] to describe time-dependent contact mechanics.

**Lemma 3.3.** *The Reeb distribution  $\mathcal{D}^R$  and the space-time distribution  $\mathcal{D}^S$  are involutive and, therefore, integrable.*

Thus, the distribution  $\mathcal{D}^R \cap \mathcal{D}^S$  is also involutive and therefore integrable. Moreover, the distribution  $\mathcal{D}^R \cap \mathcal{D}^C$  is also involutive and integrable. The following theorem characterizes a family of vector fields spanning the Reeb distribution  $\mathcal{D}^R$ .

**Theorem 3.4.** *Let  $(\mathcal{M}, \tau^\alpha, \eta^\alpha)$  be a  $k$ -cocontact manifold. Then, there exists a unique family  $R_1^x, \dots, R_k^x, R_1^s, \dots, R_k^s \in \mathfrak{X}(\mathcal{M})$  such that*

$$\begin{aligned} \iota_{R_\alpha^x} d\eta^\beta &= 0, & \iota_{R_\alpha^x} \eta^\beta &= 0, & \iota_{R_\alpha^x} \tau^\beta &= \delta_\alpha^\beta, \\ \iota_{R_\alpha^s} d\eta^\beta &= 0, & \iota_{R_\alpha^s} \eta^\beta &= \delta_\alpha^\beta, & \iota_{R_\alpha^s} \tau^\beta &= 0. \end{aligned}$$

The vector fields  $R_\alpha^x$  are called **space-time Reeb vector fields**. Vector fields  $R_\alpha^s$  are called **contact Reeb vector fields**.

Moreover, the Reeb vector fields commute and span the Reeb distribution introduced in Definition 3.1:

$$\mathcal{D}^R = \langle R_1^x, \dots, R_k^x, R_1^s, \dots, R_k^s \rangle,$$

motivating its name.

The following proposition proves the existence of a special set of coordinates, the so-called adapted coordinates.

**Proposition 3.5.** *Consider a  $k$ -cocontact manifold  $(\mathcal{M}, \tau^\alpha, \eta^\alpha)$ . Then, around every point in  $\mathcal{M}$ , there exist local coordinates  $(\alpha, z^I, s^\alpha)$  such that*

$$R_\alpha^x = \frac{\partial}{\partial x^\alpha}, \quad \tau^\alpha = dx^\alpha, \quad R_\alpha^s = \frac{\partial}{\partial s^\alpha}, \quad \eta^\alpha = ds^\alpha - f_I^\alpha(z^J) dz^I,$$

where the functions  $f_I^\alpha$  only depend on the coordinates  $z^I$ . These coordinates are called adapted coordinates.

**Example 3.6 (Canonical  $k$ -cocontact structure).** Let  $Q$  be a smooth  $n$ -dimensional manifold with coordinates  $(y^a)$  and let  $k \geq 1$ . Consider the product manifold  $\mathbf{P}^* = \mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k$  endowed with natural coordinates  $(x^\alpha; y^a, p_a^\alpha; s^\alpha)$ .

We have the canonical projections

$$\begin{array}{ccccc}
 \mathbb{R} & \xleftarrow{\pi_1^\alpha} & \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k & \xrightarrow{\pi_3^\alpha} & \mathbb{R} \\
 & & \downarrow \pi_2 & \searrow \pi_2^\alpha & \\
 & & \oplus^k \mathbb{T}^*Q & \xrightarrow{\pi^\alpha} & \mathbb{T}^*Q \\
 & \swarrow \pi_\circ & & & \\
 & & \mathbb{R}^k \times Q \times \mathbb{R}^k & & 
 \end{array}$$

Let  $\theta$  be the Liouville one-form on  $\mathbb{T}^*Q$  with local expression in natural coordinates  $\theta = p_a dy^a$ . Then, the family  $(\tau^\alpha, \eta^\alpha)$  where  $\tau^\alpha = \pi_1^{\alpha*} dx$  with  $x$  the canonical coordinate of  $\mathbb{R}$  and  $\eta^\alpha = ds^\alpha - \pi_2^{\alpha*} \theta$ , is a  $k$ -cocontact structure on  $\mathcal{M}$ . In natural coordinates,

$$\tau^\alpha = dx^\alpha, \quad \eta^\alpha = ds^\alpha - p_a^\alpha dy^a.$$

Thus, the Reeb vector fields are  $R_\alpha^x = \partial/\partial x^\alpha$  and  $R_\alpha^s = \partial/\partial s^\alpha$ .

The following theorem is an upgrade of Proposition 3.5 and states the existence of Darboux-like coordinates in a  $k$ -cocontact manifold provided the existence of a certain subdistribution  $\mathcal{V} \subset \mathcal{D}^C$ .

**Theorem 3.7 (Darboux theorem for  $k$ -cocontact manifolds).** *Let  $(\mathcal{M}, \tau^\alpha, \eta^\alpha)$  be a  $k$ -cocontact manifold with dimension  $\dim \mathcal{M} = k + n + kn + k$  such that there exists an integrable subdistribution  $\mathcal{V} \subset \mathcal{D}^C$  with  $\text{rank } \mathcal{V} = nk$ . Then, around every point of  $\mathcal{M}$  there exist local coordinates  $(x^\alpha, y^a, p_a^\alpha, s^\alpha)$ , where  $1 \leq \alpha \leq k$  and  $1 \leq a \leq n$ , such that, locally,*

$$\tau^\alpha = dx^\alpha, \quad \eta^\alpha = ds^\alpha - p_a^\alpha dy^a.$$

Using these coordinates,

$$\mathcal{D}^R = \left\langle R_\alpha^x = \frac{\partial}{\partial x^\alpha}, R_\alpha^s = \frac{\partial}{\partial s^\alpha} \right\rangle, \quad \mathcal{V} = \left\langle \frac{\partial}{\partial p_a^\alpha} \right\rangle.$$

These coordinates are called Darboux coordinates of the  $k$ -cocontact manifold  $(\mathcal{M}, \tau^\alpha, \eta^\alpha)$ .

Taking into account the previous theorem, we can consider the manifold introduced in Example 3.6 as the canonical model for  $k$ -cocontact structures.

### 3.1.2. $k$ -Cocontact Hamiltonian systems

This section introduces the notion of  $k$ -cocontact Hamiltonian system and its Hamilton–De Donder–Weyl equations. The existence of solutions to these equations is proved. We provide local expressions of the Hamilton–De Donder–Weyl equations for maps and  $k$ -vector fields in both the adapted and Darboux coordinates.

**Definition 3.8.** A *k-cocontact Hamiltonian system* is a tuple  $(\mathcal{M}, \tau^\alpha, \eta^\alpha, h)$ , where  $(\tau^\alpha, \eta^\alpha)$  is a *k-cocontact structure* on the manifold  $\mathcal{M}$  and  $h: \mathcal{M} \rightarrow \mathbb{R}$  is a *Hamiltonian function*. Given a map  $\psi: D \subset \mathbb{R}^k \rightarrow \mathcal{M}$ , the *k-cocontact Hamilton–De Donder–Weyl equations for the map  $\psi$*  are

$$\begin{cases} \iota_{\psi'_\alpha} d\eta^\alpha = (dh - (\mathcal{L}_{R_\alpha^x} h)\tau^\alpha - (\mathcal{L}_{R_\alpha^s} h)\eta^\alpha) \circ \psi, \\ \iota_{\psi'_\alpha} \eta^\alpha = -h \circ \psi, \\ \iota_{\psi'_\alpha} \tau^\beta = \delta_\alpha^\beta. \end{cases} \tag{3.1}$$

Now we are going to look at the expression in coordinates of the Hamilton–De Donder–Weyl equations (3.1). Consider first the adapted coordinates  $(x^\alpha, z^I, s^\alpha)$ , where  $t = (t^1, \dots, t^k) \in \mathbb{R}^k$ . In these coordinates,

$$\begin{aligned} R_\alpha^x &= \frac{\partial}{\partial x^\alpha}, & \tau^\alpha &= dx^\alpha, & R_\alpha^s &= \frac{\partial}{\partial s^\alpha}, \\ \eta^\alpha &= ds^\alpha - f_I^\alpha(z^J)dz^I, & d\eta^\alpha &= \frac{1}{2}\omega_{IJ}^\alpha dz^I \wedge dz^J, \end{aligned}$$

where  $\omega_{IJ}^\alpha = \frac{\partial f_I^\alpha}{\partial z^J} - \frac{\partial f_J^\alpha}{\partial z^I}$ . Consider a map  $\psi: D \subset \mathbb{R}^k \rightarrow \mathcal{M}$  with local expression  $\psi(t) = (x^\alpha(t), x^I(t), s^\alpha(t))$ . Then

$$\psi'_\alpha = \left( x^\beta, z^I, s^\beta; \frac{\partial x^\beta}{\partial t^\alpha}, \frac{\partial z^I}{\partial t^\alpha}, \frac{\partial s^\beta}{\partial t^\alpha} \right).$$

Then, the Hamilton–De Donder–Weyl equations in adapted coordinates read as follows:

$$\begin{cases} \frac{\partial x^J}{\partial t^\alpha} \omega_{JI}^\alpha = \left( \frac{\partial h}{\partial x^I} + \frac{\partial h}{\partial t^\alpha} f_I^\alpha \right) \circ \psi, \\ \frac{\partial s^\alpha}{\partial t^\alpha} - f_I^\alpha \frac{\partial x^I}{\partial t^\alpha} = -h \circ \psi, \\ \frac{\partial x^\alpha}{\partial t^\beta} = \delta_\beta^\alpha. \end{cases}$$

Furthermore, if the local expression in Darboux coordinates of a map  $\psi: D \subset \mathbb{R}^k \rightarrow \mathcal{M}$  is  $\psi(t) = (x^\alpha(t), y^a(t), p_a^\alpha(t), s^\alpha(t))$ . Then, the Hamilton–De Donder–Weyl equations in Darboux coordinates read

$$\begin{cases} \frac{\partial x^\beta}{\partial t^\alpha} = \delta_\alpha^\beta, \\ \frac{\partial y^a}{\partial t^\alpha} = \frac{\partial h}{\partial p_a^\alpha} \circ \psi, \\ \frac{\partial p_a^\alpha}{\partial t^\alpha} = - \left( \frac{\partial h}{\partial y^a} + p_a^\alpha \frac{\partial h}{\partial s^\alpha} \right) \circ \psi, \\ \frac{\partial s^\alpha}{\partial t^\alpha} = \left( p_a^\alpha \frac{\partial h}{\partial p_a^\alpha} - h \right) \circ \psi. \end{cases}$$

**Definition 3.9.** Consider a  $k$ -cocontact Hamiltonian system  $(\mathcal{M}, \tau^\alpha, \eta^\alpha, h)$ . The  $k$ -cocontact Hamilton–De Donder–Weyl equations for a  $k$ -vector field  $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(\mathcal{M})$  are

$$\begin{cases} \iota_{X_\alpha} d\eta^\alpha = dh - (\mathcal{L}_{R_\alpha^x} h)\tau^\alpha - (\mathcal{L}_{R_\alpha^s} h)\eta^\alpha, \\ \iota_{X_\alpha} \eta^\alpha = -h, \\ \iota_{X_\alpha} \tau^\beta = \delta_\alpha^\beta. \end{cases} \quad (3.2)$$

A  $k$ -vector field solution to these equations is a  $k$ -cocontact Hamiltonian  $k$ -vector field. We will denote this set of  $k$ -vector fields by  $\mathfrak{X}_{\text{Ham}}^k(\mathcal{M})$ .

**Proposition 3.10.** The  $k$ -cocontact Hamilton–De Donder–Weyl equations (3.2) admit solutions. They are not unique if  $k > 1$ .

Consider a  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(\mathcal{M})$  with local expression in adapted coordinates

$$X_\alpha = A_\alpha^\beta \frac{\partial}{\partial x^\beta} + B_\alpha^I \frac{\partial}{\partial z^I} + D_\alpha^\beta \frac{\partial}{\partial s^\beta}.$$

Thus, Eqs. (3.2) in adapted coordinates read

$$\begin{cases} A_\alpha^\beta = \delta_\alpha^\beta, \\ B_\alpha^J \omega_{JI}^\alpha = \frac{\partial h}{\partial z^I} + \frac{\partial h}{\partial s^\alpha} f_I^\alpha, \\ D_\alpha^\alpha - f_I^\alpha B_\alpha^I = -h. \end{cases}$$

On the other hand, consider a  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(\mathcal{M})$  with local expression in Darboux coordinates

$$X_\alpha = A_\alpha^\beta \frac{\partial}{\partial x^\beta} + B_\alpha^a \frac{\partial}{\partial y^a} + C_{\alpha i}^\beta \frac{\partial}{\partial p_a^\beta} + D_\alpha^\beta \frac{\partial}{\partial s^\beta}.$$

Imposing Eqs. (3.2), we get the conditions

$$\begin{cases} A_\alpha^\beta = \delta_\alpha^\beta, \\ B_\alpha^a = \frac{\partial h}{\partial p_a^\alpha}, \\ C_{\alpha i}^\alpha = - \left( \frac{\partial h}{\partial y^a} + p_a^\alpha \frac{\partial h}{\partial s^\alpha} \right), \\ D_\alpha^\alpha = p_a^\alpha \frac{\partial h}{\partial p_a^\alpha} - h. \end{cases}$$

**Proposition 3.11.** Let  $\mathbf{X} \in \mathfrak{X}^k(\mathcal{M})$  be an integrable  $k$ -vector field. Then  $\mathbf{X}$  is a solution to (3.2) if and only if every integral section of  $\mathbf{X}$  satisfies the  $k$ -cocontact Hamilton–De Donder–Weyl equations (3.1).

It is worth noting that, as in the  $k$ -symplectic and  $k$ -contact cases, Eqs. (3.1) and (3.2) are not completely equivalent since a solution to (3.1) may not be an integral section of an integrable  $k$ -vector field  $\mathbf{X}$  solution to Eqs. (3.2).

The following proposition provides an alternative way of writing the  $k$ -cocontact Hamilton–De Donder–Weyl equations for  $k$ -vector fields.

**Proposition 3.12.** *The  $k$ -cocontact Hamilton–De Donder–Weyl equations (3.2) are equivalent to*

$$\begin{cases} \mathcal{L}_{X_\alpha} \eta^\alpha = -(\mathcal{L}_{R_\alpha^s} h) \tau^\alpha - (\mathcal{L}_{R_\alpha^s} h) \eta^\alpha, \\ \iota_{X_\alpha} \eta^\alpha = -h, \\ \iota_{X_\alpha} \tau^\beta = \delta_\alpha^\beta. \end{cases}$$

### 3.2. $k$ -cocontact Lagrangian formalism

In this section, we devise the Lagrangian counterpart of the formulations introduced in the previous section. We begin by introducing the geometric structures of the phase bundle and defining the notion of second-order partial differential equation. In second place, we develop the Lagrangian formalism and introduce the  $k$ -cocontact Euler–Lagrange equations as the Hamilton–De Donder–Weyl of a  $k$ -cocontact Lagrangian system.

#### 3.2.1. Geometry of the phase bundle

The phase space for the Lagrangian counterpart of the  $k$ -cocontact formalism will be the product bundle  $\mathbf{P} = \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$  endowed with natural coordinates  $(x^\alpha, y^a, y_\alpha^a, s^\alpha)$ . We have the natural projections

$$\begin{aligned} \tau_1^\alpha: \mathbf{P} &\rightarrow \mathbb{R}, & \tau_1^\alpha(x^1, \dots, x^k, v_{q_1}, \dots, v_{q_k}, s^1, \dots, s^k) &= x^\alpha, \\ \tau_2: \mathbf{P} &\rightarrow \oplus^k \text{T}Q, & \tau_2(x^1, \dots, x^k, v_{q_1}, \dots, v_{q_k}, s^1, \dots, s^k) &= (v_{q_1}, \dots, v_{q_k}), \\ \tau_2^\alpha: \mathbf{P} &\rightarrow \text{T}Q, & \tau_2^\alpha(x^1, \dots, x^k, v_{q_1}, \dots, v_{q_k}, s^1, \dots, s^k) &= v_{q_\alpha}, \\ \tau^\alpha: \oplus^k \text{T}Q &\rightarrow \text{T}Q, & \tau^\alpha(x^1, \dots, x^k, v_{q_1}, \dots, v_{q_k}, s^1, \dots, s^k) &= v_{q_\alpha}, \\ \tau_3^\alpha: \mathbf{P} &\rightarrow \mathbb{R}, & \tau_3^\alpha(x^1, \dots, x^k, v_{q_1}, \dots, v_{q_k}, s^1, \dots, s^k) &= s^\alpha, \\ \tau_\circ: \mathbf{P} &\rightarrow \mathbb{R}^k \times Q \times \mathbb{R}^k, & \tau_\circ(x^1, \dots, x^k, v_{q_1}, \dots, v_{q_k}, s^1, \dots, s^k) & \\ & & &= (x^1, \dots, x^k, q, s^1, \dots, s^k), \end{aligned}$$

which can be summarized in the following diagram:

$$\begin{array}{ccc}
 \mathbb{R} & \xleftarrow{\tau_1^\alpha} & \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k & \xrightarrow{\tau_3^\alpha} & \mathbb{R} \\
 & & \downarrow \tau_2 & \searrow \tau_2^\alpha & \\
 & & \oplus^k \text{T}Q & \xrightarrow{\tau^\alpha} & \text{T}Q \\
 & \swarrow \tau_\circ & & & \\
 & & \mathbb{R}^k \times Q \times \mathbb{R}^k & & 
 \end{array}$$

Since the bundle  $\tau_2: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \oplus^k \text{T}Q$  is trivial, the canonical structures in  $\oplus^k \text{T}Q$ , namely the canonical  $k$ -tangent structure ( $J^\alpha$ ) and the Liouville vector field  $\Delta$ , can be extended to  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$  in a natural way. Their local expression remain the same:

$$J^\alpha = \frac{\partial}{\partial y_\alpha^a} \otimes dy^a, \quad \Delta = y_\alpha^a \frac{\partial}{\partial y_\alpha^a}.$$

These canonical structures can be used to extend the notion of second-order partial differential equation (SOPDE) to the bundle  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$ :

**Definition 3.13.** A  $k$ -vector field  $\Gamma = (\Gamma_\alpha) \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k)$  is a *second-order partial differential equation* or SOPDE if  $J^\alpha(\Gamma_\alpha) = \Delta$ .

A straightforward computation shows that the local expression of a SOPDE reads

$$\Gamma_\alpha = A_\alpha^\beta \frac{\partial}{\partial x^\beta} + y_\alpha^a \frac{\partial}{\partial y^a} + C_{\alpha\beta}^a \frac{\partial}{\partial y_\beta^a} + D_\alpha^\beta \frac{\partial}{\partial s^\beta}.$$

**Definition 3.14.** Consider a map  $\psi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times Q \times \mathbb{R}^k$  with  $\psi = (x^\alpha, \phi, s^\alpha)$ , where  $\phi: \mathbb{R}^k \rightarrow Q$ . The *first prolongation* of  $\psi$  to  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$  is the map  $\psi': \mathbb{R}^k \rightarrow \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$  given by  $\psi' = (x^\alpha, \phi', s^\alpha)$ , where  $\phi'$  is the first prolongation of  $\phi$  to  $\oplus^k \text{T}Q$ . The map  $\psi'$  is said to be *holonomic*.

Let  $\psi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times Q \times \mathbb{R}^k$  be a map with local expression  $\psi(r) = (x^\alpha(r), y^a(r), s^\alpha(r))$ , where  $r \in \mathbb{R}^k$ . Then, its first prolongation has local expression

$$\psi'(t) = \left( x^\alpha(t), y^a(t), \frac{\partial y^a}{\partial t^\alpha}(t), s^\alpha(r) \right).$$

**Proposition 3.15.** An integrable  $k$ -vector field  $\Gamma \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k)$  is a SOPDE if and only if its integral sections are holonomic.

It is important to point out that the product manifold  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$  does not have a canonical  $k$ -cocontact structure, in contrast to what happens to the manifold  $\mathbb{R}^k \times \oplus^k \text{T}^*Q \times \mathbb{R}^k$ , where we do have a natural  $k$ -cocontact structure as seen in Example 3.6. In what follows, we will show that, in favorable cases, given a

Lagrangian function  $L$  defined on  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$  one can build up a  $k$ -cocontact structure.

### 3.2.2. $k$ -Cocontact Lagrangian systems

**Definition 3.16.** A Lagrangian function on  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$  is a function  $L: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \mathbb{R}$ .

- The Lagrangian energy associated with the Lagrangian function  $L$  is the function  $E_L \in \mathcal{C}^\infty(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k)$  given by  $E_L = \Delta(L) - L$ .
- The Cartan forms associated with the Lagrangian  $L$  are

$$\begin{aligned} \theta_L^\alpha &= {}^t J^\alpha \circ dL \in \Omega^1(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k), \\ \omega_L^\alpha &= -d\theta_L^\alpha \in \Omega^2(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k), \end{aligned}$$

where  ${}^t J^\alpha$  denotes the transpose of  $J^\alpha$ .

- The contact forms associated with the Lagrangian  $L$  are

$$\eta_L^\alpha = ds^\alpha - \theta_L^\alpha \in \Omega^1(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k).$$

- The couple  $(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k, L)$  is a  $k$ -cocontact Lagrangian system.

It is clear that  $d\eta_L^\alpha = \omega_L^\alpha$ . The local expressions in natural coordinates  $(x^\alpha, y^a, y_\alpha^a, s^\alpha)$  of the objects introduced in the previous definition are

$$\begin{aligned} E_L &= y_\alpha^a \frac{\partial L}{\partial y_\alpha^a} - L, \\ \theta_L^\alpha &= \frac{\partial L}{\partial y_\alpha^a} dy^a, \\ \eta_L^\alpha &= ds^\alpha - \frac{\partial L}{\partial y_\alpha^a} dy^a, \\ d\eta_L^\alpha &= \frac{\partial^2 L}{\partial x^\beta \partial y_\alpha^a} dy^a \wedge dx^\beta + \frac{\partial^2 L}{\partial y^b \partial y_\alpha^a} dy^a \wedge dy^b + \frac{\partial^2 L}{\partial y_\beta^b \partial y_\alpha^a} dy^a \wedge dy_\beta^b \\ &\quad + \frac{\partial^2 L}{\partial s^\beta \partial y_\alpha^a} dy^a \wedge ds^\beta. \end{aligned}$$

**Definition 3.17.** Given a Lagrangian function  $L: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \mathbb{R}$ , the Legendre map of  $L$  is its fiber derivative as a function on the vector bundle  $\tau_o: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \mathbb{R}^k \times Q \times \mathbb{R}^k$ . That is, the Legendre map of a Lagrangian function

$L: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \mathbb{R}$  is the map

$$\mathcal{F}L: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \mathbb{R}^k \times \oplus^k \text{T}^*Q \times \mathbb{R}^k$$

given by

$$\mathcal{F}L(t, v_{q_1}, \dots, v_{q_k}, z) = (t, \mathcal{F}L(t, \cdot, z)(v_{q_1}, \dots, v_{q_k}), z),$$

where  $\mathcal{F}L(t, \cdot, z)$  denotes the Legendre map of the Lagrangian function with  $t$  and  $z$  “frozen”.

In natural coordinates  $(x^\alpha, y^a, y_\alpha^a, s^\alpha)$ , the Legendre map has local expression

$$\mathcal{F}L(x^\alpha, y^a, y_\alpha^a, s^\alpha) = \left( x^\alpha, y^a, \frac{\partial L}{\partial y_\alpha^a}, s^\alpha \right).$$

**Proposition 3.18.** *The Cartan forms satisfy*

$$\theta_L^\alpha = (\pi_2^\alpha \circ \mathcal{F}L)^* \theta, \quad \omega_L^\alpha = (\pi_2^\alpha \circ \mathcal{F}L)^* \omega,$$

where  $\theta \in \Omega^1(\text{T}^*Q)$  and  $\omega = -d\theta \in \Omega^2(\text{T}^*Q)$  are the Liouville and symplectic canonical forms of the cotangent bundle  $\text{T}^*Q$ .

The regularity of the Legendre map characterizes the Lagrangian functions which yield  $k$ -cocontact structures on the phase bundle  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$ .

**Proposition 3.19.** *Consider a Lagrangian function  $L: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \mathbb{R}$ . The following are equivalent:*

- (1) *The Legendre map  $\mathcal{F}L$  is a local diffeomorphism.*
- (2) *The family  $(\tau^\alpha = dx^\alpha, \eta_L^\alpha)$  is a  $k$ -cocontact structure on  $\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k$ .*

**Proof.** Taking natural coordinates  $(x^\alpha, y^a, y_\alpha^a, s^\alpha)$ , we have

$$\mathcal{F}^2 L(x^\alpha, y^a, y_\alpha^a, s^\alpha) = \left( x^\alpha, y^a, W_{ij}^{\alpha\beta}, s^\alpha \right), \quad \text{where } W_{ij}^{\alpha\beta} = \left( \frac{\partial^2 L}{\partial y_\alpha^a \partial y_\beta^b} \right).$$

The conditions in the proposition mean that the matrix  $W = (W_{ij}^{\alpha\beta})$  is everywhere nonsingular. □

**Definition 3.20.** A Lagrangian function  $L: \mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k \rightarrow \mathbb{R}$  is said to be *regular* if the equivalent statements in Proposition 3.19 hold. Otherwise,  $L$  is said to be *singular*. In addition, if the Legendre map  $\mathcal{F}L$  is a global diffeomorphism,  $L$  is a *hyperregular* Lagrangian.

Let  $(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k, L)$  be a regular  $k$ -cocontact Lagrangian system. By Theorem 3.4, the Reeb vector fields  $(R_L^x)_\alpha, (R_L^s)_\alpha \in \mathfrak{X}(\mathbb{R}^k \times \oplus^k \text{T}Q \times \mathbb{R}^k)$  are uniquely

given by the relations

$$\begin{aligned} \iota_{(R_L^x)_\alpha} d\eta_L^\beta &= 0, & \iota_{(R_L^x)_\alpha} \eta_L^\beta &= 0, & \iota_{(R_L^x)_\alpha} dx^\beta &= \delta_\alpha^\beta, \\ \iota_{(R_L^s)_\alpha} d\eta_L^\beta &= 0, & \iota_{(R_L^s)_\alpha} \eta_L^\beta &= \delta_\alpha^\beta, & \iota_{(R_L^s)_\alpha} dx^\beta &= 0. \end{aligned}$$

The local expressions of the Reeb vector fields are as follows:

$$(R_L^x)_\alpha = \frac{\partial}{\partial x^\alpha} - W_{\gamma\beta}^{ji} \frac{\partial^2 L}{\partial x^\alpha \partial y_\gamma^b} \frac{\partial}{\partial y_\beta^a}, \quad (R_L^s)_\alpha = \frac{\partial}{\partial s^\alpha} - W_{\gamma\beta}^{ji} \frac{\partial^2 L}{\partial s^\alpha \partial y_\gamma^b} \frac{\partial}{\partial y_\beta^a},$$

where  $W_{\alpha\beta}^{ij}$  is the inverse of the Hessian matrix  $W_{ij}^{\alpha\beta} = (\frac{\partial^2 L}{\partial y_\alpha^a \partial y_\beta^b})$ ; that is,

$$W_{\alpha\beta}^{ij} \frac{\partial^2 L}{\partial y_\beta^b \partial y_\gamma^k} = \delta_k^a \delta_\alpha^\gamma.$$

### 3.2.3. $k$ -Cocontact Euler–Lagrange equations

We have proved in the previous section that every regular  $k$ -cocontact Lagrangian system  $(\mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k, L)$  yields the  $k$ -cocontact Hamiltonian system  $(\mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k, \tau^\alpha = dx^\alpha, \eta^\alpha, E_L)$ . Taking this into account, we can define as follows.

**Definition 3.21.** Let  $(\mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k, L)$  be a  $k$ -cocontact Lagrangian system. The  $k$ -cocontact Euler–Lagrange equations for a holonomic map  $\psi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k$  are

$$\begin{cases} \iota_{\psi'_\alpha} d\eta_L^\alpha = \left( dE_L - (\mathcal{L}_{(R_L^x)_\alpha} E_L) dx^\alpha - (\mathcal{L}_{(R_L^s)_\alpha} E_L) \eta_L^\alpha \right) \circ \psi, \\ \iota_{\psi'_\alpha} \eta_L^\alpha = -E_L \circ \psi, \\ \iota_{\psi'_\alpha} dx^\beta = \delta_\alpha^\beta. \end{cases} \tag{3.3}$$

The  $k$ -cocontact Lagrangian equations for a  $k$ -vector field  $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k)$  are

$$\begin{cases} \iota_{X_\alpha} d\eta_L^\alpha = dE_L - (\mathcal{L}_{(R_L^x)_\alpha} E_L) dx^\alpha - (\mathcal{L}_{(R_L^s)_\alpha} E_L) \eta_L^\alpha, \\ \iota_{X_\alpha} \eta_L^\alpha = -E_L, \\ \iota_{X_\alpha} dx^\beta = \delta_\alpha^\beta. \end{cases} \tag{3.4}$$

A  $k$ -vector field  $\mathbf{X}$  solution to Eqs. (3.4) is said to be a  $k$ -cocontact Lagrangian vector field.

The next proposition states that, if the Lagrangian  $L$  is regular, the Lagrangian equations (3.4) always have solutions, although they are not unique in general.

**Proposition 3.22.** Consider a regular  $k$ -cocontact Lagrangian system  $(\mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k, L)$ . Then, the  $k$ -cocontact Lagrangian equations (3.4) admit solutions. They are not unique if  $k > 1$ .

Consider a map  $\psi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k$  with local expression in natural coordinates  $\psi(r) = (x^\alpha(r), y^a(r), y_\alpha^a(r), s^\alpha(r))$ , where  $r = (r^1, \dots, r^k) \in \mathbb{R}^k$ . Then, Eqs. (3.3) for the map  $\psi$  read

$$\begin{cases} \frac{\partial x^\beta}{\partial r^\alpha} = \delta_\alpha^\beta, \\ \frac{\partial}{\partial r^\alpha} \left( \frac{\partial L}{\partial y_\alpha^a} \circ \psi \right) = \left( \frac{\partial L}{\partial y^a} + \frac{\partial L}{\partial s^\alpha} \frac{\partial L}{\partial y_\alpha^a} \right) \circ \psi, \\ \frac{\partial(s^\alpha)}{\partial r^\alpha} = L \circ \psi. \end{cases} \quad (3.5)$$

For a  $k$ -vector field  $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \text{TQ} \times \mathbb{R}^k)$ , with local expression in natural coordinates

$$X_\alpha = A_\alpha^\beta \frac{\partial}{\partial x^\beta} + B_\alpha^a \frac{\partial}{\partial y^a} + C_{\alpha\beta}^a \frac{\partial}{\partial y_\beta^a} + D_\alpha^\beta \frac{\partial}{\partial s^\beta},$$

Eqs. (3.4) read

$$0 = A_\alpha^\beta - \delta_\alpha^\beta, \quad (3.6)$$

$$0 = (B_\alpha^b - y_\alpha^b) \frac{\partial^2 L}{\partial y_\alpha^b \partial s^\beta}, \quad (3.7)$$

$$0 = (B_\alpha^b - y_\alpha^b) \frac{\partial^2 L}{\partial y_\alpha^b \partial x^\beta}, \quad (3.8)$$

$$0 = (B_\alpha^b - y_\alpha^b) \frac{\partial^2 L}{\partial y_\beta^a \partial y_\alpha^b}, \quad (3.9)$$

$$\begin{aligned} 0 = & (B_\alpha^b - y_\alpha^b) \frac{\partial^2 L}{\partial y^a \partial y_\alpha^b} + \frac{\partial L}{\partial y^a} - \frac{\partial^2 L}{\partial x^\alpha \partial y_\alpha^a} - \frac{\partial^2 L}{\partial y^b \partial y_\alpha^a} B_\alpha^b \\ & - \frac{\partial^2 L}{\partial y_\beta^b \partial y_\alpha^a} C_{\alpha\beta}^b - \frac{\partial^2 L}{\partial s^\beta \partial y_\alpha^a} D_\alpha^\beta + \frac{\partial L}{\partial s^\alpha} \frac{\partial L}{\partial y_\alpha^a}, \end{aligned} \quad (3.10)$$

$$0 = L + \frac{\partial L}{\partial y_\alpha^a} (B_\alpha^a - y_\alpha^a) - D_\alpha^\alpha. \quad (3.11)$$

If the Lagrangian function  $L$  is regular, Eqs. (3.9) yield the conditions  $B_\alpha^a = y_\alpha^a$ , namely the  $k$ -vector field  $\mathbf{X}$  has to be a SOPDE. In this case, Eqs. (3.7) and (3.8) hold identically, and Eqs. (3.6), (3.10) and (3.11) yield

$$\begin{cases} A_\alpha^\beta = \delta_\alpha^\beta, \\ \frac{\partial L}{\partial y^a} + \frac{\partial L}{\partial s^\alpha} \frac{\partial L}{\partial y_\alpha^a} = \frac{\partial^2 L}{\partial x^\alpha \partial y_\alpha^a} + \frac{\partial^2 L}{\partial y^b \partial y_\alpha^a} y_\alpha^b + \frac{\partial^2 L}{\partial y_\beta^b \partial y_\alpha^a} C_{\alpha\beta}^b + \frac{\partial^2 L}{\partial s^\beta \partial y_\alpha^a} D_\alpha^\beta, \\ D_\alpha^\alpha = \mathcal{L}. \end{cases} \quad (3.12)$$

If the SOPDE  $\mathbf{X}$  is integrable, Eqs. (3.12) are the Euler–Lagrange equations (3.5) for its integral maps. Thus, we have proved the following.

**Proposition 3.23.** *Let  $L: \mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k \rightarrow \mathbb{R}$  be a regular Lagrangian and consider a Lagrangian  $k$ -vector field  $\mathbf{X}$ , namely a solution to Eqs. (3.4). Then  $\mathbf{X}$  is a SOPDE and if, in addition,  $\mathbf{X}$  is integrable, its integral sections are solutions to the  $k$ -cocontact Euler–Lagrange equations (3.3).*

*The SOPDE  $\mathbf{X}$  is called an Euler–Lagrange  $k$ -vector field associated with the Lagrangian function  $L$ .*

**Remark 3.24.** In the case  $k = 1$ , we recover the cocontact Lagrangian formalism presented in [21] for time-dependent contact Lagrangian systems.

### 3.3. $k$ -cocontact Hamiltonian formalism

Now, the developments stated in Sec. 3.1.2 are used to develop the Hamiltonian formalism for action-dependent field theories in this formulation.

In the  $k$ -cocontact formulation, action-dependent Hamiltonian field theories is developed in the product bundle  $\mathbf{P}^* = \mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k$  endowed with natural coordinates  $(x^\alpha, y^\alpha, p_\alpha^a, s^\alpha)$ . Then, regular or  $k$ -cocontact Hamiltonian field theories take place in the canonical  $k$ -cocontact manifold  $(\mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k, \tau^\alpha, \theta^\alpha)$ , giving a *Hamiltonian function*  $\mathcal{H} \in \mathcal{C}^\infty(\mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k)$ .

**Remark 3.25 (The canonical  $k$ -cocontact Hamiltonian system associated with a  $k$ -cocontact Lagrangian system).** Let  $(\mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k, L)$  be a  $k$ -cocontact Lagrangian system. If the Lagrangian function  $L$  is regular or hyperregular, the Legendre map  $\mathcal{FL}$  is a (local) diffeomorphism between  $\mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k$  and  $\mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k$  such that  $\mathcal{FL}^* \eta^\alpha = \eta_L^\alpha$ . In addition, there exists, at least locally, a function  $h \in \mathcal{C}^\infty(\mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k)$  such that  $h \circ \mathcal{FL} = E_L$ . Then, we have the  $k$ -cocontact Hamiltonian system  $(\mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k, \eta^\alpha, h)$ , for which  $\mathcal{FL}_*(R_L^x)_\alpha = R_\alpha^x$  and  $\mathcal{FL}_*(R_L^s)_\alpha = R_\alpha^s$ . If  $\Gamma$  is an Euler–Lagrange  $k$ -vector field associated with the Lagrangian function  $L$  in  $\mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k$ , we have that the  $k$ -vector field  $\mathbf{X} = \mathcal{FL}_* \Gamma$  is a  $k$ -cocontact Hamiltonian  $k$ -vector field associated with  $h$  in  $\mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k$ , and conversely.

## 4. Multicontact Field Theories

The *multicontact formulation* is the most general geometric framework for describing action-dependent field theories. It was first introduced in [22, 23], although a more general definition of multicontact structure has recently been proposed in [24]. (You can find all the details on this structure and its applications in these references.)

### 4.1. Multicontact structures

First, following [24], we define.

**Definition 4.1.** Let  $\mathcal{M}$  be a differentiable manifold. A form  $\Theta \in \Omega^k(\mathcal{M})$ , with  $\dim \mathcal{M} > k$ , is a *multicontact form* in  $\mathcal{M}$  if:

- (1)  $\ker \Theta \cap \ker d\Theta = \{0\}$ , and
- (2)  $\ker d\Theta \neq \{0\}$ .

Then, the pair  $(\mathcal{M}, \Theta)$  is said to be a *multicontact manifold*.

The properties of this kind of structure have been studied in [24]. Nevertheless, as is already the case with multisymplectic structures [27, 66], the existence of adapted or Darboux coordinates is not guaranteed for these multicontact forms, unless additional conditions are imposed [22].

Thus, let  $\mathcal{M}$  with  $\dim \mathcal{M} = k + N$  be a manifold, with  $N \geq k \geq 1$ ; and let  $\Theta, \omega \in \Omega^k(\mathcal{M})$  be two  $k$ -forms with constant ranks. Given a regular distribution  $\mathcal{D} \subset T\mathcal{M}$ , consider the  $\mathcal{C}^\infty(\mathcal{M})$ -module of sections  $\Gamma(\mathcal{D})$  and, for every  $m \in \mathbb{N}$ , define the set of  $m$ -forms in  $\mathcal{M}$  vanishing by the vector fields in  $\Gamma(\mathcal{D})$ ; that is,

$$\begin{aligned} \mathcal{A}^m(\mathcal{D}) &:= \{ \alpha \in \Omega^m(\mathcal{M}) \mid \iota_Z \alpha = 0 \text{ for every } Z \in \Gamma(\mathcal{D}) \} \\ &= \{ \alpha \in \Omega^m(\mathcal{M}) \mid \Gamma(\mathcal{D}) \subset \ker \alpha \}, \end{aligned}$$

where  $\ker \alpha = \{ Z \in \mathfrak{X}(\mathcal{M}) \mid \iota_Z \alpha = 0 \}$  (it is the 1-ker of a form  $\alpha \in \Omega^m(\mathcal{M})$ , with  $m > 1$ ).

**Definition 4.2.** The *Reeb distribution* associated with the pair  $(\Theta, \omega)$  is the distribution  $\mathcal{D}^{\mathfrak{R}} \subset T\mathcal{M}$  defined as

$$\mathcal{D}^{\mathfrak{R}} = \{ Z \in \ker \omega \mid \iota_Z d\Theta \in \mathcal{A}^k(\ker \omega) \}.$$

The set of sections of the Reeb distribution is denoted by  $\mathfrak{R} := \Gamma(\mathcal{D}^{\mathfrak{R}})$ , and its elements  $R \in \mathfrak{R}$  are called *Reeb vector fields*. If  $\ker \omega$  has a constant rank, then

$$\mathfrak{R} = \{ R \in \Gamma(\ker \omega) \mid \iota_R d\Theta \in \mathcal{A}^k(\ker \omega) \}.$$

Note that  $\ker \omega \cap \ker d\Theta \subset \mathcal{D}^{\mathfrak{R}}$ . Furthermore, if  $\omega \in \Omega^k(\mathcal{M})$  is a closed form and has a constant rank, then  $\mathfrak{R}$  is involutive.

**Definition 4.3.** The pair  $(\Theta, \omega)$  is a *special multicontact structure* on  $\mathcal{M}$  if  $\omega \in \Omega^k(\mathcal{M})$  is a closed form, and we have the following properties:

- (1)  $\text{rank } \ker \omega = N$ .
- (2)  $\text{rank } \mathcal{D}^{\mathfrak{R}} = k$ .
- (3)  $\text{rank}(\ker \omega \cap \ker \Theta \cap \ker d\Theta) = 0$ .
- (4)  $\mathcal{A}^{k-1}(\ker \omega) = \{ \iota_R \Theta \mid R \in \mathfrak{R} \}$ .

Then, the triple  $(\mathcal{M}, \Theta, \omega)$  is said to be a *special multicontact manifold*,  $(\Theta, \omega)$  is a *special multicontact structure*, and  $\Theta \in \Omega^k(\mathcal{M})$  is a *special multicontact form* on  $\mathcal{M}$ .

The following proposition presents an essential characteristic of special multicontact structures.

**Proposition 4.4.** *Let  $(\mathcal{M}, \Theta, \omega)$  be a special multicontact manifold, then there exists a unique 1-form  $\sigma_\Theta \in \Omega^1(\mathcal{M})$ , called the dissipation form, satisfying*

$$\sigma_\Theta \wedge \iota_R \Theta = \iota_R d\Theta \quad \text{for every } R \in \mathfrak{R}.$$

And, using this form, we introduce the following.

**Definition 4.5.** Let  $\sigma_\Theta \in \Omega^1(\mathcal{M})$  be the dissipation form. We define the operator

$$\begin{aligned} \bar{d} : \Omega^m(\mathcal{M}) &\rightarrow \Omega^{m+1}(\mathcal{M}) \\ \beta &\mapsto \bar{d}\beta = d\beta + \sigma_\Theta \wedge \beta. \end{aligned}$$

The multicontact structures corresponding to action-dependent field theories arising from the Herglotz variational principle (see [43]) satisfy the following additional requirement.

**Definition 4.6.** Let  $(\mathcal{M}, \Theta, \omega)$  be a multicontact manifold satisfying

$$\iota_X \iota_Y \Theta = 0 \quad \text{for every } X, Y \in \Gamma(\ker \omega).$$

Then  $(\mathcal{M}, \Theta, \omega)$  is a *variational multicontact manifold* and  $(\Theta, \omega)$  is said to be a *variational multicontact structure*.

The next theorem states the existence of canonical coordinates for these last kinds of multicontact structures.

**Theorem 4.7.** *Given a special multicontact manifold  $(\mathcal{M}, \Theta, \omega)$ ; around every point  $p \in \mathcal{M}$ , there exists a local chart of adapted coordinates  $(U; x^\mu, u^I, s^\mu,)$   $(1 \leq \mu \leq k, 1 \leq I \leq N - k)$  such that*

$$\ker \omega = \left\langle \frac{\partial}{\partial u^I}, \frac{\partial}{\partial s^\mu} \right\rangle, \quad \mathcal{D}^{\mathfrak{R}} = \left\langle \frac{\partial}{\partial s^\mu} \right\rangle,$$

and the coordinates  $(x^\mu)$  can be chosen such that

$$\omega|_U = dx^1 \wedge \cdots \wedge dx^k \equiv d^k x,$$

(and so we shall do henceforth). In addition, if  $(P, \Theta, \omega)$  is a variational multicontact manifold, then the local expression of the multicontact form is

$$\Theta|_U = H(x^\nu, u^J, s^\nu) d^k x + f_I^\mu(x^\nu, u^J) du^I \wedge d^{k-1} x_\mu + ds^\mu \wedge d^{k-1} x_\mu,$$

where  $d^{k-1} x_\mu = \iota_{\frac{\partial}{\partial x^\mu}} d^k x$ . Furthermore, in these coordinates,

$$\sigma_\Theta|_U = \frac{\partial H}{\partial s^\mu} dx^\mu.$$

Moreover, we have the following local characterization of the Reeb vector fields.

**Proposition 4.8.** *If  $(\mathcal{M}, \Theta, \omega)$  is a special multicontact manifold, in the above coordinate chart, there exists a unique local basis  $\{R_\mu\}$  of  $\mathfrak{R}$  such that*

$$(\iota_{R_\mu} \Theta)|_U = d^{k-1}x_\mu.$$

Moreover,  $[R_\mu, R_\nu] = 0$ .

Finally, to establish the relation between multicontact and special multicontact structures, we need following result.

**Lemma 4.9.** *Let  $(\mathcal{M}, \Theta, \omega)$  be a special multicontact manifold. Then*

- (1)  $\ker \Theta \subset \ker \omega$ .
- (2) *Moreover, if  $(\mathcal{M}, \Theta, \omega)$  is a variational multicontact manifold (see Definition 4.6), then*

$$\ker \Theta \oplus D^{\mathfrak{R}} = \ker \omega.$$

**Proof.** First, note that the condition 4 of a special multicontact structure in Definition 4.3 implies that, for every  $(m-1)$ -form  $\alpha$  that vanishes by the contraction of any element of  $\ker \omega$ , there exists a Reeb vector field  $R \in \Gamma(\mathcal{D}^{\mathfrak{R}})$  such that  $\iota_R \Theta = \alpha$ .

- (1) Assume that at a point  $p \in \mathcal{M}$  there exists an element  $Y_p \in T_p \mathcal{M}$  such that  $Y_p \in \ker \Theta_p$  but  $Y_p \notin \ker \omega_p$ . Then, there exists a  $(m-1)$  form  $\alpha$  at  $p$  that vanishes by the contraction of any element of  $\ker \omega_p$  such that  $\iota_{Y_p} \alpha \neq 0$ . Let  $R \in \mathcal{D}^{\mathfrak{R}}$  such that  $\iota_R \Theta = \alpha$ , at  $p$ . Then

$$0 = \iota_{Y_p} \iota_{R_p} \Theta_p = \iota_{Y_p} \alpha_p,$$

which is a contradiction.

- (2) Due to the previous item and the definition of the Reeb distribution, clearly

$$\ker \Theta + D^{\mathfrak{R}} \subset \ker \omega.$$

Fix a point  $p \in \mathcal{M}$  and let  $Y_p \in \ker \omega_p$ . Then  $\iota_{Y_p} \Theta_p$  vanishes by the action of any element of  $\ker \omega_p$ , because it is variational. Then, there exists  $R \in \mathcal{D}^{\mathfrak{R}}$  such that  $\iota_{R_p} \Theta_p = \iota_{Y_p} \Theta_p$ . In particular,  $\iota_{(Y_p - R_p)} \Theta_p = 0$ . Therefore, we can decompose  $Y_p = (Y_p - R_p) + R_p$  as a sum of an element of  $\ker \Theta_p$  and an element of  $D_p^{\mathfrak{R}}$ . Finally,  $\ker \Theta_p \cap D_p^{\mathfrak{R}} = \{0\}$  due to [22, Lemma 3.5].  $\square$

Then, taking this lemma into account, we have the following result:

**Proposition 4.10.** *If  $(\mathcal{M}, \Theta, \omega)$  is a special multicontact manifold, then  $(\mathcal{M}, \Theta)$  is a multicontact manifold.*

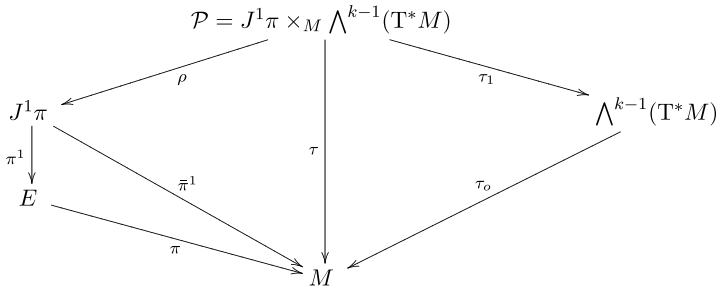
## 4.2. Multicontact Lagrangian formalism

Next, we describe the Lagrangian action-dependent field theories using multicontact structures.

4.2.1. Geometry of the phase bundle

Consider a bundle  $\pi: E \rightarrow M$ , where  $M$  is an orientable  $k$ -dimensional manifold with volume form  $\omega_M \in \Omega^m(M)$ , and let  $J^1\pi \rightarrow E \rightarrow M$  be the corresponding first-order jet bundle. If  $\dim M = k$  and  $\dim E = n + k$ , then  $\dim J^1\pi = nk + n + k$ . Natural coordinates in  $J^1\pi$  adapted to the bundle structure are  $(x^\mu, y^a, y_\mu^a)$  ( $\mu = 1, \dots, k; a = 1, \dots, n$ ), and are such that  $\omega_M = dx^1 \wedge \dots \wedge dx^k =: d^k x$ .

In the multicontact Lagrangian formalism for action-dependent field theories, the *configuration bundle* of the theory is  $E \times_M \wedge^{k-1}(T^*M) \rightarrow M$ , where  $\wedge^{k-1}(T^*M)$  denotes the bundle of  $(k - 1)$ -forms on  $M$ . The corresponding *phase bundle* is  $\mathcal{P} = J^1\pi \times_M \wedge^{k-1}(T^*M)$ . Natural coordinates in  $\mathcal{P}$  are  $(x^\mu, y^a, y_\mu^a, s^\mu)$ , and  $\dim \mathcal{P} = 2k + n + nk$ . Moreover, we also have the natural projections depicted in the following diagram:



As  $\wedge^{k-1}(T^*M)$  is a bundle of forms over  $M$ , it is endowed with a canonical structure, its “tautological form”  $\theta \in \Omega^{k-1}(\wedge^{k-1}(T^*M))$ , which is defined as usual, and whose local expression, in natural coordinates, is  $\theta = s^\mu d^{k-1}x_\mu$ .

**Definition 4.11.** The form  $\bar{S} := \tau_1^*\theta \in \Omega^{k-1}(\mathcal{P})$  is called the *canonical action form* of  $\mathcal{P}$ .

Its expression in coordinates is also  $\bar{S} = s^\mu d^{k-1}x_\mu$ .

**Definition 4.12.** A section  $\psi: M \rightarrow \mathcal{P}$  of the projection  $\tau: \mathcal{P} \rightarrow M$  is said to be a *holonomic section* in  $\mathcal{P}$  if the section  $\psi := \rho \circ \psi: M \rightarrow J^1\pi$  is holonomic in  $J^1\pi$ ; that is, there is a section  $\phi: M \rightarrow E$  of  $\pi$  such that  $\psi = j^1\phi$ . It is customary to write  $\psi = (\psi, s) = (j^1\phi, s)$ , where  $s: M \rightarrow \wedge^{k-1}(T^*M)$  is a section of the projection  $\tau_0: \wedge^{k-1}(T^*M) \rightarrow M$ ; then, we also say that  $\psi$  is the *canonical prolongation* of the section  $\phi := (\phi, s): M \rightarrow E \times_M \wedge^{k-1}(T^*M)$  to  $\mathcal{P}$ .

Now, consider the set  $\mathfrak{X}^k(\mathcal{P})$ , of multivector fields in  $\mathcal{P}$  (see Appendix A for details).

**Definition 4.13.** A  $k$ -multivector field  $\Gamma \in \mathfrak{X}^k(\mathcal{P})$  is a *holonomic  $k$ -multivector field* or a *SOPDE* in  $\mathcal{P}$  if it is  $\tau$ -transverse, integrable, and its integral sections are holonomic on  $\mathcal{P}$ .

The local expression of a SOPDE in  $\mathcal{P}$ , satisfying the transversality condition  $\iota_{\mathbf{X}}\omega = 1$ , is

$$\mathbf{X} = \bigwedge_{\mu=1}^k \left( \frac{\partial}{\partial x^\mu} + y_\mu^a \frac{\partial}{\partial y^a} + F_{\mu\nu}^a \frac{\partial}{\partial y_\nu^a} + g_\mu^\nu \frac{\partial}{\partial s^\nu} \right), \tag{4.1}$$

and its integral sections are solutions to the system of second-order partial differential equations:

$$y_\mu^a = \frac{\partial y^a}{\partial x^\mu}, \quad F_{\mu\nu}^a = \frac{\partial^2 y^a}{\partial x^\mu \partial x^\nu}.$$

**Remark 4.14.** The first-order jet bundle  $J^1\pi$  is endowed with a canonical structure, which is called the *canonical endomorphism*, and is a  $(1, 2)$ -tensor field in  $J^1\pi$ , denoted  $J$ . Its local expression in natural coordinates of  $J^1\pi$  is

$$J = (dy^a - y_\mu^a dx^\mu) \otimes \frac{\partial}{\partial y_\nu^a} \otimes \frac{\partial}{\partial x^\nu}$$

(see [34, 76]). As  $\mathcal{P} = J^1\pi \times_M \Lambda^{k-1}(\mathbb{T}^*M)$  is a trivial bundle, this canonical structure can be extended to  $\mathcal{P}$  in a natural way. This extension is denoted with the same notation  $J$ , and has the same coordinate expression.

Then, a direct calculation in coordinates leads to the following characterization of SOPDE multivector fields.

**Proposition 4.15.** *An integrable  $k$ -multivector field  $\mathbf{X} \in \mathfrak{X}^m(\mathcal{P})$  is a SOPDE if and only if*

$$\iota_{\mathbf{X}}J = 0, \tag{4.2}$$

where  $\iota_{\mathbf{X}}J$  denotes the natural inner contraction between tensor fields.

**Remark 4.16.** The  $\tau$ -transverse decomposable  $k$ -multivector fields satisfying condition (4.2), whose local expression is (4.1), are usually referred to as *semi-holonomic multivector fields*.

#### 4.2.2. Multicontact Lagrangian systems

Physical information in field theories is introduced by means of *Lagrangian densities*. A *Lagrangian density* is a  $k$ -form  $\mathcal{L} \in \Omega^k(\mathcal{P})$ ; hence  $\mathcal{L} = L d^k x$ , where  $L \in \mathcal{C}^\infty(\mathcal{P})$  is the *Lagrangian function* and  $d^k x$  is also the local expression of the form  $\omega := \tau^*\omega_M$ .

**Definition 4.17.** The *Lagrangian form* associated with  $\mathcal{L}$  is the form

$$\Theta_{\mathcal{L}} = -\iota_J d\mathcal{L} - \mathcal{L} + d\bar{S} \in \Omega^k(\mathcal{P}).$$

In natural coordinates, the coordinate expression of this form reads

$$\Theta_{\mathcal{L}} = -\frac{\partial L}{\partial y_\mu^a} dy^a \wedge d^{k-1}x_\mu + \left( \frac{\partial L}{\partial y_\mu^a} y_\mu^a - L \right) d^k x + ds^\mu \wedge d^{k-1}x_\mu, \tag{4.3}$$

where the local function  $E_{\mathcal{L}} := \frac{\partial L}{\partial y_{\mu}^a} y_{\mu}^a - L$  is called the *Lagrangian energy* associated with  $L$ .

Then, the following property holds [22].

**Proposition 4.18.** *For a Lagrangian function  $L \in \mathcal{C}^{\infty}(\mathcal{P})$ , the Lagrangian form  $\Theta_{\mathcal{L}}$  is a special (variational) multicontact form in  $\mathcal{P}$  (and hence  $(\Theta_{\mathcal{L}}, \omega)$  is a special (variational) multicontact structure) if and only if the Hessian matrix  $(W_{ij}^{\mu\nu}) = (\frac{\partial^2 L}{\partial y_{\mu}^a \partial y_{\nu}^b})$  is regular everywhere.*

Thus, we define as follows.

**Definition 4.19.** A Lagrangian function  $L \in \mathcal{C}^{\infty}(\mathcal{P})$  is said to be *regular* if the equivalent conditions of Proposition 4.18 hold. Otherwise,  $L$  is a *singular* Lagrangian.

**Definition 4.20.** If  $L \in \mathcal{C}^{\infty}(\mathcal{P})$  is a regular Lagrangian function, the triad  $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$  is called a *multicontact Lagrangian system*.

For a multicontact Lagrangian system  $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ , as  $L$  is regular, there exists the inverse  $(W_{\mu\nu}^{ab})$  of the Hessian matrix, namely  $W_{\mu\nu}^{ab} \frac{\partial^2 L}{\partial y_{\nu}^b \partial y_{\mu}^a} = \delta_{\mu}^a \delta_{\nu}^b$ . Then, from Lemma 4.8, a simple calculation in coordinates leads to the following expression for the Reeb vector fields  $(R_{\mathcal{L}})_{\mu} \in \mathfrak{X}_{\mathcal{L}}$ :

$$(R_{\mathcal{L}})_{\mu} = \frac{\partial}{\partial s^{\mu}} - W_{\gamma\nu}^{ba} \frac{\partial^2 \mathcal{L}}{\partial s^{\mu} \partial y_{\gamma}^b} \frac{\partial}{\partial y_{\nu}^a}.$$

Furthermore, bearing in mind Proposition 4.4 and Eq. (4.3), we obtain

$$\sigma_{\Theta_{\mathcal{L}}} = -\frac{\partial L}{\partial s^{\mu}} dx^{\mu}. \tag{4.4}$$

Finally, we construct the form

$$\bar{d}\Theta_{\mathcal{L}} = d\Theta_{\mathcal{L}} + \sigma_{\Theta_{\mathcal{L}}} \wedge \Theta_{\mathcal{L}} = d\Theta_{\mathcal{L}} - \frac{\partial L}{\partial s^{\nu}} dx^{\nu} \wedge \Theta_{\mathcal{L}}.$$

For a multicontact Lagrangian system  $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ , the Lagrangian field equations are derived from the *generalized Herglotz Variational Principle* [43], and are stated alternatively as follows.

**Definition 4.21.** Let  $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$  be a multicontact Lagrangian system.

(1) The *multicontact Lagrangian equations for holonomic sections*  $\psi: M \rightarrow \mathcal{P}$  are

$$\psi^* \Theta_{\mathcal{L}} = 0, \quad \psi^* \iota_X \bar{d}\Theta_{\mathcal{L}} = 0, \quad \text{for every } X \in \mathfrak{X}(\mathcal{P}), \tag{4.5}$$

or, equivalently, for the canonical prolongation  $\psi^{(k)}$  (see Appendix A)

$$\iota_{\psi^{(k)}}(\Theta_{\mathcal{L}} \circ \psi) = 0, \quad \iota_{\psi^{(k)}}(\bar{d}\Theta_{\mathcal{L}} \circ \psi) = 0. \tag{4.6}$$

- (2) The multicontact Lagrangian equations for holonomic multivector fields  $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^k(\mathcal{P})$  are

$$\iota_{\mathbf{X}_{\mathcal{L}}}\Theta_{\mathcal{L}} = 0, \quad \iota_{\mathbf{X}_{\mathcal{L}}}\bar{d}\Theta_{\mathcal{L}} = 0. \tag{4.7}$$

These holonomic multivector fields are called the *Euler–Lagrange multivector fields* associated with  $\mathcal{L}$ .

Recall that holonomic multivector fields are  $\tau$ -transverse. Note also that Eqs. (4.7) and the  $\tau$ -transversality condition,  $\iota_{\mathbf{X}_{\mathcal{L}}}\omega \neq 0$ , hold for every multivector field of the equivalence class  $\{\mathbf{X}_{\mathcal{L}}\}$  (that is, for every  $\mathbf{X}'_{\mathcal{L}} = f\mathbf{X}_{\mathcal{L}}$ , with  $f$  non-vanishing; see Appendix A). Then, the condition of  $\tau$ -transversality can be imposed simply by asking  $\iota_{\mathbf{X}_{\mathcal{L}}}\omega = 1$ .

**Theorem 4.22.** *Let  $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$  be a multicontact Lagrangian system.*

- (1) *The multicontact Lagrangian field equations for multivector fields (4.7) have solutions on  $\mathcal{P}$ . (The solutions are not unique if  $k > 1$ .)*
- (2) *Every  $k$ -multivector field  $\mathbf{X}_{\mathcal{L}}$  that is solution to Eqs. (4.7) is semi-holonomic.*
- (3) *If  $\mathbf{X}_{\mathcal{L}}$  is a holonomic multivector field (a SOPDE) solution to the Lagrangian equations (4.7), then its integral sections are the solutions to the multicontact Euler–Lagrange field equations for holonomic sections (4.5) or (4.6).*

**Proof.** In a natural chart of coordinates of  $\mathcal{P}$ , a  $\tau$ -transverse and locally decomposable  $k$ -multivector field satisfying  $\iota_{\mathbf{X}_{\mathcal{L}}}\omega = 1$ , has the local expression

$$\mathbf{X}_{\mathcal{L}} = \bigwedge_{\mu=1}^k \left( \frac{\partial}{\partial x^\mu} + (X_{\mathcal{L}})^\alpha_\mu \frac{\partial}{\partial y^a} + (X_{\mathcal{L}})^\alpha_{\mu\nu} \frac{\partial}{\partial y^\nu} + (X_{\mathcal{L}})^\nu_\mu \frac{\partial}{\partial s^\nu} \right),$$

and, bearing in mind the local expressions (4.3) and (4.4), we have that

$$\begin{aligned} \bar{d}\Theta_{\mathcal{L}} &= d \left( -\frac{\partial L}{\partial y^\alpha_\mu} dy^a \wedge d^{k-1}x_\mu + \left( \frac{\partial L}{\partial y^\alpha_\mu} y^\alpha_\mu - L \right) d^k x \right) \\ &\quad - \left( \frac{\partial L}{\partial s^\mu} \frac{\partial L}{\partial y^\alpha_\mu} dy^a - \frac{\partial L}{\partial s^\mu} ds^\mu \right) \wedge d^k x. \end{aligned}$$

Then, Eqs. (4.7) lead to

$$0 = L + \frac{\partial L}{\partial y^\alpha_\mu} ((X_{\mathcal{L}})^\alpha_\mu - y^\alpha_\mu) - (X_{\mathcal{L}})^\mu_\mu, \tag{4.8}$$

$$0 = ((X_{\mathcal{L}})^\alpha_\mu - y^\alpha_\mu) \frac{\partial^2 L}{\partial y^\alpha_\mu \partial s^\nu}, \tag{4.9}$$

$$0 = ((X_{\mathcal{L}})^\alpha_\mu - y^\alpha_\mu) \frac{\partial^2 L}{\partial y^\alpha_\mu \partial x^\nu},$$

$$0 = ((X_{\mathcal{L}})^\alpha_\mu - y^\alpha_\mu) \frac{\partial^2 L}{\partial y^\alpha_\mu \partial y^\beta_\nu}, \tag{4.10}$$

$$\begin{aligned}
 0 = & ((X_{\mathcal{L}})_{\mu}^{\alpha} - y_{\mu}^{\alpha}) \frac{\partial^2 L}{\partial y^b \partial y_{\mu}^{\alpha}} + \frac{\partial L}{\partial y^b} - \frac{\partial^2 L}{\partial x^{\mu} \partial y_{\mu}^b} - \frac{\partial^2 L}{\partial s^{\nu} \partial y_{\mu}^b} (X_{\mathcal{L}})_{\mu}^{\nu} \\
 & - \frac{\partial^2 L}{\partial y^{\alpha} \partial y_{\mu}^b} (X_{\mathcal{L}})_{\mu}^{\alpha} - \frac{\partial^2 L}{\partial y_{\nu}^{\alpha} \partial y_{\mu}^b} (X_{\mathcal{L}})_{\mu\nu}^{\alpha} + \frac{\partial L}{\partial s^{\mu}} \frac{\partial L}{\partial y_{\mu}^b},
 \end{aligned} \tag{4.11}$$

and a last group of equations which are identities when they are combined with the above ones. If  $\mathbf{X}_{\mathcal{L}}$  is a SOPDE, then it is semi-holonomic and,

$$y_{\mu}^{\alpha} = (X_{\mathcal{L}})_{\mu}^{\alpha}, \tag{4.12}$$

therefore, (4.9)–(4.10) hold identically, and (4.8) and (4.11) give

$$(X_{\mathcal{L}})_{\mu}^{\mu} = L,$$

$$\frac{\partial L}{\partial y^b} - \frac{\partial^2 L}{\partial x^{\mu} \partial y_{\mu}^b} - \frac{\partial^2 L}{\partial y^{\alpha} \partial y_{\mu}^b} y_{\mu}^{\alpha} - \frac{\partial^2 L}{\partial s^{\nu} \partial y_{\mu}^b} (X_{\mathcal{L}})_{\mu}^{\nu} - \frac{\partial^2 L}{\partial y_{\nu}^{\alpha} \partial y_{\mu}^b} (X_{\mathcal{L}})_{\mu\nu}^{\alpha} = - \frac{\partial L}{\partial s^{\mu}} \frac{\partial L}{\partial y_{\mu}^{\alpha}}. \tag{4.13}$$

Finally, for the holonomic integral sections  $\psi(x^{\nu}) = (x^{\mu}, y^{\alpha}(x^{\nu}), \frac{\partial y^{\alpha}}{\partial x^{\mu}}(x^{\nu}), s^{\mu}(x^{\nu}))$  of  $\mathbf{X}_{\mathcal{L}}$ , the last equations transform into

$$\begin{aligned}
 \frac{\partial s^{\mu}}{\partial x^{\mu}} &= L \circ \psi, \\
 \frac{\partial}{\partial x^{\mu}} \left( \frac{\partial L}{\partial y_{\mu}^b} \circ \psi \right) &= \left( \frac{\partial L}{\partial y^b} + \frac{\partial L}{\partial s^{\mu}} \frac{\partial L}{\partial y_{\mu}^b} \right) \circ \psi,
 \end{aligned} \tag{4.14}$$

which are the coordinate expression of the Lagrangian equations (4.5) or (4.6) for holonomic sections. Equations (4.14) are called the *Herglotz–Euler–Lagrange field equations*.

If  $L$  is a regular Lagrangian, Eqs. (4.10) lead to the semi-holonomy condition (4.12) and, in addition, Eqs. (4.13) always have solution since the Hessian matrix  $(\frac{\partial^2 L}{\partial y_{\nu}^b \partial y_{\mu}^{\alpha}})$  is regular everywhere. The solution is not unique, unless  $k = 1$ .  $\square$

Observe that all these equations are the same as those obtained in the  $k$ -cocontact Lagrangian formulation of non-conservative field theories [71] and also match those of the  $k$ -contact Lagrangian formalism when the Lagrangian function does not depend on the space-time variables  $x^{\mu}$  [40, 42].

### 4.3. Multicontact Hamiltonian formalism

The Hamiltonian formulation of action-dependent first-order field theories is based on the Hamiltonian multisymplectic formalism of standard classical field theories.

#### 4.3.1. Geometry of the phase bundle

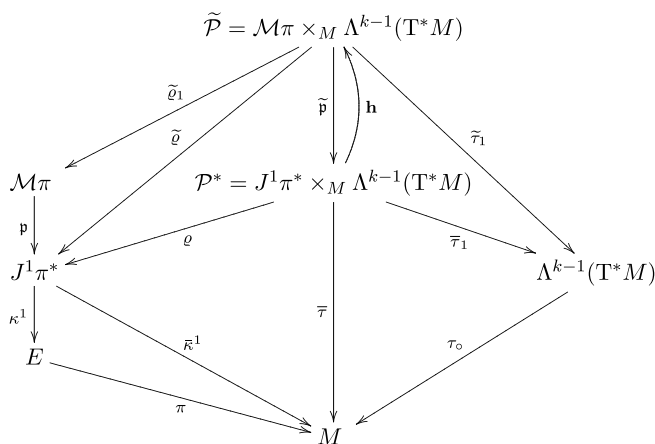
Consider a bundle  $\pi: E \rightarrow M$ , where  $M$  is an orientable  $k$ -dimensional manifold with volume form  $\omega_M \in \Omega^m(M)$ . Let  $\mathcal{M}\pi \equiv \Lambda_2^k T^*E$  denote the bundle of  $k$ -forms on  $E$  vanishing by contraction with two  $\pi$ -vertical vector fields which, in field

theories, is called the *extended multimomentum bundle*. It is endowed with natural coordinates  $(x^\nu, y^a, p_a^\nu, p)$  adapted to the bundle structure  $\mathcal{M}\pi \rightarrow E \rightarrow M$ , and such that  $\omega_M = d^k x$ ; so  $\dim \mathcal{M}\pi = nk + n + k + 1$ . Consider also the quotient manifold  $J^{1*}\pi = \mathcal{M}\pi/\pi^*\Lambda^k T^*M$  ( $\pi^*\Lambda^k T^*M$  is the bundle of  $\pi$ -basic  $k$ -forms on  $E$ ), which is called the *restricted multimomentum bundle*. Its natural coordinates are  $(x^\mu, y^a, p_a^\mu)$ , and so  $\dim J^{1*}\pi = nk + n + k$ .

Then, for the Hamiltonian formalism of action-dependent field theories, in the regular case, consider the bundles

$$\tilde{\mathcal{P}} = \mathcal{M}\pi \times_M \Lambda^{k-1}(T^*M), \quad \mathcal{P}^* = J^{1*}\pi \times_M \Lambda^{k-1}(T^*M),$$

which have natural coordinates  $(x^\mu, y^a, p_a^\mu, p, s^\mu)$  and  $(x^\mu, y^a, p_a^\mu, s^\mu)$ , respectively. We have the natural projections depicted in the following diagram:



Since  $\mathcal{M}\pi$  and  $\Lambda^{k-1}(T^*M)$  are bundles of forms, they have canonical structures, their “tautological forms”  $\Theta \in \Omega^k(\mathcal{M}\pi)$ , called the *Liouville form* of  $\mathcal{M}\pi$  (see, for instance, [17, 36] for its definition), and  $\theta \in \Omega^{k-1}(\Lambda^{k-1}(T^*M))$  whose local expressions are

$$\Theta = p_a^\mu dy^a \wedge d^{k-1}x_\mu + p d^k x, \quad \theta = s^\mu d^{k-1}x_\mu.$$

**Definition 4.23.** The *canonical (special) multicontact form* of  $\tilde{\mathcal{P}}$  is

$$\tilde{\Theta} := -\tilde{\varrho}_1^* \Theta + d(\tilde{\tau}_1^* \theta). \tag{4.15}$$

In natural coordinates of  $\tilde{\mathcal{P}}$ , the expression of this form is

$$\tilde{\Theta} = -p_a^\mu dy^a \wedge d^{k-1}x_\mu - p d^k x + ds^\mu \wedge d^{k-1}x_\mu.$$

### 4.3.2. Multicontact Hamiltonian systems

**Definition 4.24.** Let  $\mathbf{h}: \mathcal{P}^* \rightarrow \tilde{\mathcal{P}}$  be a section of  $\tilde{\mathfrak{p}}$ . It is locally determined by a function  $H \in \mathcal{C}^\infty(U)$ ,  $U \subset \mathcal{P}^*$ , such that  $\mathbf{h}(x^\mu, y^a, p_a^\mu, s^\mu) = (x^\mu, y^a, p_a^\mu, p =$

$-H(x^\nu, y^b, p_b^\nu, s^\nu), s^\mu)$ . The elements  $\mathbf{h}$  and  $H$  are called a *Hamiltonian section* and its associated *Hamiltonian function*.

Then the *Hamiltonian form* associated with  $\mathbf{h}$  is defined by

$$\Theta_{\mathcal{H}} := \mathbf{h}^* \tilde{\Theta} = -(\tilde{\varrho}_1 \circ \mathbf{h})^* \Theta + d(\bar{\tau}_1^* \theta).$$

It is a variational multicontact form, and the triad  $(\mathcal{P}^*, \Theta_{\mathcal{H}}, \omega = (\bar{\tau} \circ \tilde{\mathfrak{p}})^* \omega_M)$  is called a *multicontact Hamiltonian system*.

Obviously,  $\Theta_{\mathcal{H}}$  is a special (variational) multicontact form. In natural coordinates of  $\mathcal{P}$ , the expression of this form is

$$\Theta_{\mathcal{H}} = -p_a^\mu dy^a \wedge d^{k-1} x_\mu + H dx + ds^\mu \wedge d^{k-1} x_\mu,$$

and the dissipation form is expressed as

$$\sigma_{\mathcal{H}} = \frac{\partial H}{\partial s^\mu} dx^\mu. \tag{4.16}$$

**Definition 4.25.** Given a multicontact Hamiltonian system  $(\mathcal{P}^*, \Theta_{\mathcal{H}}, \omega)$ , the field equations can alternatively be stated as follows:

- (1) The *multicontact Hamilton–de Donder–Weyl equations for sections*  $\psi: M \rightarrow \mathcal{P}^*$ :

$$\psi^* \Theta_{\mathcal{H}} = 0, \quad \psi^* \iota_Y \bar{d} \Theta_{\mathcal{H}} = 0, \quad \text{for every } Y \in \mathfrak{X}(\mathcal{P}^*), \tag{4.17}$$

or, equivalently,

$$\iota_{\psi^{(k)}}(\Theta_{\mathcal{H}} \circ \psi) = 0, \quad \iota_{\psi^{(k)}}(\bar{d} \Theta_{\mathcal{H}} \circ \psi) = 0. \tag{4.18}$$

- (2) The *multicontact Hamilton–de Donder–Weyl equations for  $\bar{\tau}$ -transverse and integrable multivector fields*  $\mathbf{X}_{\mathcal{H}} \in \mathfrak{X}^k(\mathcal{P}^*)$ :

$$\iota_{\mathbf{X}_{\mathcal{H}}} \Theta_{\mathcal{H}} = 0, \quad \iota_{\mathbf{X}_{\mathcal{H}}} \bar{d} \Theta_{\mathcal{H}} = 0. \tag{4.19}$$

Equation (4.19) and the  $\bar{\tau}$ -transversality condition hold for every multivector field of the equivalence class  $\{\mathbf{X}_{\mathcal{H}}\}$ , and the transversality condition can be imposed by asking  $\iota_{\mathbf{X}_{\mathcal{H}}} \omega = 1$ .

In natural coordinates, for a  $\bar{\tau}$ -transverse, locally decomposable multivector field  $\mathbf{X}_{\mathcal{H}} \in \mathfrak{X}^k(\mathcal{P}^*)$ ,

$$\mathbf{X}_{\mathcal{H}} = \bigwedge_{\mu=0}^{k-1} \left( \frac{\partial}{\partial x^\mu} + (X_{\mathcal{H}})_\mu^a \frac{\partial}{\partial y^a} + (X_{\mathcal{H}})_{\mu a}^\nu \frac{\partial}{\partial p_a^\nu} + (X_{\mathcal{H}})_\mu^\nu \frac{\partial}{\partial s^\nu} \right), \tag{4.20}$$

if it is a solution to Eq. (4.19), bearing in mind the local expression Eq. (4.16), these field equations lead to

$$(X_{\mathcal{H}})_\mu^\mu = p_a^\mu \frac{\partial H}{\partial p_a^\mu} - H, \quad (X_{\mathcal{H}})_\mu^a = \frac{\partial H}{\partial p_a^\mu}, \quad (X_{\mathcal{H}})_{\mu a}^\mu = - \left( \frac{\partial H}{\partial y^a} + p_a^\mu \frac{\partial H}{\partial s^\mu} \right),$$

together with a last group of equations which are identities when the above ones are taken into account. Then, the integral sections  $\psi(x^\nu) = (x^\mu, y^a(x^\nu), p_a^\mu(x^\nu), s^\mu(x^\nu))$

of the integrable solutions  $\mathbf{X}_{\mathcal{H}}$  of (4.19) are the solutions to Eqs. (4.17) or (4.18) which read as

$$\frac{\partial s^\mu}{\partial x^\mu} = \left( p_a^\mu \frac{\partial H}{\partial p_a^\mu} - H \right) \circ \psi, \quad \frac{\partial y^a}{\partial x^\mu} = \frac{\partial H}{\partial p_a^\mu} \circ \psi, \quad \frac{\partial p_a^\mu}{\partial x^\mu} = - \left( \frac{\partial H}{\partial y^a} + p_a^\mu \frac{\partial H}{\partial s^\mu} \right) \circ \psi,$$

and are called the *Herglotz–Hamilton–de Donder–Weyl equations* for action-dependent field theories. These equations are compatible in  $\mathcal{P}^*$ .

Observe that these equations are the same as those obtained in the  $k$ -cocontact Hamiltonian formulation of non-conservative field theories and also lead to those of the  $k$ -contact Hamiltonian formalism when the Hamiltonian function does not depend on the space-time variables  $x^\mu$  [40, 42].

**Remark 4.26** (*The canonical multicontact Hamiltonian system associated with a multicontact Lagrangian system*). Let  $\mathcal{L} \in \Omega^k(\mathcal{P})$  be a Lagrangian density with  $\mathcal{L} = L\omega$ .

First, denote  $FL_s: J^1\pi \rightarrow J^{1*}\pi$  the Legendre map associated with the restriction of the Lagrangian function  $L \in \mathcal{C}^\infty(\mathcal{P})$  to the fibers of the projection  $\tau_1$  (recall diagram 4.1). Informally, it is obtained considering  $L$  with  $s^\mu$  “frozen”, which is denoted  $L_s \in \mathcal{C}^\infty(J^1\pi)$ . Then, the *restricted Legendre map* associated with the Lagrangian function  $L \in \mathcal{C}^\infty(\mathcal{P})$  is the map  $\mathcal{FL}: \mathcal{P} \rightarrow \mathcal{P}^*$  given by  $\mathcal{FL} := (FL_s, \text{Id}_{\Lambda^{k-1}(T^*M)})$ . It is locally given by

$$\mathcal{FL}(x^\mu, y^a, y_\mu^a, s^\mu) = \left( x^\mu, y^a, \frac{\partial L}{\partial y_\mu^a}, s^\mu \right).$$

Similarly, the *extended Legendre map* associated with  $L$  is the map  $\widetilde{\mathcal{FL}}: \mathcal{P} \rightarrow \widetilde{\mathcal{P}}$  given by  $\widetilde{\mathcal{FL}} := (\widetilde{FL}_s, \text{Id}_{\Lambda^{k-1}(T^*M)})$ . Its local expression is

$$\widetilde{\mathcal{FL}}(x^\mu, y^a, y_\mu^a, s^\mu) = \left( x^\mu, y^a, \frac{\partial L}{\partial y_\mu^a}, L - y_\mu^a \frac{\partial L}{\partial y_\mu^a}, s^\mu \right).$$

It is not difficult to prove that  $L$  is a regular Lagrangian function if and only if the Legendre map  $\mathcal{FL}$  is a local diffeomorphism. In particular,  $L$  is said to be *hyperregular* if  $\mathcal{FL}$  is a global diffeomorphism.

Therefore, if  $L$  is a hyperregular Lagrangian function, we can define the Hamiltonian section  $\mathbf{h} := \widetilde{\mathcal{FL}} \circ \mathcal{FL}^{-1}$ , and construct the corresponding multicontact Hamiltonian system  $(\mathcal{P}^*, \Theta_{\mathcal{H}}, \omega)$ , which is called the *canonical multicontact Hamiltonian system* associated with the multicontact Lagrangian system  $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ . If  $\mathcal{L}$  is regular, this construction is local. Bearing in mind the coordinate expressions (4.3), (4.4), and (4.16), we obtain that  $\mathcal{FL}^*\Theta_{\mathcal{H}} = \Theta_{\mathcal{L}}$  and  $\mathcal{FL}^*\bar{\mathbf{d}}\Theta_{\mathcal{H}} = \bar{\mathbf{d}}\Theta_{\mathcal{L}}$ .

## 5. Relationship Between Multicontact, $k$ -Cocontact and $k$ -Contact Structures

For the relation among multisymplectic,  $k$ -cosymplectic and  $k$ -symplectic structures in classical field theories, see [20, 69] (see also [29, 30]).

The relation among the multicontact, the  $k$ -cocontact and the  $k$ -contact structures for action-dependent field theories is done for the particular situation where  $\pi: E \rightarrow M$  is the trivial bundle  $\mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$ .

**5.1. The Hamiltonian case**

A previous result needed to establish this relation is as follows.

**Proposition 5.1.** (1) *The manifold  $\mathcal{M}\pi \equiv \Lambda_2^k T^*(\mathbb{R}^k \times Q)$  is diffeomorphic to  $\mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q$ .*

(2) *As a consequence,  $J^1\pi^*$  is diffeomorphic to  $\mathbb{R}^k \times \oplus^k T^*Q$ .*

**Proof.** (1) For  $t \in \mathbb{R}^k$ , let  $i_t: Q \hookrightarrow \mathbb{R}^k \times Q$  be the canonical embedding given by  $i_t(q) = (t, q)$ , and  $\rho_Q: \mathbb{R}^k \times Q \rightarrow Q$  the canonical submersion. Then, we can define the map

$$\begin{aligned} \bar{\Psi}: \Lambda_2^k T^*(\mathbb{R}^k \times Q) &\rightarrow \mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q \\ \xi_{(t,q)} &\mapsto (t, p, \xi_q^1, \dots, \xi_q^k), \end{aligned}$$

where, for  $X \in \mathfrak{X}(Q)$ ,

$$\begin{aligned} \xi_q^\alpha(X) &= \xi_{(t,q)} \left( \frac{\partial}{\partial x^1} \Big|_{(t,q)}, \dots, \frac{\partial}{\partial x^{\alpha-1}} \Big|_{(t,q)}, (i_t)_* X_q, \frac{\partial}{\partial x^{\alpha+1}} \Big|_{(t,q)}, \dots, \frac{\partial}{\partial x^k} \Big|_{(t,q)} \right), \\ p &= \xi_{(t,q)} \left( \frac{\partial}{\partial x^1} \Big|_{(t,q)}, \dots, \frac{\partial}{\partial x^k} \Big|_{(t,q)} \right) \end{aligned}$$

(observe that  $x^\alpha$  and  $p$  are now global coordinates in the corresponding fibers and, then, the global coordinate  $p$  can be identified with the natural projection  $p: \mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q \rightarrow \mathbb{R}$ ). The inverse of  $\bar{\Psi}$ , at a point  $(t, p, \xi_q^\alpha) \in \mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q$  is given by

$$\xi_{(t,q)} = p \, d^k x|_{(t,q)} + (\rho_Q)_*(t,q) \xi_q^\alpha \wedge d^{k-1} x_\alpha|_{(t,q)}.$$

Thus,  $\bar{\Psi}$  is a diffeomorphism (locally  $\bar{\Psi}$  is written as the identity).

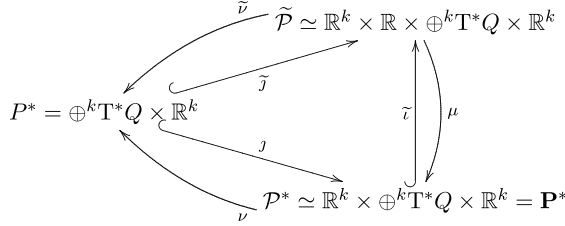
(2) Bearing in mind the identification of  $\Lambda^k(T^*M)$  with  $\mathbb{R}$ , this is a straightforward consequence of the above item since

$$J^1\pi^* = \mathcal{M}\pi/\pi^* \Lambda^k T^*M \simeq (\mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q)/\mathbb{R} \simeq \mathbb{R}^k \times \oplus^k T^*Q. \quad \square$$

Note that  $\Lambda^{k-1}(T^*M)$  can be identified with  $\mathbb{R}^k$ , and therefore, taking into account the above proposition, we can write

$$\begin{aligned} \tilde{\mathcal{P}} &= \mathcal{M}\pi \times_M \Lambda^{k-1}(T^*M) \simeq \mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q \times \mathbb{R}^k, \\ \mathcal{P}^* &= J^1\pi^* \times_M \Lambda^{k-1}(T^*M) \simeq \mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k = \mathbf{P}^*. \end{aligned}$$

The following diagram contains the projections and embeddings that we will use next.



The embeddings  $\tilde{\iota}, j$  and  $\tilde{j}$  are given by the zero-sections:

$$\begin{aligned}
 \tilde{\iota}(x^\alpha, y^a, p_a^\alpha, s^\alpha) &= (x^\alpha, p = 0, y^a, p_a^\alpha, s^\alpha), \\
 j(y^a, p_a^\alpha, s^\alpha) &= (x^\alpha = 0, y^a, p_a^\alpha, s^\alpha), \\
 \tilde{j}(y^a, p_a^\alpha, s^\alpha) &= (x^\alpha = 0, p = 0, y^a, p_a^\alpha, s^\alpha).
 \end{aligned}$$

5.1.1. Relationship between the canonical structures of

$$\tilde{\mathcal{P}} = \mathcal{M}\pi \times_M \Lambda^{k-1}(\mathbb{T}^*M) \text{ and } P^* = \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$$

If the manifold  $\tilde{\mathcal{P}} = \mathcal{M}\pi \times_M \Lambda^{k-1}(\mathbb{T}^*M)$  is diffeomorphic to  $\mathbb{R}^k \times \mathbb{R} \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$ , then it is a trivial bundle over  $\mathbb{R}^k$ . Therefore, the canonical vector fields  $\frac{\partial}{\partial x^\alpha} \in \mathfrak{X}(\mathbb{R}^k)$  can be extended to vector fields in  $\tilde{\mathcal{P}}$ , which have the same coordinate expressions.

Then, following the same pattern as in the proof of the item 1 of Proposition 5.1 and starting from the canonical special multicontact form  $\tilde{\Theta} \in \Omega^k(\tilde{\mathcal{P}})$  given in (4.15), we can define the forms  $\eta^\alpha \in \Omega^1(\oplus^k \mathbb{T}^*Q \times \mathbb{R}^k)$  by

$$\begin{aligned}
 \eta^\alpha &= (-1)^{\alpha-1} \tilde{j}^* \left( \iota_{\left(\frac{\partial}{\partial x^k}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^{\alpha+1}}\right)} \iota_{\left(\frac{\partial}{\partial x^{\alpha-1}}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^1}\right)} \tilde{\Theta} \right) \\
 &= -\tilde{j}^* \left( \iota_{\left(\frac{\partial}{\partial x^k}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^1}\right)} (\tilde{\Theta} \wedge dx^\alpha) \right).
 \end{aligned} \tag{5.1}$$

In coordinates, we have,

$$\tilde{\Theta} \wedge dx^\alpha = (-1)^{k-1} (-p_a^\alpha dy^a \wedge d^k x + ds^\alpha \wedge d^k x).$$

Then

$$\begin{aligned}
 &-\tilde{j}^* \left( \iota_{\left(\frac{\partial}{\partial x^k}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^1}\right)} (\tilde{\Theta} \wedge dx^\alpha) \right) \\
 &= (-1)^k \tilde{j}^* \left( \iota_{\left(\frac{\partial}{\partial x^k}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^1}\right)} (-p_a^\alpha dy^a \wedge d^k x + ds^\alpha \wedge d^k x) \right) \\
 &= -p_a^\alpha dy^a + ds^\alpha.
 \end{aligned}$$

Therefore,  $\{\eta^\alpha\}$  is the canonical  $k$ -contact structure in  $P^* = \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$ .

Conversely, starting from the  $k$ -contact structure  $\{\eta^\alpha\}$  in  $P^* = \oplus^k T^*Q \times \mathbb{R}^k$  we can obtain the canonical special multicontact form in  $\tilde{\mathcal{P}} \simeq (\mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q) \times \mathbb{R}^k$ , by doing

$$\tilde{\Theta} = p\omega + \tilde{\nu}^* \eta^\alpha \wedge \iota\left(\frac{\partial}{\partial x^\alpha}\right)\omega = p d^k x + \tilde{\nu}^* \eta^\alpha \wedge d^{k-1} x_\alpha. \tag{5.2}$$

The Reeb vector fields are the same for both structures and are  $\left\{\frac{\partial}{\partial s^\alpha}\right\}$ .

In summary, we have proved the following.

**Theorem 5.2.** *The canonical special multicontact form on  $\tilde{\mathcal{P}} \simeq \mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q \times \mathbb{R}^k$  and the contact forms of the canonical  $k$ -contact structure on  $P^* = \oplus^k T^*Q \times \mathbb{R}^k$  are related by (5.1) and (5.2).*

5.1.2. *Relationship between the structures of  $\mathcal{P}^* = J^{1*} \pi \times_M \Lambda^{k-1}(T^*M)$  and  $P^* = \oplus^k T^*Q \times \mathbb{R}^k$*

If  $\mu: \tilde{\mathcal{P}} \simeq \mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q \times \mathbb{R}^k \rightarrow \mathcal{P}^* \simeq \mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k$  is a trivial bundle, we can take a global Hamiltonian section  $\mathbf{h}: \mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k \rightarrow \mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q \times \mathbb{R}^k$ , specified by a (global) Hamiltonian function  $H \in \mathcal{C}^\infty(\mathcal{P}^*)$ , and then define the (non-canonical) special multicontact form  $\Theta_{\mathcal{H}} \in \Omega^k(\mathcal{P}^*)$  given in (4.16). Therefore, following the same pattern as in the above section, we can obtain the forms  $\eta^\alpha \in \Omega^1(\oplus^k T^*Q \times \mathbb{R}^k)$  given as follows:

$$\begin{aligned} \eta^\alpha &= (-1)^{\alpha-1} j^* \left( \iota\left(\frac{\partial}{\partial x^k}\right) \cdots \iota\left(\frac{\partial}{\partial x^{\alpha+1}}\right) \iota\left(\frac{\partial}{\partial x^{\alpha-1}}\right) \cdots \iota\left(\frac{\partial}{\partial x^1}\right) \Theta_{\mathcal{H}} \right) \\ &= -j^* \left( \iota\left(\frac{\partial}{\partial x^k}\right) \cdots \iota\left(\frac{\partial}{\partial x^1}\right) (\Theta_{\mathcal{H}} \wedge dx^\alpha) \right), \end{aligned} \tag{5.3}$$

that define the canonical  $k$ -contact structure in  $P^* = \oplus^k T^*Q \times \mathbb{R}^k$ .

Conversely, starting from the  $k$ -contact structure  $\{\eta^\alpha\}$  in  $P^* = \oplus^k T^*Q \times \mathbb{R}^k$  we can obtain the canonical special multicontact form in  $\tilde{\mathcal{P}} \simeq (\mathbb{R}^k \times \mathbb{R} \times \oplus^k T^*Q) \times \mathbb{R}^k$ , by doing

$$\Theta_{\mathcal{P}} = -H\omega + \nu^* \eta^\alpha \wedge \iota\left(\frac{\partial}{\partial x^\alpha}\right)\omega = -H d^k x + \nu^* \eta^\alpha \wedge d^{k-1} x_\alpha. \tag{5.4}$$

The Reeb vector fields are the same for both structures and are  $\left\{\frac{\partial}{\partial s^\alpha}\right\}$ .

Thus, we have proved the following result.

**Theorem 5.3.** *The special multicontact form on  $\mathcal{P}^* \simeq \mathbb{R}^k \times \oplus^k T^*Q \times \mathbb{R}^k$  and the contact forms of the canonical  $k$ -contact structure on  $P^* = \oplus^k T^*Q \times \mathbb{R}^k$  are related by (5.3) and (5.4).*

5.1.3. *Relationship between the canonical structures of*

$$\tilde{\mathcal{P}} = \mathcal{M}\pi \times_M \Lambda^{k-1}(\mathbb{T}^*M) \text{ and } \mathbf{P}^* = \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$$

From the canonical special multicontact form  $\tilde{\Theta} \in \Omega^1(\tilde{\mathcal{P}})$  we define the forms  $\eta^\alpha \in \Omega^1(\mathbf{P}^*)$  as in (5.1); that is,

$$\eta^\alpha = (-1)^{\alpha-1} \tilde{\iota}^* \left( \iota_{\left(\frac{\partial}{\partial x^k}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^{\alpha+1}}\right)} \iota_{\left(\frac{\partial}{\partial x^{\alpha-1}}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^1}\right)} \tilde{\Theta} \right), \quad (5.5)$$

whose coordinate expressions are  $\eta^\alpha = ds^\alpha - p_\alpha^\alpha dy^\alpha$ . In addition, we also define the forms

$$\tau^\alpha = (-1)^{k-\alpha} \tilde{\iota}^* \left( \iota_{\left(\frac{\partial}{\partial x^k}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^{\alpha+1}}\right)} \iota_{\left(\frac{\partial}{\partial x^{\alpha-1}}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^1}\right)} \iota_{\left(\frac{\partial}{\partial p}\right)} d\tilde{\Theta} \right) \quad (5.6)$$

(recall that  $p$  denotes the global canonical coordinate in  $\mathbb{R}$ ). Observe also that, since  $p$  denotes the global canonical coordinate in  $\mathbb{R}$ , the 1-forms  $\tau^\alpha = dx^\alpha$  are canonically defined on  $\mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$ .

Conversely, from the forms  $\{\eta^\alpha\}$  of the  $k$ -cocontact structure in  $\mathbf{P}^* = \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$  we can obtain the canonical special multicontact form in  $\tilde{\mathcal{P}} \simeq (\mathbb{R}^k \times \mathbb{R} \times \oplus^k \mathbb{T}^*Q) \times \mathbb{R}^k$  similarly to (5.2).

In this way, we have proved the following.

**Theorem 5.4.** *The canonical special multicontact form on  $\tilde{\mathcal{P}} \simeq \mathbb{R}^k \times \mathbb{R} \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$  and the forms of the canonical  $k$ -cocontact structure on  $\mathbf{P}^* = \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$  are related by (5.2), (5.5) and (5.6).*

The (contact) Reeb vector fields are the same for both structures and are  $\left\{ \frac{\partial}{\partial s^\alpha} \right\}$ , and the space-time Reeb vector fields in  $\mathbf{P}^*$  are  $\left\{ \frac{\partial}{\partial x^\alpha} \right\}$ .

5.1.4. *Relationship between the (non-canonical) structures of*

$$\mathcal{P}^* = J^{1*}\pi \times_M \Lambda^{k-1}(\mathbb{T}^*M) \text{ and } \mathbf{P}^* = \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$$

First, observe that, in this particular situation, the manifolds  $\mathcal{P}^*$  and  $\mathbf{P}^*$  are canonically identified with  $\mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$ . Then, as in the above two sections, we obtain that, starting from the (non-canonical) special multicontact form  $\Theta_{\mathcal{H}} \in \Omega^k(\mathcal{P}^*)$  given in (4.16), we get the forms

$$\eta^\alpha = -\iota_{\left(\frac{\partial}{\partial x^k}\right)} \cdots \iota_{\left(\frac{\partial}{\partial x^1}\right)} (\Theta_{\mathcal{H}} \wedge dx^\alpha), \quad (5.7)$$

and the 1-forms  $\tau^\alpha = dx^\alpha$  are canonically defined.

Conversely, the special multicontact form in  $\mathcal{P}^* \simeq \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$  is obtained from the forms  $\{\eta^\alpha\}$  of the  $k$ -cocontact structure in  $\mathbf{P}^* = \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$  as in (5.2), without the pullback of  $\tilde{\nu}$ .

So, we have the following.

**Theorem 5.5.** *The special multicontact form on  $\mathcal{P}^* \simeq \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$  and the forms of the canonical  $k$ -cocontact structure on  $\mathbf{P}^* = \mathbb{R}^k \times \oplus^k \mathbb{T}^*Q \times \mathbb{R}^k$  are related by (5.2) and (5.7), and the 1-forms  $\tau^\alpha$  are canonically defined.*

### 5.2. The Lagrangian case

As in the Hamiltonian case, now we have the canonical identifications

$$\mathcal{P} = J\pi \times_M \Lambda^{k-1}(T^*M) \simeq \mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k \simeq \mathbf{P},$$

and the natural embedding

$$\iota: P = \oplus^k TQ \times \mathbb{R}^k \hookrightarrow \mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k.$$

Therefore, following the same patterns as in Secs. 5.1.2 and 5.1.4, we obtain the following.

**Theorem 5.6.** *Let  $\mathcal{L}$  be an “autonomous” Lagrangian function; namely,  $\mathcal{L} = \mathcal{L}(y^\alpha, y_\alpha^a, s^\alpha)$  and  $E_{\mathcal{L}}$  its associated Lagrangian energy. The Lagrangian multicontact form on  $\mathcal{P} \simeq \mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k$  and the contact forms of the Lagrangian  $k$ -contact structure on  $P = \oplus^k TQ \times \mathbb{R}^k$  are related as follows:*

$$\begin{aligned} \eta_{\mathcal{L}}^\alpha &= \iota^* \left( \iota \left( \frac{\partial}{\partial x^k} \right) \cdots \iota \left( \frac{\partial}{\partial x^{\alpha+1}} \right) \iota \left( \frac{\partial}{\partial x^{\alpha-1}} \right) \cdots \iota \left( \frac{\partial}{\partial x^1} \right) \Theta_{\mathcal{L}} \right) \\ &= -\iota^* \left( \iota \left( \frac{\partial}{\partial x^k} \right) \cdots \iota \left( \frac{\partial}{\partial x^1} \right) (\Theta_{\mathcal{L}} \wedge dx^\alpha) \right), \\ \Theta_{\mathcal{L}} &= -E_{\mathcal{L}} \omega + \kappa_2^* \theta^\alpha \wedge \iota \left( \frac{\partial}{\partial x^\alpha} \right) \omega = -E_{\mathcal{L}} d^k t + \kappa_2^* \theta^\alpha \wedge d^{k-1} x_\alpha, \end{aligned}$$

where  $\kappa_2: \mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k \rightarrow \oplus^k TQ \times \mathbb{R}^k$  is the canonical submersion.

**Theorem 5.7.** *Let  $\mathcal{L}$  be a “non-autonomous” Lagrangian function; that is,  $\mathcal{L} = \mathcal{L}(x^\alpha, y^\alpha, y_\alpha^a, s^\alpha)$  and  $E_{\mathcal{L}}$  its associated Lagrangian energy. The Lagrangian multi-contact form on  $\mathcal{P} \simeq \mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k$  and the contact forms of the Lagrangian  $k$ -cocontact structure on  $\mathbf{P} = \mathbb{R}^k \times \oplus^k TQ \times \mathbb{R}^k$  are related as follows:*

$$\begin{aligned} \eta_{\mathcal{L}}^\alpha &= -\iota \left( \frac{\partial}{\partial x^k} \right) \cdots \iota \left( \frac{\partial}{\partial x^1} \right) (\Theta_{\mathcal{L}} \wedge dx^\alpha), \\ \Theta_{\mathcal{L}} &= -E_{\mathcal{L}} \omega + \kappa_2^* \theta^\alpha \wedge \iota \left( \frac{\partial}{\partial x^\alpha} \right) \omega = -E_{\mathcal{L}} d^k t + \kappa_2^* \theta^\alpha \wedge d^{k-1} x_\alpha, \end{aligned}$$

and the 1-forms  $\tau^\alpha$  are canonically defined.

## 6. Relation with Other Kinds of Multicontact Structures

For completeness, we briefly comment on the relations between the structures studied in this paper and other similar structures present in the literature.

A higher-dimensional version of contact distributions was proposed in [79]. There, a distribution  $D \subset T\mathcal{M}$  is called multicontact if it is maximally non-integrable. That is, the only vector field  $X \in \Gamma(D)$  with the property that  $[X, Y] \in \Gamma(D)$  for all  $Y \in \Gamma(D)$  is  $X = 0$ . This definition is so general that one cannot even assume that, locally, it is the kernel of a  $k$ -contact structure. To ensure this, it is necessary to introduce an additional condition: the existence of a family of symmetries of the distribution. The relation between  $k$ -contact geometry and

maximally non-integrable distributions has been studied in [33]. Given a  $k$ -contact structure in  $\mathcal{M}$ , consider the distribution  $\mathcal{D}^C$  defined in 2.1. Then, [33, Theorem 3.6] states that  $\mathcal{D}^C$  is maximally non-integrable. The converse is not true (see [33, Theorem 3.13] for a counterexample). The hypothesis needed for a maximally non-integrable distribution to be associated with a  $k$ -contact structure is shown in [33, Theorem 3.14].

We proceed to study the relation between  $k$ -cocontact and multicontact structures with maximally non-integrable distributions.

**Proposition 6.1.** *Let  $(\mathcal{M}, \eta^\alpha, \tau^\alpha)$  be a  $k$ -cocontact manifold. Then  $\mathcal{D}^S \cap \mathcal{D}^C$  is maximally non-integrable.*

**Proof.** Consider  $X \in \Gamma(\mathcal{D}^S \cap \mathcal{D}^C)$  such that  $[X, Y] \in \Gamma(\mathcal{D}^S \cap \mathcal{D}^C)$  for all  $Y \in \Gamma(\mathcal{D}^S \cap \mathcal{D}^C)$ . Then

$$0 = \iota_{[X, Y]}\eta^\alpha = -\iota_Y \iota_X d\eta^\alpha.$$

By a dimensional-counting argument,

$$(\mathcal{D}^S \cap \mathcal{D}^C) \oplus \mathcal{D}^R = T\mathcal{M}. \tag{6.1}$$

Since  $\iota_R \iota_X d\eta^\alpha = 0$ , for every  $R \in \Gamma(\mathcal{D}^R)$ , we conclude that  $\iota_X d\eta^\alpha$  vanishes by the action of all tangent vector fields. Therefore,  $\iota_X d\eta^\alpha = 0$ , for every  $\alpha$  and  $X \in \Gamma(\mathcal{D}^R)$ . In light of (6.1), it must be  $X = 0$ .  $\square$

**Proposition 6.2.** *Let  $(\mathcal{M}, \Theta, \omega)$  be a variational multicontact manifold. Then  $\ker \Theta$  is maximally non-integrable.*

**Proof.** Consider  $X \in \Gamma(\ker \Theta)$  such that  $[X, Y] \in \Gamma(\ker \Theta)$  for all  $Y \in \Gamma(\ker \Theta)$ . Then

$$0 = \iota_{[X, Y]}\Theta = -\iota_Y \iota_X d\Theta.$$

Moreover, if  $R \in \Gamma(\mathcal{D}^R)$ , then  $\iota_R \iota_X d\Theta = 0$  because  $X \in \ker \Theta \subset \ker \omega$ . Therefore,  $\iota_X d\Theta$  vanishes by the action of the elements of  $\ker \Theta$  and  $\mathcal{D}^R$ , which, by Lemma 4.9, are all the elements of  $\ker \omega$ . Consequently,  $X \in \Gamma(\mathcal{D}^R)$ . Since we also have  $X \in \Gamma(\ker \Theta)$ , then  $X = 0$  according to the third condition of special multicontact structures Definition 4.3 and Lemma 4.9.  $\square$

A non-variational special multicontact structure does not lead, in general, to a maximally non-integrable distribution. For instance, consider  $\mathcal{M} = \mathbb{R}^7$  with the globally defined coordinates  $(x, y, q, p^x, p^y, s^x, s^y)$  and the special multicontact structure

$$\omega = dx \wedge dy, \quad \Theta = (ds^x - p^x dq) \wedge dy - (ds^y - p^y dq) \wedge dx + dp^x \wedge dq.$$

In this case,  $\ker \Theta = \ker \omega \cap \ker \Theta = \langle \frac{\partial}{\partial p^y} \rangle$ , that is not maximally non-integrable.

### 7. Conclusions and Outlook

The covariant geometrical description of conservative classical field theories is well established, relying on finite-dimensional structures such as  $k$ -symplectic,  $k$ -cosymplectic and multisymplectic geometries, among others, which extend traditional symplectic geometry. For action-dependent (non-conservative) field theories, analogous frameworks, inspired by the aforementioned structures, are provided by  $k$ -contact,  $k$ -cocontact and multicontact structures, which extend contact geometry.

This work begins by reviewing the definitions and core features of these new geometric structures, including  $k$ -contact,  $k$ -cocontact and multicontact (distinguishing different types of the latter), and Hamiltonian systems on each case. This foundation enables a finite-dimensional covariant geometrical description of action-dependent regular classical field theories in both Lagrangian and Hamiltonian formalisms.

In the same way as in the  $k$ -(co)symplectic and multisymplectic formalisms, in action-dependent field theories, the multicontact formulation is more general than the  $k$ -(co)contact versions. This is because, unlike in the  $k$ -(co)contact case, multiphase bundles are not always trivial. Moreover, the  $k$ -symplectic and  $k$ -contact formulations are restricted to autonomous field theories. From a physical point of view, whenever we have Darboux-type coordinates, the local description of the three formalisms is equivalent.

The main contribution of the paper is the study of the relationships among these types of structure, specifically in the case where the phase bundles of the field theories are trivial. We have also compared these structures with existing alternative definitions of multicontact structures.

This research opens avenues for future extension to singular (almost-regular) action-dependent classical field theories.

### Appendix A. Multivector Fields

This appendix provides a review on the theory of multivector fields. See [15, 16, 35, 61, 65] for more details.

Let  $\mathcal{M}$  be an  $N$ -dimensional differentiable manifold. The  $k$ -multivector fields in  $\mathcal{M}$  ( $k \leq N$ ) are the sections of the  $k$ -multitangent bundle  $\bigwedge^k T\mathcal{M} := \overbrace{T\mathcal{M} \wedge \cdots \wedge T\mathcal{M}}^k$ ; that is, the skew-symmetric contravariant tensor fields of order  $k$  in  $\mathcal{M}$ . The set  $\mathfrak{X}^k(\mathcal{M})$  consists of all these multivector fields. In particular,  $\mathbf{X} \in \mathfrak{X}^k(\mathcal{M})$  is a *locally decomposable multivector field* if there exist  $X_1, \dots, X_k \in \mathfrak{X}(U)$  such that  $\mathbf{X}|_U = X_1 \wedge \cdots \wedge X_k$ .

Locally decomposable  $k$ -multivector fields are locally associated with  $k$ -dimensional distributions  $D \subset T\mathcal{M}$ , and this splits  $\mathfrak{X}^k(\mathcal{M})$  into *equivalence classes*  $\{\mathbf{X}\} \subset \mathfrak{X}^k(\mathcal{M})$  which are made of locally decomposable multivector fields associated with the same distribution. If  $\mathbf{X}, \mathbf{X}' \in \{\mathbf{X}\}$  then, for  $U \subset \mathcal{M}$ , there exists a non-vanishing function  $f \in \mathcal{C}^\infty(U)$  such that  $\mathbf{X}' = f\mathbf{X}$  on  $U$ . In particular, an

integrable multivector field is a locally decomposable multivector field whose associated distribution is integrable; that is, involutive.

If  $\Omega \in \Omega^k(\mathcal{M})$  and  $\mathbf{X} \in \mathfrak{X}^k(\mathcal{M})$ , the contraction between  $\mathbf{X}$  and  $\Omega$  is the natural contraction between tensor fields; in particular, it gives zero when  $m < k$ , and if  $m \geq k$ , and for locally decomposable multivector fields is

$$\iota_{\mathbf{X}}\Omega|_U := \iota_{(X_1 \wedge \dots \wedge X_k)}\Omega = \iota_{X_k} \dots \iota_{X_1}\Omega.$$

Now, let  $\varrho: \mathcal{M} \rightarrow M$  be a fiber bundle, where  $M$  is an oriented manifold with volume form  $\omega_M \in \Omega^k(M)$ . A multivector field  $\mathbf{X} \in \mathfrak{X}^k(\mathcal{M})$  is  $\varrho$ -transverse if, for every  $\beta \in \Omega^k(M)$  such that  $\beta_{\varrho(p)} \neq 0$ , at every point  $p \in \mathcal{M}$ , we have  $(\iota_{\mathbf{X}}(\varrho^*\beta))_p \neq 0$ . Then, if  $\mathbf{X} \in \mathfrak{X}^k(\mathcal{M})$  is integrable and  $\varrho$ -transverse, its integral manifolds are local sections of the projection  $\varrho: \mathcal{M} \rightarrow M$ .

Therefore, the canonical prolongation of a section  $\psi: U \subset M \rightarrow \mathcal{M}$  to  $\Lambda^k\text{T}\mathcal{M}$  is the section  $\psi^{(k)}: U \subset M \rightarrow \Lambda^k\text{T}\mathcal{M}$  defined as  $\psi^{(k)} := \Lambda^k\text{T}\psi \circ \mathbf{Y}_{\omega_M}$ ; where  $\Lambda^m\text{T}\psi: \Lambda^m\text{T}\mathcal{M} \rightarrow \Lambda^m\text{T}M$  is the natural extension of  $\psi$  to the corresponding multitangent bundles, and  $\mathbf{Y}_{\omega_M} \in \mathfrak{X}^k(M)$  is the unique  $k$ -multivector field on  $M$  such that  $\iota_{\mathbf{Y}_{\omega_M}}\omega = 1$ . Then,  $\psi$  is an integral section of  $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$  if and only if  $\mathbf{X} \circ \psi = \psi^{(m)}$ .

Thus, if  $(U; z^i)$  is a chart of coordinates in  $\mathcal{M}$ , and  $x^\alpha$  are the coordinates in  $\mathbb{R}^k$ ; given a decomposable multivector field  $\mathbf{X} = X_1 \wedge \dots \wedge X_k$ , with local expression  $X_\alpha = X_\alpha^i \frac{\partial}{\partial z^i}$ , a section  $\psi: M \rightarrow \mathcal{M}$ , with  $\psi = (\psi^\alpha)$ , is an integral map of  $\mathbf{X}$  if it satisfies the set of partial differential equations

$$\frac{\partial \psi^i}{\partial x^\alpha} = X_\alpha^i \circ \psi. \tag{A.1}$$

In addition, the stronger integrability condition  $[X_\alpha, X_\beta] = 0$ , for every  $\alpha, \beta = 1, \dots, k$ , must be imposed; and this is precisely the integrability condition of the PDE (A.1) (see [63]).

### Authors' Contributions

All authors contributed to the study conception and design. The manuscript was written and revised by all authors. All authors read and approved the final version.

### Competing Interests

The authors have no competing interests to declare.


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## ORCID

Jordi Gaset Rifà  <https://orcid.org/0000-0001-8796-3149>

Xavier Rivas  <https://orcid.org/0000-0002-4175-5157>

Narciso Román-Roy  <https://orcid.org/0000-0003-3663-9861>

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