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The evolution operator connecting the Lagrangian and Hamiltonian formalisms for contact systems

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Abstract

Some mechanical systems with dissipation can be described within the framework of the so-called contact mechanics: a modified form of the Euler–Lagrange equations stemming from Herglotz's variational principle, which admits a geometric formulation in terms of contact geometry. On the other hand, the study of singular Lagrangian systems and Dirac's theory of constraints can be enhanced by using the evolution operator K that connects the Lagrangian and Hamiltonian formalisms. The main purpose of this paper is to transpose this evolution operator to the case of contact mechanics, and to study some of its main properties. In particular, we show that it provides a geometric description of the evolution equations and it relates the Hamiltonian and Lagrangian constraints. To illustrate the theory, we provide examples of singular contact systems based on modified versions of the simple pendulum and the Cawley Lagrangian.

Keywords Contact structure · Lagrangian mechanics · Hamiltonian mechanics · Singular Lagrangian · Constraint · Evolution operator

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1 Introduction

In analytical mechanics, it is well known that a regular Lagrangian function leads to uniquely determined dynamics in the velocity space $\mathrm{T}Q$ (Lagrangian formalism) and in the phase space T^*Q (Hamiltonian formalism), which are equivalent through the Legendre transformation. This is no longer true when the Lagrangian is singular, which is the case for relativistic invariant theories. Moreover, a Hamiltonian formulation of mechanics or field theory is highly convenient if one aims for a quantum theory. This led P.A.M. Dirac and P.G. Bergmann, near 1950 [2, 24], to start the study of the Hamiltonian formulation of singular Lagrangian theories, introducing the fundamental notions of constraints and gauge freedom.

Differential equations can be written for singular Lagrangian and Hamiltonian dynamics, but there is no guarantee that their solutions are unique, upon fixation of initial conditions. What is remarkable, however, is that there is still a kind of well-defined evolution, but at the price of involving two different spaces. Let us explain this point.

If a time evolution on a manifold M is ruled by a differential equation defined by a vector field X, the dynamical trajectories obey $\gamma' = X \circ \gamma$, in coordinates $\dot{x}^i = X^i(x(t))$, then the evolution of any function (an observable, in physical terminology) can be computed based on the chain rule, and is given by $\mathrm{D}(f \circ \gamma) = (\mathscr{L}_X f) \circ \gamma$, in coordinates $\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x^i}\dot{x}^i = \frac{\partial f}{\partial x^i} X^i$.

Now consider a Lagrangian system given by a Lagrangian function $L \colon \mathrm{T}Q \to \mathbb{R}$. Using natural coordinates (q^i, v^i) in $\mathrm{T}Q$ and (q^i, p_i) in T^*Q, the definition of the Legendre map $\mathcal{F}L \colon \mathrm{T}Q \to \mathrm{T}^*Q$, in coordinates $p_i = \frac{\partial L}{\partial v^i}$, and the Euler–Lagrange equations, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial q^i}$, give us

$$\dot{q}^i = v^i , \qquad \dot{p}_i = \frac{\partial L}{\partial q^i} .$$

So, using the chain rule, we can compute the time derivative of an arbitrary function $g(q, p)$, and obtain an expression like

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial q^i} v^i + \frac{\partial g}{\partial p_i} \frac{\partial L}{\partial q^i} .$$

That is: even if we do not know the time-evolution of a function $g(q, p)$ on the phase space, we *know it* as expressed in terms of functions on the velocity space. Maybe the reader has noticed that we loosely mixed velocity and momentum coordinates (as, for instance, P.A.M. Dirac did in his 1950 paper, or K. Kamimura in 1982 [47]). In geometric terms, one may work on the direct sum $\mathrm{T}^*Q \oplus \mathrm{T}Q$ (sometimes called the Pontryagin bundle), see, for instance, [56]. Notwithstanding that, the right-hand side of the preceding equation can be correctly written using the pullback operator, and this led to the introduction of the operator K in a 1986 paper by Batlle et al. [4]:

$$K \cdot g = \mathcal{F}L^* \left(\frac{\partial g}{\partial q^i} \right) v^i + \mathcal{F}L^* \left(\frac{\partial g}{\partial p_i} \right) \frac{\partial L}{\partial q^i} .$$

Hence, K was introduced as a kind of differential operator $K \colon \mathscr{C}^\infty(\mathrm{T}^*Q) \to \mathscr{C}^\infty(\mathrm{T}Q)$. The same authors used it intensively to develop the Lagrangian and Hamiltonian constraint algorithms, as well as to study the relationship between the different types of constraints in singular theories [4, 5, 52].

A geometrical description of this differential operator was presented in [15] by working on the Pontryagin bundle, but shortly afterward it was realized that K is indeed a *vector field along the Legendre map* [37]. In this last paper, two different but equivalent constructions of K were presented, one that is reminiscent of the definition of the symplectic gradient (Hamiltonian vector field) on a symplectic manifold, and another one involving Tulczyjew's diffeomorphism $\mathrm{T}^*\mathrm{T}Q \to \mathrm{T}\mathrm{T}^*Q$ [57].

In addition to writing equations of motion and relating the Lagrangian and Hamiltonian constraint algorithms, the operator K has also been used to study Noether's theorem for singular systems, as well as the relation between the generators of gauge and rigid symmetries both in the Lagrangian and Hamiltonian formalisms [33, 36, 39, 41, 42]. It has also been used in the study of the Hamilton–Jacobi equation [12], and for Lagrangian systems whose Legendre map degenerates on a hypersurface [54].

The operator K can be defined for higher-order Lagrangian systems [6, 16, 43], where it has the same applications as in the first-order case, see also [40, 44]. Some work for the case of field theories can be found in [25, 55].

In any formulation of the Euler–Lagrange equations, either with the help of the Lagrangian 2-form, or with the operator K, or in coordinates, it is clear that the non-regularity of the equations is due to a singular linear factor (in coordinates the Hessian matrix $\frac{\partial^2 L}{\partial v^i \partial v^j}$) in front of the highest-order derivative. This led to the consideration of *linearly singular systems* in [38], as a particular class of implicit differential equations, those that can be written in coordinates as $A(x)\dot{x} = b(x)$, with $A(x)$ a singular matrix.

Lagrangian and Hamiltonian systems are essentially conservative. Nevertheless, certain dissipative systems can be described through G. Herglotz's variational principle [45, 46], similar to Hamilton's variational principle, but with an additional variable s whose time-derivative is the Lagrangian, so that s is essentially the action:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial s}\frac{\partial L}{\partial \dot{q}^i}\,, \qquad \dot{s} = L\,.$$

The resulting equations are the so-called Herglotz–Euler–Lagrange equations, and they can be expressed in terms of contact geometry [3, 11, 28]. In fact, contact geometry is a very suitable tool to describe these kinds of system, both in the Hamiltonian [8–10, 14, 22, 23, 30] and Lagrangian descriptions [19, 21, 27, 51].

The aim of this paper is to define and study the properties of the time-evolution operator K for dissipative systems in the context of contact mechanics. We show how the main features of this operator in ordinary mechanics extend to the contact case. In particular, we show that it allows us to write the evolution equations even when the Lagrangian is singular, and we prove that its application to constraint functions on the Hamiltonian formalism yields constraint functions on the Lagrangian one. We also provide necessary and sufficient conditions for the Lagrangian constraint functions obtained in this way to be $\mathcal{F}L$-projectable. To obtain these results, we extend some of the theory of almost-regular Lagrangian functions to the framework of contact mechanics.

We also introduce two new, equivalent formulations of the contact Hamiltonian equations, namely Eqs. (2) and (3). These formulations do not depend on the Reeb vector field and are thus particularly well-suited for cases in which the Reeb field is not well-defined. In particular, these results allow us to compute the examples presented in Sect. 5.

The paper is organized as follows. In Sect. 2, we briefly review the definition and main properties of the evolution operator in the context of ordinary mechanics. Section 3 is devoted to contact dynamics. We introduce the main concepts of contact manifolds, and we define contact Hamiltonian systems and the corresponding contact Hamiltonian equations, presenting them in several equivalent forms, one of which notably does not involve the Reeb vector field. We provide an account of the contact Lagrangian formalism and its associated contact Hamiltonian formulation. We also present some results for almost-regular contact Lagrangian functions. In Sect. 4, we provide an intrinsic characterization of the evolution operator K for contact systems and give several equivalent constructions, one of which is related to Tulczyjew's triples. We also analyze its the main properties. Finally, Sect. 5 contains two illustrative examples: the simple pendulum and Cawley's Lagrangian, both with an additional dissipation term.

Throughout the paper, all the manifolds and maps are smooth. Sum over crossed repeated indices is understood.

2 The evolution operator of Lagrangian/Hamiltonian mechanics

The definition of the evolution operator K is based on the concept of vector field along a map. The usage of vector fields along a path is ubiquitous in differential geometry, since the velocity γ' of a path γ is a vector field along γ. Other examples are provided by parallel transport and the Frenet frame in Riemannian geometry. Less widespread is the notion of vector field along a map, that is, a lift of a map to the tangent bundle [7, section 8.6], [48, p. 376] or, more generally, that of a section along a map [53, p. 36] —see [13] for a review on their applications.

In particular, a vector field Z along a map $F\colon M \to N$ is a map $Z\colon M \to \mathrm{T}N$ such that $\tau_N \circ Z = F$, where $\tau_N\colon \mathrm{T}N \to N$ denotes the canonical projection. The set of such maps is denoted as $\mathfrak{X}(F)$, and is a $\mathscr{C}^\infty(M)$-module. Trivial examples of vector fields along F are $Y \circ F$ and $\mathrm{T}F \circ X$, for vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. In a similar way we can speak of differential forms along a map. The inner contraction between $Z \in \mathfrak{X}(F)$ and $\alpha \in \Omega^k(F)$ can be defined in an obvious way. See [17] for more examples of operations with sections along a map.

Let $L \in \mathscr{C}^\infty(\mathrm{T}Q)$ be a Lagrangian function and $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$ its associated Legendre map. The *time-evolution operator* K associated with L is the vector field along the Legendre map, $K\colon \mathrm{T}Q \to \mathrm{T}(\mathrm{T}^*Q)$, satisfying the following conditions:

1. (Dynamical condition): Given the canonical 2-form $\omega_Q \in \Omega^2(\mathrm{T}^*Q)$ and the Lagrangian energy $E_L \in \mathscr{C}^\infty(\mathrm{T}Q)$ defined by L, then,

$$\mathcal{F}L^*(i_K(\omega_Q \circ \mathcal{F}L)) = \mathrm{d}E_L$$

2. (Second-order condition): If $\pi_Q\colon \mathrm{T}^*Q \to Q$ is the canonical projection, then,

$$\mathrm{T}\pi_Q \circ K = \mathrm{Id}_{\mathrm{T}Q}\,.$$

The existence and uniqueness of this operator is discussed in [37], and it can be shown that it can be alternatively defined as $K = \chi \circ \mathrm{d}L$, where $\chi\colon \mathrm{T}^*\mathrm{T}Q \to \mathrm{T}\mathrm{T}^*Q$ is the canonical isomorphism introduced in [57]. If (q^i, v^i) and (q^i, p_i) are natural local coordinates on $\mathrm{T}Q$ and T^*Q, respectively, the local expression of K is

$$K(q, v) = v^i \left.\frac{\partial}{\partial q^i}\right|_{\mathcal{F}L(q,v)} + \frac{\partial L}{\partial q^i} \left.\frac{\partial}{\partial p_i}\right|_{\mathcal{F}L(q,v)}\,.$$

By definition, $\varphi\colon I \subset \mathbb{R} \to \mathrm{T}Q$ is an integral curve of K if $\mathrm{T}\mathcal{F}L \circ \dot\varphi = K \circ \varphi$. Furthermore, as a consequence of the second-order condition (in the item 2), we have that $\varphi = \dot\phi$, for $\phi\colon \mathbb{R} \to Q$, that is, φ is a holonomic curve. We have the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{T}(\mathrm{T}Q) \arrow[r, "\mathrm{T}(\mathcal{F}L)"] \arrow[d, "\tau_{\mathrm{T}Q}"] & \mathrm{T}(\mathrm{T}^*Q) \arrow[d, "\tau_{\mathrm{T}^*Q}"] \\
\mathbb{R} \arrow[ur, "\dot\varphi"] \arrow[r, "\varphi"] \arrow[dr, "\phi"'] & \mathrm{T}Q \arrow[ur, "K"] \arrow[r, "\mathcal{F}L"] \arrow[d, "\tau_Q"] & \mathrm{T}^*Q \\
 & Q &
\end{tikzcd}$$

In coordinates, the integral curves of K are the solutions to the equations,

$$\dot{q} = v\,, \qquad \dot{p} = \frac{\partial L}{\partial q} \qquad \text{and} \qquad p = \frac{\partial L}{\partial v}\,,$$

which are obviously equivalent to the Euler–Lagrange equations for L.

The Lagrangian and Hamiltonian descriptions of a dynamical Lagrangian system can be unified by means of the operator K, as set out in the following properties [4, 5, 15, 37]:

- Let X_L be a holonomic vector field on $\mathrm{T}Q$ (that is, a *second-order differential equation* or SODE) which is a solution to the Lagrangian equation $i_{X_L}\omega_L = \mathrm{d}E_L$, where $\omega_L \in \Omega^2(\mathrm{T}Q)$ is the Lagrangian 2-form associated with L. Then $\varphi : \mathbb{R} \to \mathrm{T}Q$ is an integral curve of X_L if, and only if, it is an integral curve of K.
  As a direct consequence of this fact, the relation between K and X_L is
  $$\mathrm{T}\mathcal{F}L \circ X_L = K\,.$$
- Let X_H be a vector field on $\mathcal{F}L(\mathrm{T}Q) \subseteq \mathrm{T}^*Q$ which is a solution to the Hamilton equations in the Hamiltonian formalism associated with the Lagrangian system $(\mathrm{T}Q, L)$. Then the path $\psi : \mathbb{R} \to \mathrm{T}^*Q$ is an integral curve of X_H if, and only if,
  $$\dot{\psi} = K \circ \mathrm{T}\pi_Q \circ \dot{\psi}\,,$$
  and, as a consequence, in the final constraint submanifold S_f the relation between K and X_H is
  $$X_H \circ \mathcal{F}L = K\,.$$

In this way, the equivalence between the sets of solutions of Euler–Lagrange equations and Hamilton equations is established via the evolution operator.

- The complete classification of Lagrangian and Hamiltonian constraints appearing in the constraint algorithms for singular dynamical systems is also achieved using the operator K. In fact, all Lagrangian constraints can be obtained from the Hamiltonian ones using the time-evolution operator since, if $\xi \in \mathscr{C}^\infty(\mathrm{T}^*Q)$ is a Hamiltonian constraint, then $\mathscr{L}_K\xi = i_K\mathrm{d}\xi$ is a Lagrangian constraint, and all Lagrangian constraints are recovered in this way. Additionally, if ξ is a first-class constraint (resp. a second-class constraint), then $\mathscr{L}_K\xi$ is a dynamical constraint (resp. a SODE constraint).

3 Contact dynamics

3.1 Hamiltonian systems

A *contact manifold* is a pair (M, C) such that M is a $(2n+1)$-dimensional manifold and C is a corank-one maximally non-integrable distribution on M. We call C a *contact distribution* on M. Note that C can be locally described, on an open neighborhood U

of each point $x \in M$, as the kernel of a 1-form $\eta \in \Omega^1(U)$ such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form on U.

A *co-orientable contact manifold* is a pair (M, η), where η is a 1-form on M such that $(M, \operatorname{Ker} \eta)$ is a contact manifold. Then η is called a *contact form*. Since we are interested in local properties of contact manifolds and related structures, we will hereafter restrict ourselves to co-oriented contact manifolds. To simplify terminology, co-oriented contact manifolds will be called contact manifolds as in the standard modern literature on contact geometry [8, 22, 27]. Thus, (M, η) is a contact manifold if $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form.

A contact 1-form $\eta \in \Omega^1(M)$ induces a decomposition of the tangent bundle $\mathrm{T}M$:

$$\mathrm{T}M = \operatorname{Ker} \eta \oplus \operatorname{Ker} \mathrm{d}\eta \,.$$

Note that if η is a contact form on M, then $f\eta$ is also a contact form on M for every nowhere-vanishing function $f \in \mathscr{C}^\infty(M)$, and $\operatorname{Ker} \eta = \operatorname{Ker} f\eta$.

Let (M, η) be a contact manifold. There exists a vector bundle isomorphism $B \colon \mathrm{T}M \to \mathrm{T}^*M$ given by

$$B(v) = i_v(\mathrm{d}\eta)_x + (i_v\eta_x)\eta_x \,, \qquad \forall v \in \mathrm{T}_x M, \quad \forall x \in M \,. \tag{1}$$

This isomorphism can be extended to a $\mathscr{C}^\infty(M)$-module isomorphism $B \colon \mathfrak{X}(M) \to \Omega^1(M)$ in the natural way.

Given a contact manifold (M, η), there exists a unique vector field $R \in \mathfrak{X}(M)$, called the *Reeb vector field*, such that $i_R \mathrm{d}\eta = 0$ and $i_R\eta = 1$, or equivalently, $R = B^{-1}(\eta)$. We have that $\mathscr{L}_R\eta = 0$ and, therefore, $\mathscr{L}_R \mathrm{d}\eta = 0$.

Theorem 3.1 (Darboux theorem [1, 50]) *Given a contact manifold (M, η) with* $\dim M = 2n + 1$, *around every point $x \in M$ there exist local coordinates (q^i, p_i, s), with $i = 1, \ldots, n$, called* Darboux coordinates, *such that*

$$\eta = \mathrm{d}s - p_i \mathrm{d}q^i \,.$$

In these coordinates, $R = \partial/\partial s$.

Example 3.2 *(Canonical contact manifold)* Consider the product manifold $M = \mathrm{T}^*Q \times \mathbb{R}$, where Q is an n-dimensional manifold. Given a local chart (q^i) on Q, there exist canonical coordinates (q^i, p_i) in the cotangent bundle T^*Q. In addition, $\mathbb{R}$ has a natural coordinate s. This gives rise to a coordinate system (q^i, p_i, s) on $\mathrm{T}^*Q \times \mathbb{R}$. Then $\eta_Q = \mathrm{d}s - \theta_Q$, where θ_Q is the pull-back of the Liouville 1-form $\theta \in \Omega^1(\mathrm{T}^*Q)$ relative to the canonical projection $\mathrm{T}^*Q \times \mathbb{R} \to \mathrm{T}^*Q$, is a contact form on M. In canonical coordinates,

$$\eta_Q = \mathrm{d}s - p_i \mathrm{d}q^i \,, \qquad R = \frac{\partial}{\partial s} \,.$$

Note that the coordinates (q^i, p_i, s) are Darboux coordinates on M.

A *contact Hamiltonian system* [9, 22, 27] is a triple (M, η, H), where (M, η) is a contact manifold and $H \in \mathscr{C}^\infty(M)$ is called a *Hamiltonian function*. Given a contact Hamiltonian system (M, η, H), there exists a unique vector field $X_H \in \mathfrak{X}(M)$, called the *contact Hamiltonian vector field* of H, satisfying any of the following equivalent conditions:

(1) $i_{X_H} \mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_R H)\eta \quad \text{and} \quad i_{X_H}\eta = -H$,
(2) $\mathscr{L}_{X_H}\eta = -(\mathscr{L}_R H)\eta \quad \text{and} \quad i_{X_H}\eta = -H$,
(3) $B(X_H) = \mathrm{d}H - (\mathscr{L}_R H + H)\eta$.

A vector field $X \in \mathfrak{X}(M)$ is said to be *Hamiltonian* relative to the contact form η if it is the Hamiltonian vector field of a function $H \in \mathscr{C}^\infty(M)$. Unlike in the case of symplectic mechanics, a Hamiltonian function H may not be preserved along the integral curves of its associated contact Hamiltonian vector field X_H. More precisely,

$$\mathscr{L}_{X_H} H = -(\mathscr{L}_R H)H\,.$$

A function $f \in \mathscr{C}^\infty(M)$ such that $\mathscr{L}_{X_H} f = -(\mathscr{L}_R H) f$ is called a *dissipated quantity* [27]. In Darboux coordinates, the contact Hamiltonian vector field X_H reads

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial s}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial s}\,.$$

Its integral curves, $\gamma(t) = (q^i(t), p_i(t), s(t))$, satisfy the system of differential equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}\,, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial s}\right)\,, \quad \frac{\mathrm{d}s}{\mathrm{d}t} = p_j \frac{\partial H}{\partial p_j} - H\,, \qquad i = 1, \ldots, n\,.$$

Proposition 3.3 *Given a contact Hamiltonian system (M, η, H) and a vector field $X \in \mathfrak{X}(M)$, the following statements are equivalent:*

(1) The vector field X is the contact Hamiltonian vector field of H.
(2) The vector field X satisfies

$$(i_X \mathrm{d}\eta) \wedge \eta = (\mathrm{d}H) \wedge \eta \quad \textit{and} \quad i_X\eta = -H\,. \tag{2}$$

(3) The vector field X satisfies

$$i_X(\eta \wedge \mathrm{d}\eta) = \Omega\,, \tag{3}$$

where Ω is the 2-form on M defined as $\Omega := -H\mathrm{d}\eta + \mathrm{d}H \wedge \eta$.

Proof It is clear that (2) implies (3). The converse follows from the fact that if we take the wedge by η on both sides of (3), we obtain

$$(H + i_X\eta)\eta \wedge \mathrm{d}\eta = 0\,,$$

which implies that $i_X\eta = -H$, because $\eta \wedge \mathrm{d}\eta \neq 0$ as a consequence of η being a contact form.

In order to prove that (3) implies any of the other equivalent forms of the contact Hamiltonian stated before, we just need to contract (3) with the Reeb vector field. We obtain

$$i_R i_X(\eta \wedge \mathrm{d}\eta) = -i_X \mathrm{d}\eta = i_R \Omega = (\mathscr{L}_R H)\eta - \mathrm{d}H\,.$$

The converse comes directly from wedging with the 1-form η on both sides of the equation $i_{X_H}\mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_R H)\eta$. □

Remark 3.4 The reason why these two last equivalent conditions have been explicitly separated from the rest is because they do not use the Reeb vector field. This is particularly important when dealing with *precontact systems*, where Reeb vector fields may not exist, or be unique.

In [20, 21, 49], singular contact Lagrangian functions and precontact systems were studied, and a constraint algorithm for these types of systems was developed. However, those works only consider the case in which the precontact structure admits a Reeb vector field, an assumption that is not always satisfied. In [35], a more general definition of precontact form was presented and studied avoiding hypothesis of existence of Reeb vector fields. Thus, a 1-form η on a manifold is a *precontact form* if

- η is nowhere-vanishing, and
- the dimension of $\operatorname{Ker}\eta \cap \operatorname{Ker}\mathrm{d}\eta$ is constant.

These conditions are equivalent to $\eta \wedge (\mathrm{d}\eta)^r \neq 0$ and $\eta \wedge (\mathrm{d}\eta)^{r+1} = 0$, for some r, and the rank of $\mathrm{d}\eta$ being constant. A *precontact manifold* is a pair (M, η), where M is a manifold and η is a precontact form on M. Note that if $\operatorname{Ker}\eta \cap \operatorname{Ker}\mathrm{d}\eta = \{0\}$ the 1-form η is a contact form.

One can define dynamics on precontact manifolds using Eq. (2) (see [35]). Precontact structures defined as hyperplane fields have also been studied, see for instance [31].

3.2 Lagrangian systems

Let Q be an n-dimensional manifold and consider the product manifold $\mathrm{T}Q \times \mathbb{R}$ equipped with adapted coordinates (q^i, v^i, s) and the canonical projections,

$$s\colon \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}\,, \qquad \tau_1\colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}Q\,, \qquad \tau_0\colon \mathrm{T}Q \times \mathbb{R} \to Q \times \mathbb{R}\,.$$

Note that the projections τ_1 and τ_0 are the projection maps of two vector bundle structures. We will mostly use the latter. In fact, the bundle given by τ_0 is the pull-back of the tangent bundle $\mathrm{T}Q \to Q$ with respect to the map $Q \times \mathbb{R} \to Q$.

The next step is to extend the usual geometric structures of the tangent bundle, namely the tangent structure and the Liouville vector field, to the product $\mathrm{T}Q \times \mathbb{R}$. Note that we can write

$$\mathrm{T}(\mathrm{T}Q \times \mathbb{R}) = (\mathrm{TT}Q \times \mathbb{R}) \oplus_{Q\times\mathbb{R}} (\mathrm{T}Q \times \mathrm{T}\mathbb{R})\,,$$

so any operation acting on vectors tangent to $\mathrm{T}Q$ also acts on vectors tangent to $\mathrm{T}Q\times\mathbb{R}$. In particular, the vertical endomorphism of $\mathrm{TT}Q$ yields a *vertical endomorphism* on $\mathrm{T}(\mathrm{T}Q\times\mathbb{R})$, $\mathcal{J}\colon \mathrm{T}(\mathrm{T}Q\times\mathbb{R})\to\mathrm{T}(\mathrm{T}Q\times\mathbb{R})$. Analogously, the Liouville vector field $\Delta\in\mathfrak{X}(\mathrm{T}Q)$ yields a *Liouville vector field* $\Delta\in\mathfrak{X}(\mathrm{T}Q\times\mathbb{R})$. In fact, this is the Liouville vector field of the vector bundle structure given by τ_0. In adapted coordinates, these objects read

$$\mathcal{J}=\frac{\partial}{\partial v^i}\otimes \mathrm{d}q^i\,,\qquad \Delta=v^i\,\frac{\partial}{\partial v^i}\,.$$

Given a path $\mathbf{c}\colon\mathbb{R}\to Q\times\mathbb{R}$ with $\mathbf{c}=(\mathbf{c}_1,\mathbf{c}_2)$, its *prolongation* to $\mathrm{T}Q\times\mathbb{R}$ is the path

$$\widetilde{\mathbf{c}}=(\mathbf{c}_1',\mathbf{c}_2)\colon\mathbb{R}\longrightarrow\mathrm{T}Q\times\mathbb{R}\,,$$

where $\mathbf{c}_1'$ is the velocity of $\mathbf{c}_1$. The path $\widetilde{\mathbf{c}}$ is said to be *holonomic*. A vector field $\Gamma\in\mathfrak{X}(\mathrm{T}Q\times\mathbb{R})$ is said to satisfy the *second-order condition* (for short: it is a SODE) when its integral curves are holonomic, or equivalently, if $\mathcal{J}\circ\Gamma=\Delta$. In adapted coordinates, if $\mathbf{c}(t)=(c^i(t),s(t))$, then

$$\widetilde{\mathbf{c}}(t)=\left(c^i(t),\,\frac{\mathrm{d}c^i}{\mathrm{d}t}(t),\,s(t)\right).$$

The local expression of a SODE is

$$\Gamma=v^i\,\frac{\partial}{\partial q^i}+f^i\,\frac{\partial}{\partial v^i}+g\,\frac{\partial}{\partial s}\,.$$

Definition 3.5 A *Lagrangian function* is a function $L\colon\mathrm{T}Q\times\mathbb{R}\to\mathbb{R}$. The *Lagrangian energy* associated with L is the function $E_L:=\Delta(L)-L\in\mathscr{C}^\infty(\mathrm{T}Q\times\mathbb{R})$. The *Cartan forms* associated with L are defined as

$$\theta_L={}^t\mathcal{J}\circ\mathrm{d}L\in\Omega^1(\mathrm{T}Q\times\mathbb{R})\,,\quad \omega_L=-\mathrm{d}\theta_L\in\Omega^2(\mathrm{T}Q\times\mathbb{R})\,.$$

The *Lagrangian 1-form* is

$$\eta_L=\mathrm{d}s-\theta_L\in\Omega^1(\mathrm{T}Q\times\mathbb{R})\,,$$

and satisfies that $\mathrm{d}\eta_L=\omega_L$.

The pair $(\mathrm{T}Q\times\mathbb{R},L)$ is a *Lagrangian system.*

If we take natural coordinates (q^i,v^i,s) in $\mathrm{T}Q\times\mathbb{R}$, the Lagrangian 1-form η_L is written as

$$\eta_L=\mathrm{d}s-\frac{\partial L}{\partial v^i}\,\mathrm{d}q^i\,,$$

and, consequently,

$$\mathrm{d}\eta_L=-\frac{\partial^2 L}{\partial s\partial v^i}\mathrm{d}s\wedge\mathrm{d}q^i-\frac{\partial^2 L}{\partial q^j\partial v^i}\mathrm{d}q^j\wedge\mathrm{d}q^i-\frac{\partial^2 L}{\partial v^j\partial v^i}\mathrm{d}v^j\wedge\mathrm{d}q^i\,.$$

The next structure to be defined is the Legendre map. This map is studied in more detail in Sect. 3.4.

Definition 3.6 Given a Lagrangian $L\colon \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$, its *Legendre map* is the fibre derivative of L, considered as a function on the vector bundle $\tau_0\colon \mathrm{T}Q \times \mathbb{R} \to Q \times \mathbb{R}$; that is, the map $\mathcal{F}L\colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ given by

$$\mathcal{F}L(v_q, s) = \big(\mathcal{F}L(\cdot, s)(v_q), s\big) \,,$$

where $L(\cdot, s)$ is the Lagrangian with s frozen.

Proposition 3.7 *For a Lagrangian function, L the following conditions are equivalent:*

(1) The Legendre map $\mathcal{F}L$ is a local diffeomorphism.
(2) The fibre Hessian $\mathcal{F}^2L\colon \mathrm{T}Q \times \mathbb{R} \longrightarrow (\mathrm{T}^*Q \times \mathbb{R}) \otimes_{Q\times\mathbb{R}} (\mathrm{T}^*Q \times \mathbb{R})$ of L is everywhere non-degenerate.
(3) The pair $(\mathrm{T}Q \times \mathbb{R}, \eta_L)$ is a contact manifold.

The proof of this result follows from the expressions in natural coordinates, where

$$\mathcal{F}L : (q^i, v^i, s) \longrightarrow \left(q^i, \frac{\partial L}{\partial v^i}, s\right) \,,$$

$$\mathcal{F}^2L(q^i, v^i, s) = (q^i, W_{ij}, s)\,, \quad \text{with } W_{ij} = \left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) .$$

The conditions in the proposition are also equivalent to the matrix $W = (W_{ij})$ being everywhere non-singular.

Definition 3.8 A Lagrangian function L is said to be *regular* if the equivalent conditions in Proposition 3.7 hold. Otherwise, L is called a *singular* Lagrangian. In particular, L is said to be *hyperregular* if $\mathcal{F}L$ is a global diffeomorphism.

Remark 3.9 As a result of the preceding definitions and results, every *regular* Lagrangian system has associated the contact Hamiltonian system $(\mathrm{T}Q \times \mathbb{R}, \eta_L, E_L)$.

Given a regular Lagrangian system $(\mathrm{T}Q \times \mathbb{R}, L)$, the *Reeb vector field* $R_L \in \mathfrak{X}(\mathrm{T}Q \times \mathbb{R})$ is uniquely determined by the conditions

$$i_{R_L}\mathrm{d}\eta_L = 0\,, \qquad i_{R_L}\eta_L = 1\,.$$

Its local expression is

$$R_L = \frac{\partial}{\partial s} - W^{ji}\frac{\partial^2 L}{\partial s \partial v^j}\frac{\partial}{\partial v^i}\,,$$

where (W^{ij}) is the inverse of the partial Hessian matrix, namely $W^{ij}W_{jk} = \delta^i_k$.

Note that the Reeb vector field does not appear in the simplest form $\partial/\partial s$. This is due to the fact that the natural coordinates in $\mathrm{T}Q \times \mathbb{R}$ are not Darboux coordinates for η_L.

For a regular Lagrangian, there exists a unique *Lagrangian vector field* such that

$$i_{X_L}\mathrm{d}\eta_L = \mathrm{d}E_L - (\mathscr{L}_{R_L}E_L)\eta_L\,, \qquad i_{X_L}\eta_L = -E_L\,. \tag{4}$$

Proposition 3.10 *If the Lagrangian L is regular, then X_L is a second-order differential vector field and its integral curves satisfy the so-called Herglotz–Euler–Lagrange equations.*

In local coordinates, this means that $X_L = v^i\dfrac{\partial}{\partial q^i} + f^i\dfrac{\partial}{\partial v^i} + g\dfrac{\partial}{\partial s}$, *with*

$$g = L\,,$$
$$\frac{\partial^2 L}{\partial v^j\partial v^i}f^j + \frac{\partial^2 L}{\partial q^j\partial v^i}v^j + \frac{\partial^2 L}{\partial s\partial v^i}L - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i}$$

Note that the Herglotz–Euler–Lagrange equations, since they come from a variational principle, exist regardless of the regularity of the Lagrangian. As a matter of fact, we can write these equations in a geometric way even for singular Lagrangians. In general, the solutions to the Herglotz–Euler–Lagrange equations can be seen as the integral curves of the second-order vector fields X_L satisfying the equations

$$i_{X_L}\mathrm{d}\eta_L = \mathrm{d}E_L + \left(\frac{\partial L}{\partial s}\right)\eta_L\,, \qquad i_{X_L}\eta_L = -E_L\,.$$

These are well-defined since $\partial L/\partial s$ is defined canonically in $\mathrm{T}Q \times \mathbb{R}$, and they are equivalent to (4), as one can check that, in the regular case, $\mathscr{L}_{R_L}E_L = -\partial L/\partial s$. If the Lagrangian is singular, these equations do not directly imply that the vector field is second-order, and the condition must be added. Also, in the singular case, solutions may not exist at every point, and if they do, they might not be unique. This requires a more careful study of both the Lagrangian and Hamiltonian formalisms in the singular case.

3.3 Hamiltonian formalism for contact Lagrangian systems

Let us analyze how to construct a contact Hamiltonian formalism that is related to the contact Lagrangian formalism we presented before. That is, we describe a contact Hamiltonian system on $\mathrm{T}^*Q \times \mathbb{R}$ which is in correspondence, via the Legendre map, with the Lagrangian system $(\mathrm{T}Q \times \mathbb{R}, L)$.

Recall, from Example 3.2, that $\mathrm{T}^*Q \times \mathbb{R}$ is endowed with a natural contact form $\eta_Q = \mathrm{d}s - \theta_Q$. A direct computation shows that

$$\eta_L = \mathcal{F}L^*(\eta_Q)\,.$$

If the Lagrangian is regular, then the energy E_L is, at least locally, $\mathcal{F}L$-projectable (since $\mathcal{F}L$ is a local diffeomorphism). Namely, we can (locally) always find a *contact Hamiltonian function* $H\colon \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R}$ such that $E_L = \mathcal{F}L^*(H) = H \circ \mathcal{F}L$. In the hyperregular case, the energy is globally $\mathcal{F}L$-projectable and the system $(\mathrm{T}^*Q \times \mathbb{R}, \eta_Q, H)$ satisfies

$$\mathcal{F}L_*(R_L) = R\,, \qquad \text{and} \qquad \mathcal{F}L_*(X_L) = X_H\,,$$

for $R, X_H \in \mathfrak{X}(\mathrm{T}^*Q \times \mathbb{R})$, respectively, the Reeb vector field of η_Q and the contact Hamiltonian vector field associated with the system. This means that the two formalisms are $\mathcal{F}L$-related, and the map $\xi \mapsto \mathcal{F}L \circ \widetilde{\xi}$, where $\xi \colon I \to Q \times \mathbb{R}$ is a solution to the Herglotz–Euler–Lagrange equations, sends solutions to solutions. Conversely, if $\psi \colon I \to \mathrm{T}^*Q \times \mathbb{R}$ is an integral path of X_H, then the map $\psi \mapsto \pi_0 \circ \psi$, with $\pi_0 \colon \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ the canonical projection, also sends solutions of the contact Hamiltonian formalism to solutions of the Herglotz–Euler–Lagrange equations.

Let us point out that, regardless of the regularity of the Lagrangian, any solution on $\mathrm{T}^*Q \times \mathbb{R}$ which is $\mathcal{F}L$-related to a solution to the Herglotz–Euler–Lagrange equations fulfills the equations

$$\begin{cases} p = \dfrac{\partial L}{\partial v}\,, \\ \dot{p} = \dfrac{\partial L}{\partial s}\dfrac{\partial L}{\partial v} + \dfrac{\partial L}{\partial q}\,, \\ \dot{s} = L\,, \end{cases} \tag{5}$$

called the *Herglotz–Dirac* equations.

3.4 Almost-regular Lagrangian functions

In this section, we present some results on singular Lagrangian functions. Namely, we extend the results presented in [42] to the case of contact mechanics. We refer the reader to the aforementioned article for a detailed presentation of the results and their proofs.

Let us start by stating some preliminary results about fibre derivatives. Recall that, given a (not necessarily linear) bundle map $f \colon E \to F$ between two vector bundles over a manifold B, the fibre derivative of f is the map $\mathcal{F}f \colon E \to \mathrm{Hom}(E, F) \approx F \otimes E^*$ obtained by restricting f to the fibres, $f_b \colon E_b \to F_b$, and computing the usual derivative of a map between vector spaces: $\mathcal{F}f(e_b) = \mathrm{D}f_b(e_b)$ (see [34] for a detailed account). This applies in particular when the second vector bundle is trivial of rank 1, that is, for a function $f \colon E \to \mathbb{R}$; then $\mathcal{F}f \colon E \to E^*$. This map also has a fibre derivative $\mathcal{F}^2 f \colon E \to E^* \otimes E^*$, which can be called the fibre hessian of f: for every $e_b \in E$, $\mathcal{F}^2 f(e_b)$ can be considered as a symmetric bilinear form on E_b. One can check that $\mathcal{F}f$ is a local diffeomorphism at $e \in E$ if, and only if, $\mathcal{F}^2 f(e)$ is non-degenerate. Indeed, it is a direct consequence of the fact that $\operatorname{Ker} \mathrm{T}(\mathcal{F}f) \subset V(E)$ and

$$v_x \in \operatorname{Ker} \mathcal{F}^2 f(e_x) \iff \mathrm{vl}_E(e_x, v_x) \in \operatorname{Ker} \mathrm{T}_{e_x}(\mathcal{F}f)\,,$$

which can be easily seen with the local expressions of the objects involved.

Another interesting result involving the fibre derivative that will be useful later is the following. Let $\xi\colon E \to E$ be a bundle map with associated vertical field $X = \xi^v = \mathrm{vl}_E \circ \xi$ on E, and let $f\colon E \to \mathbb{R}$ be a function, then

$$X \cdot f = \langle \mathcal{F} f, \xi \rangle \,.$$

In particular, if we take $\xi = \mathrm{Id}_E$, we have

$$(\Delta_E \cdot f)(e_x) = \langle \mathcal{F} f(e_x), e_x \rangle \,, \tag{6}$$

where $\Delta_E = \mathrm{vl}_E(\mathrm{Id}_E)$ is the Liouville vector field of the vector bundle E. By applying the chain rule to this last expression, we also obtain

$$\mathcal{F}(\Delta_E \cdot f)(e_x) = \mathcal{F} f(e_x) + \mathcal{F}^2 f(e_x) \cdot e_x \,. \tag{7}$$

In the case of contact Lagrangian systems, our framework involves two real vector bundles over the same base, namely $\tau_0\colon \mathrm{T}Q \times \mathbb{R} \to Q \times \mathbb{R}$ and $\pi_0\colon \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$. Taking this into account, one can readily see that most of the theory of singular Lagrangian systems extends to contact mechanics. In particular, the fibre derivative of a Lagrangian function $L\colon \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$ is its Legendre map.

As in the case of ordinary mechanics, to obtain some general results in the singular case, we still need to impose some weak regularity conditions on the Lagrangian functions. Namely, following [29], we say that a Lagrangian function $L \in \mathscr{C}^\infty(\mathrm{T}Q \times \mathbb{R})$ is *almost-regular* if: (i) the image of the Legendre map $P_0 := \mathcal{F}L(\mathrm{T}Q \times \mathbb{R}) \overset{j}{\hookrightarrow} \mathrm{T}^*Q \times \mathbb{R}$ is a closed submanifold, called the *primary Hamiltonian constraint submanifold*, (ii) the induced map $\mathcal{F}L_0\colon \mathrm{T}Q \times \mathbb{R} \to P_0$ is a submersion and has connected fibres. From a local point of view, it suffices to assume that $\mathcal{F}L$ has constant rank.

Since $P_0 := \mathcal{F}L(\mathrm{T}Q \times \mathbb{R})$ is a closed submanifold, it is locally defined by the vanishing of an independent set of functions $\{\phi_\mu\}_{\mu=1,\dots,m}$, with linearly independent differentials $\mathrm{d}\phi_\mu$ at every point. We call these functions *primary Hamiltonian constraint functions*. Now, we recall the following two lemmas:

Lemma 3.11 *Let $\alpha\colon N \to P$ be a submersion at $y \in N$. Then, there exists an open neighborhood $V \subset N$ of y such that $\alpha(V) \subset P$ is a submanifold. Additionally, we have:*

1. $\mathrm{Im}\,\mathrm{T}_y\alpha = \mathrm{T}_{\alpha(y)}(\alpha(V))$.
2. *If $\alpha(V) \subset P$ is the submanifold defined locally by the vanishing of a set of functions ϕ_μ, with linearly independent differentials* $\mathrm{d}\phi_\mu$ *at every point, then* $\mathrm{Ker}\,{}^t\mathrm{T}_y\alpha$ *admits* $\mathrm{d}_{\alpha(y)}\phi_\mu$ *as a basis, and* $\mathrm{Ker}\,{}^t\mathrm{T}\alpha$ *admits* $\mathrm{d}\phi_\mu \circ \alpha$ *as a frame over V.*

Lemma 3.12 *Let $P_0 \overset{j}{\hookrightarrow} P$ be a submanifold defined by the vanishing of $\{\phi_\mu\}_{\mu=1,\dots,m}$, such that their differentials* $\mathrm{d}\phi_\mu$ *are linearly independent at every point of P_0. Then,* $\mathrm{Ker}\,{}^t\mathrm{T}(j)$ *admits* $\{\mathrm{d}\phi_\mu|_{P_0}\}_{\mu=1,\dots,r}$ *as a frame. Also, at every point $x \in P_0$ a tangent vector $v_x \in \mathrm{T}_x P$ is in $\mathrm{T}_x P_0$ if, and only if, for every ϕ_μ, it satisfies* $\mathrm{d}\phi_\mu(v_x) = 0$.

We can apply Lemmas 3.11 and 3.12 to show that the $\mathrm{d}\phi_\mu \circ \mathcal{F}L$ form a (local) frame of the vector subbundle defined by $\operatorname{Ker} {}^t\mathrm{T}(\mathcal{F}L) \subset \mathrm{T}^*(\mathrm{T}^*Q \times \mathbb{R})$.

For every given function $g \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$, we can define the map

$$\gamma_g = \mathcal{F}g \circ \mathcal{F}L \colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}Q \times \mathbb{R}\,,$$

where we denote by $\mathcal{F}L$ the fibre derivative of L, considered as a function on $\tau_0 \colon \mathrm{T}Q \times \mathbb{R} \to Q \times \mathbb{R}$, and $\mathcal{F}g$ the fibre derivative of g, considered as a function on the vector bundle $\pi_0 \colon \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$.

If we apply the natural bijection between fibre bundle maps and vertical vector fields; that is, the vertical lift $\mathrm{vl}_{\mathrm{T}Q\times\mathbb{R}}$, we obtain the vertical vector field

$$\Gamma_g := \gamma_g^v = \mathrm{vl}_{\mathrm{T}Q\times\mathbb{R}}(\mathrm{Id}_{\mathrm{T}Q\times\mathbb{R}}, \mathcal{F}g \circ \mathcal{F}L)\,.$$

In local coordinates, these objects have the following expressions:

$$\gamma_g \colon (q, v, s) \mapsto \left(q, \frac{\partial g}{\partial p}(\mathcal{F}L(q, v, s)), s\right), \qquad \Gamma_g = \mathcal{F}L^*\left(\frac{\partial g}{\partial p}\right)\frac{\partial}{\partial v}\,.$$

With these objects, we can prove the following result:

Proposition 3.13 *The vector fields $\Gamma_\mu = \gamma_{\phi_\mu}^v$, constructed from the primary Hamiltonian constraint functions ϕ_μ, form a local frame for* $\operatorname{Ker} \mathrm{T}(\mathcal{F}L)$. *Their local expression is*

$$\Gamma_\mu = \gamma_\mu^i \frac{\partial}{\partial v^i}\,,$$

where the functions

$$\gamma_\mu^i = \mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial p_i}\right)$$

form a basis of the kernel of the Hessian matrix $W = \left(\dfrac{\partial^2 L}{\partial v^i \partial v^j}\right)$.

This last result follows directly from applying the chain rule, since

$$\mathcal{F}(g \circ \mathcal{F}L) = \mathcal{F}^2 L \bullet \gamma_g\,,$$

where the symbol $\bullet$ denotes the composition of the images of both maps. This implies that if g vanishes on the image $\mathcal{F}L(\mathrm{T}Q \times \mathbb{R}) \subset \mathrm{T}^*Q \times \mathbb{R}$, then γ_g is in the kernel of $\mathcal{F}^2L$. Since, by definition, we take the functions ϕ_μ to be linearly independent, the result follows immediately. The result can also be easily proved using the local coordinate expressions.

Now we give the most important result for the characterization of a Hamiltonian formalism in this context [4, 29].

Proposition 3.14 *If the Legendre map $\mathcal{F}L$ is a submersion, then the Lagrangian energy function E_L is locally projectable. That is, locally, there exists a function H such that $E_L = H \circ \mathcal{F}L$.*

Again, this proposition can be proved using the local expressions of the objects involved. We can also use the properties of the Liouville vector field (6), (7), as they imply

$$E_L(v_q, s) = (\Delta_{\mathrm{T}Q\times\mathbb{R}} \cdot L)(v_q, s) - L(v_q, s) = \langle \mathcal{F}L(v_q, s), (v_q, s)\rangle - L(v_q, s)\,,$$

$$\mathcal{F}E_L(v_q, s) = \mathcal{F}^2 L(v_q, s) \cdot (v_q, s)\,. \tag{8}$$

Hence, one has

$$\Gamma_\mu \cdot E_L = \langle \mathcal{F}E_L, \gamma_\mu\rangle = 0\,,$$

which is equivalent to the Lagrangian energy being locally projectable via $\mathcal{F}L$, by Proposition 3.13.

Thus, if the Lagrangian function L is almost-regular, then E_L is *globally* $\mathcal{F}L$-projectable at P_0, that is, there exists a *unique* function $H_0\colon P_0 \to \mathbb{R}$, the *Hamiltonian function*, such that $\mathcal{F}L^*(H_0) = E_L$. Also, as P_0 is assumed to be closed, the function H_0 can be extended (non-uniquely) to a function H defined on $\mathrm{T}^*Q \times \mathbb{R}$. Note that, for a local study, it is enough to just assume that the Legendre map $\mathcal{F}L$ is a submersion.

Finally, let us prove a result which relates the vector fields Γ_H and Γ_μ, defined, respectively, from a choice of Hamiltonian function and of primary Hamiltonian constraints, with the Liouville vector field.

Proposition 3.15 *Given an almost regular Lagrangian L, the choice of a Hamiltonian function H and a set of primary Hamiltonian constraints ϕ_μ allows us to write, locally, the identity map of* $\mathrm{T}Q \times \mathbb{R}$ *as*

$$\mathrm{Id}_{\mathrm{T}Q\times\mathbb{R}} = \gamma_H + \sum_\mu \lambda^\mu \gamma_{\phi_\mu}\,, \tag{9}$$

where the $\lambda^\mu \in \mathscr{C}^\infty(\mathrm{T}Q \times \mathbb{R})$ *are uniquely determined functions.*

The proof of this result comes from applying the chain rule to the definition of H, i.e., $E_L = H \circ \mathcal{F}L$, which yields

$$\mathcal{F}E_L(v_q, s) = \mathcal{F}^2\, L(v_q, s) \cdot \gamma_H(v_q, s)\,.$$

Applying (8), we obtain

$$\mathcal{F}^2\, L(v_q, s) \cdot ((v_q, s) - \gamma_H(v_q, s)) = 0\,,$$

which proves that $(v_q, s) - \gamma_H(v_q, s) = \sum_\mu \lambda^\mu(v_q, s)\gamma_\mu(v_q, s)$, as we wanted to see.

If we apply the natural bijection between fibre bundle maps and vertical vector fields to (9), we obtain the following formula for the Liouville vector field

$$\Delta_{\mathrm{T}Q\times\mathbb{R}} = \Gamma_H + \sum_\mu \lambda^\mu \Gamma_\mu\,. \tag{10}$$

Note that, in coordinates, each component of (10) yields

$$v^i = \gamma_H^i + \sum_\mu \lambda^\mu \gamma_\mu^i \,,$$

where, in local coordinates, $\gamma_H^i = \mathcal{F}L^*\left(\frac{\partial H}{\partial p_i}\right)$ and $\gamma_\mu^i = \mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial p_i}\right)$. Applying the vector fields $\Gamma_\nu = \gamma_\nu^i \frac{\partial}{\partial v^i}$ to this last expression, we obtain

$$\gamma_\nu^i = \sum_{\mu=1}^m (\Gamma_\nu \cdot \lambda^\mu)\gamma_\mu^i \,,$$

and, by the linear independency of the γ_μ, this implies that

$$\Gamma_\nu \cdot \lambda^\mu = \delta_\nu^\mu \,. \tag{11}$$

In particular, this shows that the functions λ^μ are not $\mathcal{F}L$-projectable. As we will show later in the examples, these functions correspond to the velocities that cannot be written in terms of momenta via the Legendre map.

4 The evolution operator in contact mechanics

In this section, we present the definition of the evolution operator K for contact Lagrangian systems, several equivalent characterizations of K, and some of its most relevant properties.

4.1 Construction of the evolution operator *K*

Let $L\colon \mathrm{T}Q \to \mathbb{R}$ be a contact Lagrangian function. To define the evolution operator in the setting of contact mechanics, we will follow the same structure we presented in Sect. 2. We define the evolution operator as the unique vector field along the Legendre map that satisfies three key conditions: the second-order condition and two dynamical conditions.

The condition of K being a vector field along the Legendre map is equivalent to the diagram

$$\begin{tikzcd} & \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R}) \arrow[d, "\tau_{\mathrm{T}^*Q\times\mathbb{R}}"] \\ \mathrm{T}Q \times \mathbb{R} \arrow[ur, "K"] \arrow[r, "\mathcal{F}L"'] & \mathrm{T}^*Q \times \mathbb{R} \end{tikzcd}$$

being commutative. In other words, the map $K\colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})$ satisfies

$$\tau_{\mathrm{T}^*Q\times\mathbb{R}} \circ K = \mathcal{F}L \,.$$

This equation is sometimes referred to as the structural equation, and it implies that, in local coordinates, K is expressed as

$$K(q, v, s) = \left(q, \frac{\partial L}{\partial v}, s\,;\; a(q, v, s), b(q, v, s), c(q, v, s)\right),$$

or, alternatively, as

$$\begin{aligned} K(q, v, s) = a^i(q, v, s) \frac{\partial}{\partial q^i}\bigg|_{\mathcal{F}L(q,v,s)} &+ b_i(q, v, s) \frac{\partial}{\partial p_i}\bigg|_{\mathcal{F}L(q,v,s)} \\ &+ c(q, v, s) \frac{\partial}{\partial s}\bigg|_{\mathcal{F}L(q,v,s)}, \end{aligned}$$

where a^i, b_i and c are functions yet to be determined.

Now, the so-called *second-order condition* for the time-evolution operator K is given by

$$\mathrm{T}\pi_0^1 \circ K = \tau_1 , \tag{12}$$

where $\pi_0^1 \colon \mathrm{T}^*Q \times \mathbb{R} \to Q$ and $\tau_1 \colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}Q$ are canonical projections. Writing this in coordinates, we find that

$$\mathrm{T}\pi_0^1(q, p, s, v, u, z) = (q, v) ,$$

and so,

$$\mathrm{T}\pi_0^1 \circ K = (q, a^i) .$$

Thus, Equation (12) determines that, in coordinates, we have $a^i = v^i$. Note that this condition is equivalent to $\rho \circ \mathrm{T}\pi_0 \circ K = \mathrm{Id}_{\mathrm{T}Q\times\mathbb{R}}$, where $\pi_0 \colon \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ and $\rho \colon \mathrm{T}(Q \times \mathbb{R}) \to \mathrm{T}Q \times \mathbb{R}$ are the natural projections.

The last step to fully characterize the time-evolution operator K is via the so-called *dynamical conditions*. They are

$$\begin{cases} \mathcal{F}L^*(i_K(\mathrm{d}\eta_Q \circ \mathcal{F}L)) = \mathrm{d}E_L + \dfrac{\partial L}{\partial s}\eta_L , \\ i_K(\eta_Q \circ \mathcal{F}L) = -E_L , \end{cases} \tag{13}$$

where η_Q is the canonical contact form on $\mathrm{T}^*Q \times \mathbb{R}$.

In coordinates, these conditions define the functions b_i and c. Let us see this, by writing in local coordinates all the expressions involved. First, we have

$$i_K(\mathrm{d}\eta_Q \circ \mathcal{F}L) = -b_i \mathrm{d}q^i\big|_{\mathcal{F}L(q,v,s)} + v^i \mathrm{d}p_i\big|_{\mathcal{F}L(q,v,s)} ,$$

and therefore,

$$\mathcal{F}L^*(i_K(\mathrm{d}\eta_Q \circ \mathcal{F}L)) = \left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - b_i\right) \mathrm{d}q^i + v^i \frac{\partial^2 L}{\partial v^i \partial v^j} \mathrm{d}v^i + v^i \frac{\partial^2 L}{\partial v^i \partial s} \mathrm{d}s .$$

On the right-hand side, we have

$$\mathrm{d}E_L + \frac{\partial L}{\partial s}\eta_L = \left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i} \right) \mathrm{d}q^i + v^i \frac{\partial^2 L}{\partial v^i \partial v^j} \mathrm{d}v^i + v^i \frac{\partial^2 L}{\partial v^i \partial s} \mathrm{d}s \,,$$

and so, equating both expressions, we obtain

$$b_i = \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i} \,.$$

Now, as

$$i_K(\eta_Q \circ \mathcal{F}L) = c(q, v, s) - v^i \frac{\partial L}{\partial v^i} \,,$$

the second dynamical condition shows that $c = L$.

Summing up, we have proved the following result.

Theorem 4.1 *Let $L \colon \mathrm{T}Q \to \mathbb{R}$ be a contact Lagrangian function. There exists a unique vector field along the Legendre map satisfying conditions (12) and (13). In local coordinates it reads*

$$K(q, v, s) = v^i \left(\frac{\partial}{\partial q^i} \circ \mathcal{F}L \right) + \left(\frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i} \right) \left(\frac{\partial}{\partial p_i} \circ \mathcal{F}L \right) + L \left(\frac{\partial}{\partial s} \circ \mathcal{F}L \right) .$$

This vector field along the Legendre map defines an operator

$$K \colon \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R}) \longrightarrow \mathscr{C}^\infty(\mathrm{T}Q \times \mathbb{R})$$

that takes functions defined in the phase space and gives their time derivative in the velocity space. In local coordinates, this derivation is given by

$$(K \cdot f)(q, v, s) = v^i \mathcal{F}L^* \left(\frac{\partial f}{\partial q^i} \right) + \left(\frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i} \right) \mathcal{F}L^* \left(\frac{\partial f}{\partial p_i} \right) + L\, \mathcal{F}L^* \left(\frac{\partial f}{\partial s} \right) .$$

Note that we can equivalently write $K \cdot f \in \mathscr{C}^\infty(\mathrm{T}Q \times \mathbb{R})$ as the function defined by

$$(K \cdot f)(q, v, s) = i_K(\mathrm{d}f \circ \mathcal{F}L) = \langle \mathrm{d}f(\mathcal{F}L(q, v, s)), K(q, v, s) \rangle \,. \qquad (14)$$

The evolution operator can also be characterized with only one dynamical condition, using the isomorphism $B \colon \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R}) \to \mathrm{T}^*(\mathrm{T}^*Q \times \mathbb{R})$ defined by the canonical contact 1-form on $\mathrm{T}^*Q \times \mathbb{R}$ (see Equation (1)).

Proposition 4.2 *The evolution operator K can be equivalently characterized as the unique vector field along the Legendre map such that*

$$\mathrm{T}\pi_0^1 \circ K = \tau_1 \,,$$

and

$$\mathcal{F}L^*(B \circ K) = \mathrm{d}E_L + \left(\frac{\partial L}{\partial s} - E_L\right)\eta_L\,. \tag{15}$$

Proof We can prove this in coordinates in a similar way as above. The first equation is the second-order condition, and as we saw above, it implies $a^i = v^i$. For the second equation, on the left-hand side we have

$$\begin{aligned}\mathcal{F}L^*(B \circ K) = {}^t\mathrm{T}\mathcal{F}L \circ (i_K(\mathrm{d}\eta_Q \circ \mathcal{F}L) + (i_K(\eta_Q \circ \mathcal{F}L))(\eta_Q \circ \mathcal{F}L)) = \\ \left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - b_i - c\frac{\partial L}{\partial v^i} + v^j \frac{\partial L}{\partial v^j}\frac{\partial L}{\partial v^i}\right)\mathrm{d}q^i \\ + v^i \frac{\partial^2 L}{\partial v^i \partial v^j}\mathrm{d}v^i + \left(v^i\left(\frac{\partial^2 L}{\partial v^i \partial s} - \frac{\partial L}{\partial v^i}\right) + c\right)\mathrm{d}s\,,\end{aligned}$$

and on the right-hand side,

$$\begin{aligned}\mathrm{d}E_L + \left(\frac{\partial L}{\partial s} - E_L\right)\eta_L = \left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i} + v^j \frac{\partial L}{\partial v^j}\frac{\partial L}{\partial v^i} - L\frac{\partial L}{\partial v^i}\right)\mathrm{d}q^i \\ + v^i \frac{\partial^2 L}{\partial v^i \partial v^j}\mathrm{d}v^i + \left(v^i\left(\frac{\partial^2 L}{\partial v^i \partial s} - \frac{\partial L}{\partial v^i}\right) + L\right)\mathrm{d}s\,,\end{aligned}$$

equating both sides directly yields the local expression of the evolution operator. □

We can also give a construction of the evolution operator K in terms of the so-called *Tulczyjew's triples*. We can use the canonical diffeomorphism $\chi\colon \mathrm{T}^*\mathrm{T}Q \longrightarrow \mathrm{T}\mathrm{T}^*Q$ (see [57]) to define a diffeomorphism between $\mathrm{T}^*(\mathrm{T}Q \times \mathbb{R})$ and $\mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})$, using the natural identification of $\mathrm{T}(M \times M')$ with $\mathrm{T}M \times \mathrm{T}M'$ and of both $\mathrm{T}\mathbb{R}$ and $\mathrm{T}^*\mathbb{R}$ with $\mathbb{R} \times \mathbb{R}$ (using the fixed canonical coordinate s). In coordinates, it reads

$$\begin{aligned}\widetilde{\chi}\colon \mathrm{T}^*(\mathrm{T}Q \times \mathbb{R}) &\longrightarrow \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})\,,\\ (q, v, s, u, p, z) &\longmapsto (q, p, s, v, u, z)\,,\end{aligned}$$

and then

$$K = \widetilde{\chi} \circ \left(\mathrm{d}L + L\mathrm{d}s - \frac{\partial L}{\partial s}\eta_L\right)\,.$$

4.2 The evolution operator and the equations of motion

This section studies how the Lagrangian and Hamiltonian formalisms can be recovered from the time-evolution operator.

Proposition 4.3 *Let $\xi\colon I \to \mathrm{T}Q \times \mathbb{R}$ be a path, and $\dot{\xi}\colon I \to \mathrm{T}(\mathrm{T}Q \times \mathbb{R})$ its canonical lift. Then, ξ is a solution to the Herglotz–Euler–Lagrange equations for a given Lagrangian L if, and only if,*

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi\,.$$

Proof It is enough to see this in local canonical coordinates. Assume that the path is given by $\xi = (q, v, s)$. Its canonical lift is $\dot{\xi} = (q, v, s; \dot{q}, \dot{v}, \dot{s})$. Thus, we have

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = \left(q, \frac{\partial L}{\partial v}, s\, ;\ \dot{q}, \dot{q}\frac{\partial^2 L}{\partial v \partial q} + \dot{v}\frac{\partial^2 L}{\partial v \partial v} + \dot{s}\frac{\partial^2 L}{\partial v \partial s}, \dot{s}\right),$$

and

$$K \circ \xi = \left(q, \frac{\partial L}{\partial v}, s\, ;\ v, \frac{\partial L}{\partial q} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v}, L\right).$$

Equating both expressions,

$$\begin{cases} \dot{q} = v\,, \\ v\dfrac{\partial^2 L}{\partial v \partial q} + \dot{v}\dfrac{\partial^2 L}{\partial v \partial v} + \dot{s}\dfrac{\partial^2 L}{\partial v \partial s} = \dfrac{\partial L}{\partial q} + \dfrac{\partial L}{\partial s}\dfrac{\partial L}{\partial v}\,, \\ \dot{s} = L\,, \end{cases}$$

which are precisely the second-order condition and the Herglotz–Euler–Lagrange equations. □

Note that, if a path $\xi : I \to \mathrm{T}Q \times \mathbb{R}$ satisfies $\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi$ then, necessarily, it can be obtained as the prolongation of a path $\zeta : I \to Q \times \mathbb{R}$. Therefore, the equations of motion defined by the evolution operator K incorporate the second-order condition, regardless of the regularity of the Lagrangian function.

A solution of the Herglotz–Euler–Lagrange equations satisfies $\dot{\xi} = X_L \circ \xi$, with X_L a second-order Lagrangian vector field defined on an appropriate submanifold of $\mathrm{T}Q \times \mathbb{R}$. Thus, an immediate consequence of the last proposition is that we can write

$$\mathrm{T}(\mathcal{F}L) \circ X_L \circ \xi = K \circ \xi\,.$$

If S_f is the final constraint submanifold, since we have solutions at every point, we can write

$$K|_{S_f} = \mathrm{T}(\mathcal{F}L) \circ X_L|_{S_f}\,.$$

If the Lagrangian is regular, then the Legendre transformation is a local diffeomorphism, and the Lagrangian vector field X_L is uniquely determined. Therefore, in the regular case, we obtain

$$X_L = \mathrm{T}(\mathcal{F}L^{-1}) \circ K\,.$$

Proposition 4.4 *Let $\psi : I \to \mathrm{T}^*Q \times \mathbb{R}$ be a path in the extended cotangent bundle and consider its canonical lift $\dot{\psi} : I \to \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})$. Then ψ is a solution to the Herglotz–Dirac equations (5) if, and only if,*

$$\dot{\psi} = K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}\,,$$

where $\pi_0 : \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ and $\rho : \mathrm{T}(Q \times \mathbb{R}) \to \mathrm{T}Q \times \mathbb{R}$ are the canonical projections.

Proof It is enough to explicitly write both sides of the equation in coordinates. If $\psi = (q, p, s)$, then its canonical lift is expressed as $\dot{\psi} = (q, p, s; \dot{q}, \dot{p}, \dot{s})$. Therefore,

$$K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi} = \left(q, \frac{\partial L}{\partial v}, s\,;\ \dot{q}, \frac{\partial L}{\partial q} + \frac{\partial L}{\partial v}\frac{\partial L}{\partial s}, L \right) .$$

If we compare both expressions, we obtain

$$\begin{cases} \dot{s} = L\,, \\ p = \dfrac{\partial L}{\partial v}\,, \\ \dot{p} = \dfrac{\partial L}{\partial q} + \dfrac{\partial L}{\partial v}\dfrac{\partial L}{\partial s}\,, \end{cases}$$

which are precisely the Herglotz–Dirac equations (5) for the Lagrangian L. □

Since K is an embedding, there is a correspondence between the solutions of the Herglotz–Euler–Lagrange and the Herglotz–Dirac equations. The map $\xi \mapsto \mathcal{F}L \circ \xi$ sends solutions to solutions, and its inverse is $\psi \mapsto \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}$. Thus, if there exists a solution to the Herglotz–Dirac equations $\psi\colon I \to \mathrm{T}^*Q \times \mathbb{R}$, it can be expressed as $\psi = \mathcal{F}L \circ \xi$, where $\xi\colon I \to \mathrm{T}Q \times \mathbb{R}$ is a solution of the Herglotz–Euler–Lagrange equations, and is obtained as the prolongation of the projected path $\pi_0 \circ \psi$. By Propositions 4.3 and 4.4, we have

$$\dot{\psi} = \mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi = K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}.$$

Now, suppose that we have a Hamiltonian vector field X_H, defined on the image of the Legendre map $\mathcal{F}L(\mathrm{T}Q \times \mathbb{R}) \subseteq \mathrm{T}^*Q \times \mathbb{R}$, for the Hamiltonian formalism associated with the Lagrangian system $(\mathrm{T}Q \times \mathbb{R}, L)$. Then, the solutions to Hamilton's equations are its integral curves $\dot{\psi} = X_H \circ \psi$, and we can write

$$K \circ \xi = \dot{\psi} = X_H \circ \psi = X_H \circ \mathcal{F}L \circ \xi\,. \tag{16}$$

In the final constraint submanifold S_f, this gives the relation

$$K|_{S_f} = X_H \circ \mathcal{F}L|_{S_f}\,.$$

And, if the Lagrangian is regular, we have

$$X_H = K \circ \mathcal{F}L^{-1}\,.$$

Lastly, assume that we have two $\mathcal{F}L$-related solutions ξ and ψ of the Herglotz–Euler–Lagrange and the Herglotz–Dirac equations, respectively, i.e., $\psi = \mathcal{F}L \circ \xi$ and $\xi = \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}$. Then, for a given function $f \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \psi) = \langle \mathrm{d}f \circ \psi, \dot{\psi} \rangle = \langle \mathrm{d}f \circ (\mathcal{F}L \circ \xi), K \circ \xi \rangle = (K \cdot f) \circ \xi\,,$$

where we used Equations (14) and (16).

Corollary 4.5 *Suppose that $f \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$ is a Hamiltonian constant of motion, such as a Hamiltonian constraint. Then $K \cdot f$ is a Lagrangian constraint.*

4.3 Relating the constraint algorithms

Let $L \in \mathscr{C}^\infty(\mathrm{T}Q \times \mathbb{R})$ be an almost-regular Lagrangian function, so we can apply the results presented in Sect. 3.4. Recall that $P_0 := \mathcal{F}L(\mathrm{T}Q \times \mathbb{R})$ is a closed submanifold, (locally) described by the vanishing of m functions $\{\phi_\mu\}_{\mu=1,\dots,m}$, with linearly independent differentials $\mathrm{d}\phi_\mu$ at every point of P_0. Also, the vertical vector fields $\Gamma_\mu \in \mathfrak{X}(\mathrm{T}Q \times \mathbb{R})$, locally given by

$$\Gamma_\mu = \gamma_\mu \frac{\partial}{\partial v} = \mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial p}\right)\frac{\partial}{\partial v}\,,$$

provide a frame for the vector subbundle $\operatorname{Ker}\mathrm{T}(\mathcal{F}L)$.

Since the Lagrangian is almost-regular, there exists a unique function $H_0 \in \mathscr{C}^\infty(P_0)$ such that $E_L = H_0 \circ \mathcal{F}L$ on P_0. The function H_0 can be extended to a function $H \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$ defined on the whole extended phase space $\mathrm{T}^*Q \times \mathbb{R}$. This function H satisfies $\mathcal{F}L^*(H) = E_L$ and defines a (unique) contact Hamiltonian vector field X_H.

Our aim is to write the evolution operator K in terms of the contact Hamiltonian vector fields associated with a choice of Hamiltonian function and a set of primary Hamiltonian constraints.

Proposition 4.6 *Let $H \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$ be a fixed Hamiltonian function such that $\mathcal{F}L^*(H) = E_L$, and $\{\phi_\mu\}_\mu$ be a set of primary constraint Hamiltonian functions. Then, there locally exist m uniquely determined functions λ^μ such that*

$$K = X_H \circ \mathcal{F}L + \sum_{\mu=1}^{m} \lambda^\mu (X_{\phi_\mu} \circ \mathcal{F}L)\,.$$

These functions λ^μ are the same as those appearing in Proposition 3.15.

Proof Using $\mathcal{F}L^*(H) = E_L$, we can write (15) as

$${}^t\mathrm{T}\mathcal{F}L\left(B \circ K - \mathrm{d}H \circ \mathcal{F}L + (H \circ \mathcal{F}L)(\eta_Q \circ \mathcal{F}L) - \frac{\partial L}{\partial s}(\eta_Q \circ \mathcal{F}L)\right) = 0\,.$$

Hence, we have a 1-form along the Legendre map that belongs to $\operatorname{Ker}{}^t\mathrm{T}(\mathcal{F}L)$. Recall that $\{\mathrm{d}\phi_\mu \circ \mathcal{F}L\}_\mu$ is a frame for $\operatorname{Ker}{}^t\mathrm{T}(\mathcal{F}L)$, and so

$$B \circ K - \mathrm{d}H \circ \mathcal{F}L + (H \circ \mathcal{F}L)(\eta_Q \circ \mathcal{F}L) - \frac{\partial L}{\partial s}(\eta_Q \circ \mathcal{F}L) = \sum_\mu \alpha^\mu \mathrm{d}\phi_\mu \circ \mathcal{F}L\,,$$

where the α^μ are some uniquely determined functions.

Applying B^{-1} to both sides and rearranging terms, we obtain

$$K = B^{-1}(\mathrm{d}H) \circ \mathcal{F}L + \left(\frac{\partial L}{\partial s} - H \circ \mathcal{F}L\right) R \circ \mathcal{F}L + \sum_\mu \alpha^\mu B^{-1}(\mathrm{d}\phi_\mu) \circ \mathcal{F}L\,.$$

Using Equation (9) together with the condition $\mathrm{Id}_{\mathrm{T}Q\times\mathbb{R}} = \rho \circ \mathrm{T}\pi_0 \circ K$, one can check, in local coordinates, that the functions α^μ coincide with the functions λ^μ from Proposition 3.15. Thus, we have

$$K = B^{-1}(\mathrm{d}H) \circ \mathcal{F}L + \left(\frac{\partial L}{\partial s} - H \circ \mathcal{F}L\right) R \circ \mathcal{F}L + \sum_\mu \lambda^\mu B^{-1}(\mathrm{d}\phi_\mu) \circ \mathcal{F}L\,. \tag{17}$$

Note that

$$K \cdot s = L = \mathcal{F}L^*\left(p_i \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial s}\right) - H \circ \mathcal{F}L + \frac{\partial L}{\partial s} + \sum_\mu \lambda^\mu \mathcal{F}L^*\left(p_i \frac{\partial \phi_\mu}{\partial p_i} + \frac{\partial \phi_\mu}{\partial s}\right),$$

where we have used the local expression of the vector field defined by $B^{-1}(\mathrm{d}f)$, for any function $f \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$.

Since $E_L = H \circ \mathcal{F}L$ and $\mathcal{F}L^*(p_i) = \partial L/\partial v^i$, from the last expression we obtain

$$\begin{aligned} E_L + L = \Delta(L) = \mathcal{F}L^*\left(\frac{\partial H}{\partial p_i}\right)\frac{\partial L}{\partial v^i} + \mathcal{F}L^*\left(\frac{\partial H}{\partial s}\right) + \frac{\partial L}{\partial s} \\ + \sum_\mu \lambda^\mu \left(\mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial p_i}\right)\frac{\partial L}{\partial v^i} + \mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial s}\right)\right), \end{aligned}$$

and, combining this with (10), which yields

$$\Delta(L) = \Gamma_H(L) + \sum_\mu \lambda^\mu \Gamma_\mu(L) = \mathcal{F}L^*\left(\frac{\partial H}{\partial p_i}\right)\frac{\partial L}{\partial v^i} + \sum_\mu \lambda^\mu \mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial p_i}\right)\frac{\partial L}{\partial v^i},$$

we immediately obtain the expression

$$-\frac{\partial L}{\partial s} = \mathcal{F}L^*\left(\frac{\partial H}{\partial s}\right) + \sum_\mu \lambda^\mu \mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial s}\right). \tag{18}$$

Now, the result follows directly from Equations (17) and (18), and the fact that for every primary Hamiltonian constraint we have $\phi_\mu \circ \mathcal{F}L = 0$, by definition. □

With this result, for every $f \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$, we can write $K \cdot f \in \mathscr{C}^\infty(\mathrm{T}Q \times \mathbb{R})$ as

$$K \cdot f = \mathcal{F}L^*(X_H \cdot f) + \sum_{\mu=1}^m \lambda^\mu \mathcal{F}L^*(X_{\phi_\mu} \cdot f)\,.$$

In particular, from the local expression of K we have

$$v = K \cdot q = \mathcal{F}L^*(X_H \cdot q) + \sum_{\mu=1}^{m} \lambda^\mu \mathcal{F}L^*(X_{\phi_\mu} \cdot q)\,,$$
$$\frac{\partial L}{\partial q} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v} = K \cdot p = \mathcal{F}L^*(X_H \cdot p) + \sum_{\mu=1}^{m} \lambda^\mu \mathcal{F}L^*(X_{\phi_\mu} \cdot p)\,,$$
$$L = K \cdot s = \mathcal{F}L^*(X_H \cdot s) + \sum_{\mu=1}^{m} \lambda^\mu \mathcal{F}L^*(X_{\phi_\mu} \cdot s)\,.$$

Proposition 4.7 *Let f be a function on the phase space. The function $K \cdot f$ is $\mathcal{F}L$-projectable if, and only if, $X_{\phi_\mu} \cdot f$ vanishes on P_0 for all primary Hamiltonian constraints ϕ_μ.*

Proof We have

$$\Gamma_\mu \cdot (K \cdot f) = \sum_{\nu=1}^{m} \mathcal{F}L^*(X_{\phi_\nu} \cdot f)(\Gamma_\mu \cdot \lambda^\nu)\,,$$

and, applying (11), we get

$$\Gamma_\mu \cdot (K \cdot f) = \mathcal{F}L^*(X_{\phi_\mu} \cdot f)\,.$$

Thus, $\Gamma_\mu \cdot (K \cdot f) = 0$ for every $\mu = 1, \ldots, m$ if, and only if, $X_{\phi_\mu} \cdot f$ vanishes on P_0, as we wanted to see. □

4.4 Related constructions

When dealing with a singular Lagrangian function, the equations of motion cannot be written in normal form, so they define an implicit system of differential equations. In geometric terms, an implicit system is described as a subset of the tangent bundle of some manifold; in our case, a subset $D \subseteq \mathrm{T}(\mathrm{T}Q \times \mathbb{R})$. From a geometric perspective, the main task is to establish the underlying geometric structures required to investigate this implicit differential equation, and, if possible, its “Hamiltonian” counterpart, which in this case is a subset of $\mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})$.

To achieve this, a standard approach in the literature relies on the usage of different morphisms between the various bundles associated with the problem. This is the approach of the so-called Tulczyjew’s triples, which for the case of contact mechanics have been studied both in [26] and [32]. Therein it is proved the existence of a diffeomorphism

$$\alpha^c \colon \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R}) \times \mathbb{R} \longrightarrow \mathrm{T}^*(\mathrm{T}Q \times \mathbb{R}) \times \mathbb{R}\,,$$

that in natural coordinates reads

$$\alpha^c(q^i, p_i, s, \dot q^i, \dot p_i, \dot s, r) = (q^i, \dot q^i, s, r p_i + \dot p_i, p_i, -r, \dot s)\,.$$

Now, given a Lagrangian function $L\colon \mathrm{T}Q \times \mathbb{R}$ its 1-jet prolongation is

$$\begin{aligned} j^1 L\colon\ \mathrm{T}Q \times \mathbb{R} &\longrightarrow J^1(\mathrm{T}Q \times \mathbb{R}, \mathbb{R}) \cong \mathrm{T}^*(\mathrm{T}Q \times \mathbb{R}) \times \mathbb{R} \\ (q^i, \dot{q}^i, s) &\longmapsto \left(q^i, \dot{q}^i, s, \tfrac{\partial L}{\partial q^i}, \tfrac{\partial L}{\partial \dot{q}^i}, \tfrac{\partial L}{\partial s}, L\right) \end{aligned}$$

and so one has

$$(\alpha^c)^{-1}(j^1 L)(q^i, \dot{q}^i, s) = \left(q^i, \frac{\partial L}{\partial \dot{q}^i}, s, \dot{q}^i, \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial \dot{q}^i}, L, -\frac{\partial L}{\partial s}\right).$$

Hence, one can write the evolution operator we presented in Theorem 4.1 explicitly as

$$K = \mathrm{pr}_1 \circ (\alpha^c)^{-1} \circ j^1 L\,,$$

where $\mathrm{pr}_1\colon \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R}) \times \mathbb{R} \to \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})$ is the canonical projection.

5 Examples

This last section is devoted to the study of a couple interesting examples: the simple pendulum and Cawley's Lagrangian, both with an extra dissipation term.

Note that, in these examples, we obtain precontact 1-forms in the sense of [35]. Hence, we can use Equation (2) to compute the (pre)contact Hamiltonian vector fields. The constraints are then obtained by imposing the usual tangency condition at each step.

5.1 The simple pendulum with damping

Consider a damped pendulum of length ℓ and mass m. Its position in the plane is given using polar coordinates (r, θ), where $\theta = 0$ is the rest position. The Lagrangian phase space has coordinates $(r, \theta, \mu; v_r, v_\theta, v_\mu; s)$ and the Lagrangian function of the system $L\colon \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$ is given by

$$L = \frac{1}{2}m(v_r^2 + r^2 v_\theta^2) - mg(\ell - r\cos\theta) + \mu(r - \ell) - \gamma s\,,$$

where $\gamma \in \mathbb{R}$ is a damping coefficient. The Legendre map associated with this Lagrangian function is

$$\mathcal{F}L(r, \theta, \mu; v_r, v_\theta, v_\mu; s) = \left(r, \theta, \mu;\ p_r = m v_r,\ p_\theta = m r^2 v_\theta,\ p_\mu = 0, s\right),$$

giving the constraint

$$\phi_0 = p_\mu\,.$$

The Lagrangian energy is

$$E_L = \frac{1}{2}m(v_r^2 + r^2 v_\theta^2) + mg(\ell - r\cos\theta) - \mu(r-\ell) + \gamma s\,;$$

so we can take as a possible Hamiltonian function

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + mg(\ell - r\cos\theta) - \mu(r-\ell) + \gamma s$$

We can compute the corresponding Hamiltonian vector fields, with respect to the 1-form defined in $P_0 :- \mathcal{F}L(\mathrm{T}Q \times \mathbb{R})$ by taking $\eta_0 := j^*(\eta_Q)$, where $j\colon P_0 \hookrightarrow \mathrm{T}^*Q \times \mathbb{R}$. In coordinates, we have

$$\eta_0 = \mathrm{d}s - p_r\mathrm{d}r - p_\theta\mathrm{d}\theta\,.$$

Note that this is not a contact form, but a *precontact form* (see [35] for more details). Using Equations (2), we obtain

$$\begin{aligned} X_H = \frac{p_r}{m}\frac{\partial}{\partial r} + \frac{p_\theta}{mr^2}\frac{\partial}{\partial\theta} + f_1\frac{\partial}{\partial\mu} + \left(\frac{p_\theta^2}{mr^3} + mg\cos\theta + \mu - p_r\gamma\right)\frac{\partial}{\partial p_r} \\ + (-mgr\sin\theta - p_\theta\gamma)\frac{\partial}{\partial p_\theta} + f_2\frac{\partial}{\partial p_\mu} + \left(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - H\right)\frac{\partial}{\partial s}\,, \end{aligned}$$

where f_1, f_2 are any arbitrary functions. We also obtain, as a necessary condition, the constraint

$$\phi_1 := r - \ell\,.$$

Now, when we perform the constraint algorithm, the tangency condition $0 = X_H \cdot \phi_0 = f_2$ determines the coefficient f_2. We also obtain the new constraint

$$\phi_2 := X_H \cdot \phi_1 = \frac{p_r}{m}\,.$$

If we keep performing the algorithm, this last constraint yields

$$\phi_3 := X_H \cdot \phi_2 = \frac{1}{m}\left(\frac{p_\theta^2}{mr^3} + mg\cos\theta + \mu - p_r\gamma\right),$$

and, finally, the tangency condition $X_H \cdot \phi_3 = 0$ determines the coefficient f_1. The constraint algorithm ends here.

The evolution operator associated with this Lagrangian function has local expression,

$$K = v_r \frac{\partial}{\partial r}\Big|_{\mathcal{F}L} + v_\theta \frac{\partial}{\partial \theta}\Big|_{\mathcal{F}L} + v_\mu \frac{\partial}{\partial \mu}\Big|_{\mathcal{F}L} + \left(mrv_\theta^2 + mg\cos\theta + \mu - \gamma m v_r\right)\frac{\partial}{\partial p_r}\Big|_{\mathcal{F}L}$$
$$+ \left(-mgr\sin\theta - \gamma m r^2 v_\theta\right)\frac{\partial}{\partial p_\theta}\Big|_{\mathcal{F}L} + (r-\ell)\frac{\partial}{\partial p_\mu} + L\frac{\partial}{\partial s}\Big|_{\mathcal{F}L}.$$

We can apply the evolution operator to the Hamiltonian constraints to obtain Lagrangian ones. Namely, we obtain the following constraints,

$$\begin{cases} \chi_1 := K\cdot\phi_0 = r-\ell\,, \\ \chi_2 := K\cdot\phi_1 = v_r\,, \\ \chi_3 := K\cdot\phi_2 = rv_\theta^2 + g\cos\theta + \mu/m - \gamma v_r\,, \\ \chi_4 := K\cdot\phi_3 = -3v_\theta(g\sin\theta + \gamma r v_\theta) - 3v_r v_\theta^2 + \dfrac{v_\mu}{m} - \dfrac{\gamma}{m}(mg\cos\theta + \mu - \gamma m v_r)\,. \end{cases}$$

If one performs the constraint algorithm directly for the Lagrangian formalism, one can check that these same constraints are obtained.

We can also find the functions λ, from (10). We have

$$\Gamma_H = v_r\frac{\partial}{\partial v_r} + v_\theta\frac{\partial}{\partial v_\theta}\,, \qquad \Gamma_0 = \frac{\partial}{\partial v_\mu},$$

and so,

$$\Delta = \Gamma_H + v_\mu\Gamma_0\,,$$

which implies that $\lambda^0 = v_\mu$ in this case.

If we find X_H, X_{ϕ_0}, the contact Hamiltonian vector fields, now with respect to the canonical contact structure $\eta_Q \in \Omega^1(\mathrm{T}^*Q\times\mathbb{R})$, we obtain

$$X_H = \frac{p_r}{m}\frac{\partial}{\partial r} + \frac{p_\theta}{mr^2}\frac{\partial}{\partial\theta} + \left(\frac{p_\theta^2}{mr^3} + mg\cos\theta + \mu - p_r\gamma\right)\frac{\partial}{\partial p_r}$$
$$+ (-mgr\sin\theta - p_\theta\gamma)\frac{\partial}{\partial p_\theta} + (r-\ell-p_\mu\gamma)\frac{\partial}{\partial p_\mu} + \left(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - H\right)\frac{\partial}{\partial s},$$

and

$$X_{\phi_0} = \frac{\partial}{\partial\mu}\,.$$

One can easily check that

$$K = X_H\circ\mathcal{F}L + v_\mu X_{\phi_0}\circ\mathcal{F}L\,.$$

5.2 Cawley's Lagrangian with dissipation

Cawley's Lagrangian is an academic model introduced by R. Cawley to study some features of singular Lagrangians in Dirac's theory of constraint systems [18]. In this example we introduce a velocity-dependent dissipation term. Consider the manifold $\mathrm{T}\mathbb{R}^3 \times \mathbb{R}$ with canonical coordinates $(x, y, z; v_x, v_y, v_z; s)$ and the Lagrangian function

$$L = v_x v_z + \frac{1}{2} y z^2 - \gamma s v_y ,$$

where γ is a non-zero damping coefficient. The Legendre map is

$$\mathcal{F}L\colon (x, y, z; v_x, v_y, v_z; s) \longmapsto (x, y, z;\ p_x = v_z,\ p_y = -\gamma s,\ p_z = v_x;\ s) .$$

The primary Hamiltonian constraint is $\phi_0 = p_y + \gamma s$, and L is an almost-regular Lagrangian.

The Lagrangian energy is

$$E_L = v_x v_z - \frac{1}{2} y z^2 ,$$

therefore, we can take as a Hamiltonian function

$$H = p_x p_z - \frac{1}{2} y z^2 .$$

Now, we can compute the corresponding Hamiltonian vector fields with respect to the 1-form defined on $P_0 := \mathcal{F}L(\mathrm{T}Q \times \mathbb{R})$ by taking $\eta_0 := j^*(\eta_Q)$, where $j\colon P_0 \hookrightarrow \mathrm{T}^*Q \times \mathbb{R}$. In coordinates, we have

$$\eta_0 = \mathrm{d}s - p_x \mathrm{d}x + \gamma s \mathrm{d}y - p_z \mathrm{d}z .$$

One can check that this 1-form is a precontact form which *does not define any Reeb vector field*, and so the usual contact Hamiltonian equations cannot be used to find the corresponding Hamiltonian vector fields in this case. However, if we apply the alternative Hamiltonian equations (2), we obtain

$$X_H = p_z \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + p_x \frac{\partial}{\partial z} - \gamma b p_x \frac{\partial}{\partial p_x} + c \frac{\partial}{\partial p_y} + (yz - \gamma b p_z) \frac{\partial}{\partial p_z} + \left(p_x p_z + \frac{1}{2} y z^2 - \gamma b s \right) \frac{\partial}{\partial s} ,$$

where b and c are arbitrary functions. We also obtain as a necessary condition the constraint

$$\phi_1 := \frac{1}{2} z^2 + \gamma \left(p_x p_z + \frac{1}{2} y z^2 \right) .$$

Now, demanding the tangency to the first constraint submanifold

$$X_H \cdot \phi_0 = c + \gamma \left(p_x p_z + \frac{1}{2} y z^2 - \gamma b s \right) = 0$$

we obtain an expression for c, in terms of the function b. Again, the same can be done for the constraint ϕ_1, and the condition $X_H \cdot \phi_1 = 0$ will determine the function b. The constraint algorithm ends here.

The evolution operator is given by

$$\begin{aligned} K = {} & v_x \frac{\partial}{\partial x}\Big|_{\mathcal{F}L} + v_y \frac{\partial}{\partial y}\Big|_{\mathcal{F}L} + v_z \frac{\partial}{\partial z}\Big|_{\mathcal{F}L} - \gamma v_y v_z \frac{\partial}{\partial p_x}\Big|_{\mathcal{F}L} \\ & + \left(\frac{1}{2} z^2 + \gamma^2 s v_y \right) \frac{\partial}{\partial p_y}\Big|_{\mathcal{F}L} + (yz - \gamma v_x v_y) \frac{\partial}{\partial p_z}\Big|_{\mathcal{F}L} + L \frac{\partial}{\partial s}\Big|_{\mathcal{F}L} . \end{aligned}$$

We can apply the evolution operator to the Hamiltonian constraints to obtain Lagrangian ones. Namely, we obtain the following constraints,

$$\begin{cases} \chi_1 := K \cdot \phi_0 = \frac{1}{2} z^2 + \gamma \left(v_x v_z + \frac{1}{2} y z^2 \right) , \\ \chi_2 := K \cdot \phi_1 = z v_z (1 + 2\gamma y) + \gamma v_y \left(\frac{1}{2} z^2 - 2\gamma v_x v_z \right) . \end{cases}$$

6 Conclusions and outlook

In this paper, we have introduced and studied the evolution operator K, for singular contact Lagrangian systems. To this end, we first reviewed the characterization of the evolution operator K in ordinary mechanics. We also reviewed the basic facts of contact geometry and contact Hamiltonian systems, for which we obtained alternative formulations of the dynamical equations that do not rely on the Reeb vector field.

We presented a definition of the evolution operator K within the framework of contact mechanics. This operator exhibits properties analogous to those in ordinary symplectic mechanics, and provides a fundamental link between the Lagrangian and Hamiltonian descriptions in the singular contact setting. To illustrate the theoretical developments, we examined two explicit examples: the simple pendulum and the Cawley Lagrangian, both with extra damping terms.

There are several aspects of singular contact dynamics that deserve further investigation. For singular Lagrangian systems, the operator K provides a tight connection between the Lagrangian and Hamiltonian formalisms. We plan to explore in detail these connections for contact Lagrangian systems in the future. This also includes the development of constraint algorithms for precontact Lagrangian and Hamiltonian systems.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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