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A K-CONTACT LAGRANGIAN FORMULATION FOR NONCONSERVATIVE FIELD THEORIES

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Dynamical systems with dissipative behaviour can be described in terms of contact manifolds and a modified version of Hamilton's equations. Dissipation terms can also be added to field equations, as showed in a recent paper where we introduced the notion of k-contact structure, and obtained a modified version of the De Donder–Weyl equations of covariant Hamiltonian field theory. In this paper we continue this study by presenting a k-contact Lagrangian formulation for nonconservative field theories. The Lagrangian density is defined on the product of the space of k-velocities times a k-dimensional Euclidean space with coordinates s^α, which are responsible for the dissipation. We analyze the regularity of such Lagrangians; only in the regular case we obtain a k-contact Hamiltonian system. We study several types of symmetries for k-contact Lagrangian systems, and relate them with dissipation laws, which are analogous to conservation laws of conservative systems. Several examples are discussed: we find contact Lagrangians for some kinds of second-order linear partial differential equations, with the damped membrane as a particular example, and we also study a vibrating string with a magnetic-like term.

Keywords: contact structure, field theory, Lagrangian system, dissipation, k-symplectic structure, k-contact structure.

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1. Introduction

In the last years the methods of differential geometry have been used to develop an intrinsic framework to describe dissipative or damped systems, in particular using contact geometry [2, 17, 24]. It has been applied to give both the Hamiltonian and the Lagrangian descriptions of mechanical systems with dissipation

[3, 5, 7–9, 13, 16, 25, 27]. Contact geometry has other physical applications, as for instance thermodynamics, quantum mechanics, circuit theory, control theory, etc. (see [4, 8, 20, 24, 28], among others). All of them are described by ordinary differential equations to which some terms that account for the dissipation or damping have been added.

These geometric methods have been also used to give intrinsic descriptions of the Lagrangian and Hamiltonian formalisms of field theory; in particular, those of multisymplectic and k-symplectic geometry (see, for instance, [6, 12, 14, 18, 29, 31] and references therein). Nevertheless, all these methods are developed, in general, to model systems of variational type; that is, without dissipation or damping.

In a recent paper [15] we have introduced a generalization of both contact geometry and k-symplectic geometry to describe field theories with dissipation, and more specifically their Hamiltonian (De Donder–Weyl) covariant formulation. This new formalism is inspired by contact Hamiltonian mechanics, where the addition of a "contact variable" s allows to describe dissipation terms; geometrically this new variable comes from a contact form instead of the usual symplectic form of Hamiltonian mechanics. In the field theory case, if k is the number of independent variables (usually space-time variables), we add k new dependent variables s^α to introduce dissipation terms in the De Donder–Weyl equations. These new variables can be obtained geometrically from the notion of *k-contact structure*: a family of k differential 1-forms η^α satisfying certain properties. Then a *k-contact Hamiltonian system* is a manifold endowed with a k-contact structure and a Hamiltonian function $\mathcal{H}$. With these elements we can state the k-contact Hamilton equations, which indeed add dissipation terms to the usual Hamiltonian field equations. The study of their symmetries also allows to obtain some dissipation laws. This formalism was applied to two relevant examples: the damped vibrating string and Burgers' equation.

The aim of this paper is to extend the above study, developing the Lagrangian formalism of field theories with dissipation, mainly in the regular case. For this purpose, the aforementioned k-contact structure will be used to generalize the Lagrangian formalism of the contact mechanics presented in [9, 16] and the Lagrangian k-symplectic formulation of classical field theories [12, 29]. In this new formalism the phase bundle is $\oplus^k \mathrm{T}Q \times \mathbb{R}^k = (\mathrm{T}Q \oplus \overset{k}{\dots} \oplus \mathrm{T}Q) \times \mathbb{R}^k$. Then, given a Lagrangian function $\mathcal{L}\colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}$, one defines k differential 1-forms $\eta^\alpha_{\mathcal{L}}$ which, when $\mathcal{L}$ is *regular*, constitute a k-contact structure on the phase bundle. The k-contact Lagrangian field equations are then defined as the k-contact Hamiltonian field equations for the Lagrangian energy $E_{\mathcal{L}}$. When written in coordinates they are the Euler–Lagrange equations for $\mathcal{L}$ with some additional terms which account for dissipation.

We also study several types of symmetries for these Lagrangian field theories, as well as their associated dissipation laws, which are characteristic of dissipative systems, and are analogous to the conservation laws for conservative systems.

As examples of this formalism we study the k-contact Lagrangian formulation for second-order elliptic and hyperbolic partial differential equations, finding contact Lagrangians for some specific kinds of these equations. This procedure is exemplified with the equation of the damped vibrating membrane, which is given a 2-contact Lagrangian description. In another example we deal with a two-dimensional vibration of a string illustrating the difference between the linear terms that appear in the equations arising from magnetic-like terms and those coming from a k-contact formulation.

The paper is organized as follows. Section 2 is devoted to briefly review several preliminary concepts on k-symplectic manifolds, k-contact geometry and k-contact Hamiltonian systems for field theories with dissipation. In Section 3 we introduce the notion of k-contact Lagrangian system, and set the geometric framework for the Lagrangian formalism of field theories with dissipation, stating the geometric form of the contact Euler–Lagrange equations in several equivalent ways, as well as the Legendre transformation and the associated canonical Hamiltonian formalism. In Section 4 we study several types of Lagrangian symmetries and the relations between them, as well as the corresponding dissipation laws. Finally, some examples are given in Section 5.

Throughout the paper all the manifolds and mappings are assumed to be smooth. Sum over crossed repeated indices is understood.

2. Preliminaries

2.1. k-tangent bundle, k-vector fields and geometric structures

(See [12, 29] for more details.)

Let Q be a manifold and consider $\oplus^k \mathrm{T}Q = \mathrm{T}Q \oplus \overset{k}{\ldots} \oplus \mathrm{T}Q$ (it is called the *k-tangent bundle*, or bundle of k^1-velocities, of Q), which is endowed with the natural projections to each direct summand and to the base manifold,

$$\tau_\alpha \colon \oplus^k \mathrm{T}Q \to \mathrm{T}Q, \quad \tau_Q^1 \colon \oplus^k \mathrm{T}Q \to Q.$$

A point of $\oplus^k \mathrm{T}Q$ is $\mathbf{w}_q = (v_{1q}, \ldots, v_{kq}) \in \oplus^k \mathrm{T}Q$, where $(v_i)_q \in \mathrm{T}_q Q$.

A *k-vector field* on Q is a section $\mathbf{X} \colon Q \longrightarrow \oplus^k \mathrm{T}Q$ of the projection τ_Q^1. It is specified by giving k vector fields $X_1, \ldots, X_k \in \mathfrak{X}(Q)$, obtained as $X_\alpha = \tau_\alpha \circ \mathbf{X}$; for $1 \leq \alpha \leq k$, and it is denoted $\mathbf{X} = (X_1, \ldots, X_k)$.

Given a map $\phi \colon D \subset \mathbb{R}^k \to Q$, the *first prolongation* of ϕ to $\oplus^k \mathrm{T}Q$ is the map $\phi' \colon D \subset \mathbb{R}^k \to \oplus^k \mathrm{T}Q$ defined by

$$\phi'(t) = \left(\phi(t), \mathrm{T}\phi \left(\left. \frac{\partial}{\partial t^1} \right|_t \right), \ldots, \mathrm{T}\phi \left(\left. \frac{\partial}{\partial t^k} \right|_t \right) \right) \equiv (\phi(t); \phi'_\alpha(t)),$$

where $t = (t^1, \ldots, t^k)$ are the canonical coordinates of $\mathbb{R}^k$. A map $\varphi \colon D \subset \mathbb{R}^k \to \oplus^k \mathrm{T}Q$ is said to be *holonomic* if it is the first prolongation of a map $\phi \colon D \subset \mathbb{R}^k \to Q$.

A map $\phi \colon D \subset \mathbb{R}^k \to Q$ is an *integral map* of a k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ when

$$\phi' = \mathbf{X} \circ \phi. \tag{1}$$

Equivalently, $\mathrm{T}\phi \circ \dfrac{\partial}{\partial t^\alpha} = X_\alpha \circ \phi$, for every α. A k-vector field $\mathbf{X}$ is *integrable* if every point of Q is in the image of an integral map of $\mathbf{X}$.

In coordinates, if $X_\alpha = X^i_\alpha \dfrac{\partial}{\partial x^i}$, then ϕ is an integral map of $\mathbf{X}$ if, and only if, it is a solution to the following system of partial differential equations,

$$\frac{\partial \phi^i}{\partial t^\alpha} = X^i_\alpha(\phi).$$

A k-vector field $\mathbf{X} = (X_1, \dots, X_k)$ is integrable if, and only if, $[X_\alpha, X_\beta] = 0$, for every α, β [26]; these are the necessary and sufficient conditions for the integrability of the above system of partial differential equations.

As in the case of the tangent bundle, local coordinates (q^i) in $U \subset Q$ induce natural coordinates (q^i, v^i_α) in $(\tau^1_Q)^{-1}(U) \subset \oplus^k \mathrm{T}Q$, with $1 \le i \le n$ and $1 \le \alpha \le k$.

Given α and $\mathbf{w}_q \in \oplus^k \mathrm{T}Q$, there exists a natural map $(\Lambda^{\mathbf{w}_q}_q)^\alpha \colon \mathrm{T}_q Q \to \mathrm{T}_{\mathbf{w}_q}(\oplus^k \mathrm{T}Q)$, called the *$\alpha$-vertical lift* from q to $\mathbf{w}_q$, defined as

$$(\Lambda^{\mathbf{w}_q}_q)^\alpha(u_q) = \frac{d}{d\lambda}(v_{1q}, \dots, v_{\alpha-1q}, v_{\alpha q} + \lambda u_q, v_{\alpha+1q}, \dots, v_{kq})_{|\lambda=0}.$$

In coordinates, if

$$u_q = a^i \frac{\partial}{\partial q^i}\bigg|_q,$$

we have

$$(\Lambda^{\mathbf{w}_q}_q)^\alpha(u_q) = a^i \frac{\partial}{\partial v^i_\alpha}\bigg|_{w_q}.$$

Observe that these α-vertical lifts are τ^1_Q-vertical vectors. These vertical lifts extend to vector fields in a natural way; that is, if $X \in \mathfrak{X}(Q)$, then its α-vertical lift, $\Lambda^\alpha(X) \in \mathfrak{X}(\oplus^k \mathrm{T}Q)$, is given by $(\Lambda^\alpha(X))_{\mathbf{w}_q} := (\Lambda^{\mathbf{w}_q}_q)^\alpha(X_q)$.

The *canonical k-tangent structure* on $\oplus^k \mathrm{T}Q$ is the set $(J^1, \dots, J^k)$ of tensor fields of type $(1, 1)$ in $\oplus^k \mathrm{T}Q$ defined as

$$J^\alpha_{\mathbf{w}_q} := (\Lambda^{\mathbf{w}_q}_q)^\alpha \circ \mathrm{T}_{\mathbf{w}_q}\tau^1_Q.$$

In natural coordinates we have

$$J^\alpha = \frac{\partial}{\partial v^i_\alpha} \otimes dq^i.$$

The *Liouville vector field* $\Delta \in \mathfrak{X}(\oplus^k \mathrm{T}Q)$ is the infinitesimal generator of the flow $\psi \colon \mathbb{R} \times \oplus^k \mathrm{T}Q \longrightarrow \oplus^k \mathrm{T}Q$, given by $\psi(t; v_{1q}, \dots, v_{kq}) = (e^t v_{1q}, \dots, e^t v_{kq})$. Observe that $\Delta = \Delta_1 + \dots + \Delta_k$, where each $\Delta_\alpha \in \mathfrak{X}(\oplus^k \mathrm{T}Q)$ is the infinitesimal generator of the flow $\psi^\alpha \colon \mathbb{R} \times \oplus^k \mathrm{T}Q \longrightarrow \oplus^k \mathrm{T}Q$,

$$\psi^\alpha(s; v_{1q}, \dots, v_{kq}) = (v_{1q}, \dots, v_{(\alpha-1)q}, e^t v_{\alpha q}, v_{(\alpha+1)q}, \dots, v_{kq}).$$

In coordinates, $\Delta = v^i_\alpha \dfrac{\partial}{\partial v^i_\alpha}$.

Given a map $\Phi\colon M \to N$, there exists a natural extension $\oplus^k \mathrm{T}\Phi\colon \oplus^k \mathrm{T}M \to \oplus^k \mathrm{T}N$, defined by

$$\oplus^k \mathrm{T}\Phi(v_{1q}, \dots, v_{kq}) := (\mathrm{T}_q\Phi(v_{1q}), \dots, \mathrm{T}_q\Phi(v_{kq})).$$

By definition, a k-vector field $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ in $\oplus^k \mathrm{T}Q$ is a section of the projection

$$\tau^1_{\oplus^k \mathrm{T}Q}\colon \mathrm{T}(\oplus^k \mathrm{T}Q) \oplus \overset{k}{\dots} \oplus \mathrm{T}(\oplus^k \mathrm{T}Q) \to \oplus^k \mathrm{T}Q.$$

Then, we say that Γ is a *second-order partial differential equation (*SOPDE*)* if it is also a section of the projection

$$\oplus^k \mathrm{T}\tau^1_Q\colon \mathrm{T}(\oplus^k \mathrm{T}Q) \oplus \overset{k}{\dots} \oplus \mathrm{T}(\oplus^k \mathrm{T}Q) \to \oplus^k \mathrm{T}Q;$$

that is, $\oplus^k \mathrm{T}\tau^1_Q \circ \Gamma = \mathrm{Id}_{\oplus^k \mathrm{T}Q} = \tau^1_{\oplus^k \mathrm{T}Q} \circ \Gamma$. Notice that a k-vector field Γ in $\oplus^k \mathrm{T}Q$ is a SOPDE if, and only if, $J^\alpha(\Gamma_\alpha) = \Delta$.

In addition, an integrable k-vector field $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ in $\oplus^k \mathrm{T}Q$ is a SOPDE if, and only if, its integrable maps are holonomic.

In natural coordinates, the expression of the components of a SOPDE is $\Gamma_\alpha = v^i_\alpha \dfrac{\partial}{\partial q^i} + \Gamma^i_{\alpha\beta} \dfrac{\partial}{\partial v^i_\beta}$. Then, if $\psi\colon \mathbb{R}^k \to \oplus^k \mathrm{T}Q$, locally given by $\psi(t) = (\psi^i(t), \psi^i_\beta(t))$, is an integral map of an integrable SOPDE, from (1) we have that

$$\left.\frac{\partial \psi^i}{\partial t^\alpha}\right|_t = \psi^i_\alpha(t), \qquad \left.\frac{\partial \psi^i_\beta}{\partial t^\alpha}\right|_t = \Gamma^i_{\alpha\beta}(\psi(t)) \,.$$

Furthermore, $\psi = \phi'$, where ϕ' is the first prolongation of the map $\phi = \tau \circ \psi\colon \mathbb{R}^k \xrightarrow{\psi} \oplus^k \mathrm{T}Q \xrightarrow{\tau} Q$, and hence ϕ is a solution to the system of second-order partial differential equations

$$\frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta}(t) = \Gamma^i_{\alpha\beta}\left(\phi^i(t), \frac{\partial \phi^i}{\partial t^\gamma}(t)\right). \tag{2}$$

Observe that, from (2) we obtain that, if Γ is an integrable SOPDE, then $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$.

2.2. k-symplectic manifolds

(See [1, 10–12, 29] for more details).

Let M be a manifold of dimension $N = n + kn$. A *k-symplectic structure* on M is a family $(\omega^1, \dots, \omega^k; V)$, where ω^α $(\alpha = 1, \dots, k)$ are closed 2-forms, and V is an integrable nk-dimensional tangent distribution on M such that

$$\text{(i)}\ \ \omega^\alpha|_{V\times V} = 0 \ \text{(for every } \alpha), \qquad \text{(ii)}\ \bigcap_{\alpha=1}^{k} \ker \omega^\alpha = \{0\} \,.$$

Then (M, ω^α, V) is called a *k-symplectic manifold.*

For every point of M there exist a neighbourhood U and local coordinates (q^i, p^α_i) $(1 \leq i \leq n,\ 1 \leq \alpha \leq k)$ such that, on U,

$$\omega^\alpha = dq^i \wedge dp^\alpha_i\ , \quad V = \left\langle \frac{\partial}{\partial p^1_i}, \ldots, \frac{\partial}{\partial p^k_i} \right\rangle .$$

These are the so-called *Darboux* or *canonical coordinates* of the k-symplectic manifold [1].

The canonical model for k-symplectic manifolds is $\oplus^k \mathrm{T}^*Q = \mathrm{T}^*Q \oplus \overset{k}{\ldots} \oplus \mathrm{T}^*Q$, with natural projections

$$\pi^\alpha \colon \oplus^k \mathrm{T}^*Q \to \mathrm{T}^*Q\ , \quad \pi^1_Q \colon \oplus^k \mathrm{T}^*Q \to Q\ .$$

As in the case of the cotangent bundle, local coordinates (q^i) in $U \subset Q$ induce natural coordinates (q^i, p^α_i) in $(\pi^1_Q)^{-1}(U)$. If θ and $\omega = -d\theta$ are the canonical forms of T^*Q, then $\oplus^k \mathrm{T}^*Q$ is endowed with the canonical forms

$$\theta^\alpha = (\pi^\alpha)^*\theta, \quad \omega^\alpha = (\pi^\alpha)^*\omega = -(\pi^\alpha)^* d\theta = -d\theta^\alpha, \tag{3}$$

and in natural coordinates we have that $\theta^\alpha = p^\alpha_i dq^i$ and $\omega^\alpha = dq^i \wedge dp^\alpha_i$. Thus, the triple $(\oplus^k \mathrm{T}^*Q, \omega^\alpha, V)$, where $V = \ker \mathrm{T}\pi^1_Q$, is a k-symplectic manifold, and the natural coordinates in $\oplus^k \mathrm{T}^*Q$ are Darboux coordinates.

2.3. k-contact structures

The definition of k-contact structure has been recently introduced in [15], where the reader can find more details.

Remember that, if M is a smooth manifold of dimension m, a (generalized) distribution on M is a subset $D \subset \mathrm{T}M$ such that, for every $x \in M$, $D_x \subset \mathrm{T}_x M$ is a vector subspace. The distribution D is smooth when it can be locally spanned by a family of smooth vector fields, and is regular when it is smooth and has locally constant rank. A codistribution on M is a subset $C \subset \mathrm{T}^*M$ with similar properties. The annihilator D° of a distribution D is a codistribution.

A (smooth) differential 1-form $\eta \in \Omega^1(M)$ generates a smooth codistribution that we denote by $\langle \eta \rangle \subset \mathrm{T}^*M$; it has rank 1 at every point where η does not vanish. Its annihilator is a distribution $\langle \eta \rangle^\circ \subset \mathrm{T}M$; it can be described also as the kernel of the vector bundle morphism $\widehat{\eta} \colon \mathrm{T}M \to M \times \mathbb{R}$ defined by η. This distribution has corank 1 at every point where η does not vanish.

Now, given k differential 1-forms $\eta^1, \ldots, \eta^k \in \Omega^1(M)$, let:

$$\begin{aligned}
&\mathcal{C}^{\mathrm{C}} = \langle \eta^1, \ldots, \eta^k \rangle \subset \mathrm{T}^*M, \\
&\mathcal{D}^{\mathrm{C}} = \left(\mathcal{C}^{\mathrm{C}}\right)^\circ = \ker \widehat{\eta^1} \cap \ldots \cap \ker \widehat{\eta^k} \subset \mathrm{T}M, \\
&\mathcal{D}^{\mathrm{R}} = \ker \widehat{d\eta^1} \cap \ldots \cap \ker \widehat{d\eta^k} \subset \mathrm{T}M, \\
&\mathcal{C}^{\mathrm{R}} = \left(\mathcal{D}^{\mathrm{R}}\right)^\circ \subset \mathrm{T}^*M.
\end{aligned}$$

DEFINITION 1. A *k-contact structure* on M is a family of k differential 1-forms $\eta^\alpha \in \Omega^1(M)$ such that, with the preceding notations,

(i) $\mathcal{D}^{\mathrm{C}} \subset \mathrm{T}M$ is a regular distribution of corank k; or, what is equivalent, $\eta^1 \wedge \ldots \wedge \eta^k \neq 0$, at every point.
(ii) $\mathcal{D}^{\mathrm{R}} \subset \mathrm{T}M$ is a regular distribution of rank k.
(iii) $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}} = \{0\}$ or, what is equivalent, $\displaystyle\bigcap_{\alpha=1}^{k} \left(\ker \widehat{\eta^\alpha} \cap \ker \widehat{d\eta^\alpha}\right) = \{0\}$.

We call $\mathcal{C}^{\mathrm{C}}$ the *contact codistribution*; $\mathcal{D}^{\mathrm{C}}$ the *contact distribution*; $\mathcal{D}^{\mathrm{R}}$ the *Reeb distribution*; and $\mathcal{C}^{\mathrm{R}}$ the *Reeb codistribution*.
A *k-contact manifold* is a manifold endowed with a k-contact structure.

REMARK 1. If conditions (i) and (ii) hold, then (iii) is equivalent to

(iii′) $\mathrm{T}M = \mathcal{D}^{\mathrm{C}} \oplus \mathcal{D}^{\mathrm{R}}$.

For $k = 1$ we recover the definition of contact structure.

THEOREM 1. *Let (M, η^α) be a k-contact manifold.*

1. *The Reeb distribution $\mathcal{D}^{\mathrm{R}}$ is involutive, and therefore integrable.*
2. *There exist k vector fields $\mathcal{R}_\alpha \in \mathfrak{X}(M)$, the* Reeb vector fields, *uniquely defined by the relations*

$$i(\mathcal{R}_\beta)\eta^\alpha = \delta^\alpha_\beta, \quad i(\mathcal{R}_\beta)d\eta^\alpha = 0. \tag{4}$$

3. *The Reeb vector fields commute, $[\mathcal{R}_\alpha, \mathcal{R}_\beta] = 0$, and they generate $\mathcal{D}^{\mathrm{R}}$.*

There are coordinates $(x^I; s^\alpha)$ such that

$$\mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha}, \quad \eta^\alpha = ds^\alpha - f^\alpha_I(x)\, dx^I,$$

where $f^\alpha_I(x)$ are functions depending only on the x^I; we call them *adapted coordinates* (to the k-contact structure).

EXAMPLE 1. Given $k \geq 1$, the manifold $(\oplus^k \mathrm{T}^*Q) \times \mathbb{R}^k$ has a canonical k-contact structure defined by the 1-forms

$$\eta^\alpha = ds^\alpha - \theta^\alpha,$$

where s^α is the α-th Cartesian coordinate of $\mathbb{R}^k$, and θ^α is the pull-back of the canonical 1-form of T^*Q with respect to the projection $(\oplus^k \mathrm{T}^*Q) \times \mathbb{R}^k \to \mathrm{T}^*Q$ to the α-th direct summand. Using coordinates q^i on Q and natural coordinates (q^i, p^α_i) on each T^*Q, their local expressions are

$$\eta^\alpha = ds^\alpha - p^\alpha_i\, dq^i,$$

from which $d\eta^\alpha = dq^i \wedge dp^\alpha_i$, and the Reeb vector fields are

$$\mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha}.$$

The following result ensures the existence of canonical coordinates for a particular kind of k-contact manifolds.

THEOREM 2 (k-contact Darboux theorem). *Let (M, η^α) be a k-contact manifold of dimension $n+kn+k$ such that there exists an integrable subdistribution $\mathcal{V}$ of $\mathcal{D}^{\mathrm{C}}$ with* $\operatorname{rank}\mathcal{V} = nk$. *Around every point of M, there exists a local chart of coordinates $(U; q^i, p_i^\alpha, s^\alpha)$, $1 \le \alpha \le k$, $1 \le i \le n$, such that*

$$\eta^\alpha|_U = ds^\alpha - p_i^\alpha\, dq^i \ .$$

In these coordinates,

$$\mathcal{D}^{\mathrm{R}}|_U = \left\langle \mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha} \right\rangle, \quad \mathcal{V}|_U = \left\langle \frac{\partial}{\partial p_i^\alpha} \right\rangle.$$

These are the so-called canonical *or* Darboux coordinates *of the k-contact manifold.*

This theorem allows us to consider the manifold presented in Example 1 as the canonical model for these kinds of k-contact manifolds.

2.4. k-contact Hamiltonian systems

Together with k-contact structures, k-contact Hamiltonian systems have also been defined in [15].

A *k-contact Hamiltonian system* is a family $(M, \eta^\alpha, \mathcal{H})$, where (M, η^α) is a k-contact manifold, and $\mathcal{H} \in \mathscr{C}^\infty(M)$ is called a *Hamiltonian function*. The *k-contact Hamilton–de Donder–Weyl equations* for a map $\psi \colon D \subset \mathbb{R}^k \to M$ are

$$\begin{cases} i(\psi'_\alpha)d\eta^\alpha = \big(d\mathcal{H} - (\mathscr{L}_{\mathcal{R}_\alpha}\mathcal{H})\eta^\alpha\big) \circ \psi, \\ i(\psi'_\alpha)\eta^\alpha = -\mathcal{H} \circ \psi. \end{cases} \tag{5}$$

The *k-contact Hamilton–de Donder–Weyl equations* for a k-vector field $\mathbf{X} = (X_1, \dots, X_k)$ in M are

$$\begin{cases} i(X_\alpha)d\eta^\alpha = d\mathcal{H} - (\mathscr{L}_{\mathcal{R}_\alpha}\mathcal{H})\eta^\alpha, \\ i(X_\alpha)\eta^\alpha = -\mathcal{H}. \end{cases} \tag{6}$$

Their solutions are called *Hamiltonian k-vector fields*. These equations are equivalent to

$$\begin{cases} \mathscr{L}_{X_\alpha}\eta^\alpha = -(\mathscr{L}_{\mathcal{R}_\alpha}\mathcal{H})\eta^\alpha, \\ i(X_\alpha)\eta^\alpha = -\mathcal{H}. \end{cases} \tag{7}$$

Solutions to these equations always exist, although they are neither unique, nor necessarily integrable.

If $\mathbf{X}$ is an *integrable* k-vector field in M, then every integral map $\psi \colon D \subset \mathbb{R}^k \to M$ of $\mathbf{X}$ satisfies the k-contact equation (5) if, and only if, $\mathbf{X}$ is a solution to (6). Notice, however, that equations (5) and (6) are not, in general, fully equivalent, since a solution to (5) may not be an integral map of some integrable k-vector field in M solution to (6).

An alternative, partially equivalent, expression for the Hamilton–De Donder–Weyl equations, which does not use the Reeb vector fields $\mathcal{R}_\alpha$, can be given as follows. Consider the 2-forms $\Omega^\alpha = -\mathcal{H}\, d\eta^\alpha + d\mathcal{H} \wedge \eta^\alpha$. On the open set $\mathcal{O} = \{p \in M \mid \mathcal{H}(p) \neq 0\}$, if a k-vector field $\mathbf{X} = (X_\alpha)$ satisfies

$$\begin{cases} i(X_\alpha)\Omega^\alpha = 0, \\ i(X_\alpha)\eta^\alpha = -\mathcal{H}, \end{cases} \tag{8}$$

then $\mathbf{X}$ is a solution of the Hamilton–De Donder–Weyl equations (6)). Any integral map ψ of such a k-vector field is a solution to

$$\begin{cases} i(\psi'_\alpha)\Omega^\alpha = 0\,, \\ i(\psi'_\alpha)\eta^\alpha = -\mathcal{H} \circ \psi\,. \end{cases} \tag{9}$$

REMARK 2. If the family (M, η^α) does not hold some of the conditions of Definition 1, then (M, η^α) is called a *k-precontact manifold* and $(M, \eta^\alpha, \mathcal{H})$ is said to be a *k-precontact Hamiltonian system*. In this case, the Reeb vector fields are not uniquely defined. However, as it happens in other similar situations (precosymplectic mechanics, k-precosymplectic field theories or precontact mechanics) [9, 23], it could be proved that Eqs. (5) and (6) do not depend on the used Reeb vector fields and, thus, the equations are still valid.

In canonical coordinates, if $\psi(t) = (q^i(t), p_i^\alpha(t), s^\alpha(t))$, then its first prolongation has components

$$\psi'_\beta = \left(q^i, p_i^\alpha, s^\alpha, \frac{\partial q^i}{\partial t^\beta}, \frac{\partial p_i^\alpha}{\partial t^\beta}, \frac{\partial s^\alpha}{\partial t^\beta}\right),$$

and the k-contact Hamilton–de Donder–Weyl equations read

$$\begin{cases} \dfrac{\partial q^i}{\partial t^\alpha} = \dfrac{\partial \mathcal{H}}{\partial p_i^\alpha} \circ \psi, \\[2ex] \dfrac{\partial p_i^\alpha}{\partial t^\alpha} = -\left(\dfrac{\partial \mathcal{H}}{\partial q^i} + p_i^\alpha \dfrac{\partial \mathcal{H}}{\partial s^\alpha}\right) \circ \psi, \\[2ex] \dfrac{\partial s^\alpha}{\partial t^\alpha} = \left(p_i^\alpha \dfrac{\partial \mathcal{H}}{\partial p_i^\alpha} - \mathcal{H}\right) \circ \psi. \end{cases} \tag{10}$$

If $\mathbf{X} = (X_\alpha)$ is a k-vector field solution to (8) and in canonical coordinates we have that

$$X_\alpha = X_\alpha^\beta \frac{\partial}{\partial s^\beta} + X_\alpha^i \frac{\partial}{\partial q^i} + X_{\alpha i}^\beta \frac{\partial}{\partial p_i^\beta},$$

then

$$\begin{cases} X^i_\alpha = \dfrac{\partial \mathcal{H}}{\partial p^\alpha_i}, \\ X^\alpha_{\alpha i} = -\left(\dfrac{\partial \mathcal{H}}{\partial q^i} + p^\alpha_i \dfrac{\partial \mathcal{H}}{\partial s^\alpha} \right), \\ X^\alpha_\alpha = p^\alpha_i \dfrac{\partial \mathcal{H}}{\partial p^\alpha_i} - \mathcal{H}, \end{cases} \tag{11}$$

3. k-contact Lagrangian field theory

3.1. k-contact Lagrangian systems

Using the geometric framework introduced in Section 2.1, we are ready to deal with Lagrangian systems with dissipation in field theories. First we need to enlarge the bundle in order to include the dissipation variables. Then, consider the bundle $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ with canonical projections

$$\bar{\tau}_1 \colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}Q, \quad \bar{\tau}^k \colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathrm{T}Q, \quad s^\alpha \colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}.$$

Natural coordinates in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ are $(q^i, v^i_\alpha, s^\alpha)$.

As $\oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}Q$ is a trivial bundle, the canonical structures in $\oplus^k \mathrm{T}Q$ (the canonical k-tangent structure and the Liouville vector field described above) can be extended to $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ in a natural way, and are denoted with the same notation (J^α) and Δ. Then, using these structures, we can extend also the concept of SOPDE k-vector fields to $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ as follows.

DEFINITION 2. A k-vector field $\mathbf{\Gamma} = (\Gamma_\alpha)$ in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ is a *second-order partial differential equation* (SOPDE) if $J^\alpha(\Gamma_\alpha) = \Delta$.

The local expression of a SOPDE is

$$\Gamma_\alpha = v^i_\alpha \frac{\partial}{\partial q^i} + \Gamma^i_{\alpha\beta} \frac{\partial}{\partial v^i_\beta} + g^\beta_\alpha \frac{\partial}{\partial s^\beta}. \tag{12}$$

DEFINITION 3. Let $\psi \colon \mathbb{R}^k \to Q \times \mathbb{R}^k$ be a section of the projection $Q \times \mathbb{R}^k \to \mathbb{R}^k$; with $\psi = (\phi, s^\alpha)$, where $\phi \colon \mathbb{R}^k \to Q$. The *first prolongation* of ψ to $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ is the map $\boldsymbol{\sigma} \colon \mathbb{R}^k \to \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ given by $\boldsymbol{\sigma} = (\phi', s^\alpha)$. The map $\boldsymbol{\sigma}$ is said to be *holonomic*.

The following property is a straightforward consequence of the above definitions and the results about SOPDE in the bundle $\oplus^k \mathrm{T}Q$ given in Section 2.1:

PROPOSITION 1. *A k-vector field $\mathbf{\Gamma}$ in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ is a* SOPDE *if, and only if, its integral maps are holonomic.*

Now we can state the Lagrangian formalism of field theories with dissipation.

DEFINITION 4. A *Lagrangian function* is a function $\mathcal{L} \in \mathscr{C}^\infty(\oplus^k \mathrm{T}Q \times \mathbb{R}^k)$. The *Lagrangian energy* associated with $\mathcal{L}$ is the function defined by $E_\mathcal{L} := \Delta(\mathcal{L}) - \mathcal{L} \in \mathscr{C}^\infty(\oplus^k \mathrm{T}Q \times \mathbb{R}^k)$.
The *Cartan forms* associated with $\mathcal{L}$ are

$$\theta_\mathcal{L}^\alpha = {}^t(J^\alpha) \circ d\mathcal{L} \in \Omega^1(\oplus^k \mathrm{T}Q \times \mathbb{R}^k), \quad \omega_\mathcal{L}^\alpha = -d\theta_\mathcal{L}^\alpha \in \Omega^2(\oplus^k \mathrm{T}Q \times \mathbb{R}^k).$$

Finally, we can define the forms

$$\eta_\mathcal{L}^\alpha = ds^\alpha - \theta_\mathcal{L}^\alpha \in \Omega^1(\oplus^k \mathrm{T}Q \times \mathbb{R}^k), \quad d\eta_\mathcal{L}^\alpha = \omega_\mathcal{L}^\alpha \in \Omega^2(\oplus^k \mathrm{T}Q \times \mathbb{R}^k).$$

The couple $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$ is said to be a *k-contact Lagrangian system.*

In natural coordinates $(q^i, v_\alpha^i, s^\alpha)$ of $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$, the local expressions of these elements are

$$E_\mathcal{L} = v_\alpha^i \frac{\partial \mathcal{L}}{\partial v_\alpha^i} - \mathcal{L}, \quad \eta_\mathcal{L}^\alpha = ds^\alpha - \frac{\partial \mathcal{L}}{\partial v_\alpha^i} dq^i.$$

Before introducing the Legendre map, remember that, given a bundle map $f \colon E \to F$ between two vector bundles over a manifold B, the fibre derivative of f is the map $\mathcal{F}f \colon E \to \mathrm{Hom}(E_\mathcal{L}, F) \approx F \otimes E^*$ obtained by restricting f to the fibres, $f_b \colon E_b \to F_b$, and computing the usual derivative of a map between two vector spaces: $\mathcal{F}f(e_b) = \mathrm{D}f_b(e_b)$. This applies in particular when the second vector bundle is trivial of rank 1, that is, for a function $f \colon E \to \mathbb{R}$; then $\mathcal{F}f \colon E \to E^*$. This map also has a fibre derivative $\mathcal{F}^2 f \colon E \to E^* \otimes E^*$, which is usually called the fibre Hessian of f. For every $e_b \in E$, $\mathcal{F}^2 f(e_b)$ can be considered as a symmetric bilinear form on E_b. It is easy to check that $\mathcal{F}f$ is a local diffeomorphism at a point $e \in E$ if, and only if, the Hessian $\mathcal{F}^2 f(e)$ is nondegenerate (see [21] for details).

DEFINITION 5. The *Legendre map* associated with a Lagrangian $\mathcal{L} \in \mathscr{C}^\infty(\oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ is the fibre derivative of $\mathcal{L}$, considered as a function on the vector bundle $\oplus^k \mathrm{T}Q \times \mathbb{R}^k \to Q \times \mathbb{R}^k$; that is, the map $\mathcal{FL} \colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ given by

$$\mathcal{FL}(v_{1q}, \dots, v_{kq}; s^\alpha) = \big(\mathcal{FL}(\cdot, s^\alpha)(v_{1q}, \dots, v_{kq}), s^\alpha\big) \; ; \quad (v_{1q}, \dots, v_{kq}) \in \oplus^k \mathrm{T}Q,$$

where $\mathcal{L}(\cdot, s^\alpha)$ denotes the Lagrangian with s^α freezed.

This map is locally given by $\mathcal{FL}(q^i, v_\alpha^i, s^\alpha) = \Big(q^i, \dfrac{\partial \mathcal{L}}{\partial v_\alpha^i}, s^\alpha\Big)$.

REMARK 3. The Cartan forms can also be defined as

$$\theta_\mathcal{L}^\alpha = \mathcal{FL}^* \theta^\alpha, \quad \omega_\mathcal{L}^\alpha = \mathcal{FL}^* \omega^\alpha,$$

where θ^α and ω^α are given in (3).

PROPOSITION 2. *For a Lagrangian function $\mathcal{L}$ the following conditions are equivalent:*

1. *The Legendre map $\mathcal{F}\mathcal{L}$ is a local diffeomorphism.*
2. *The fibre Hessian $\mathcal{F}^2\mathcal{L}\colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \longrightarrow (\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k) \otimes (\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k)$ of $\mathcal{L}$ is everywhere nondegenerate. (The tensor product is of vector bundles over $Q \times \mathbb{R}^k$.)*
3. *$(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \eta^\alpha_{\mathcal{L}})$ is a k-contact manifold.*

Proof: The proof can be easily done using natural coordinates, bearing in mind that

$$F\mathcal{L}(q^i, v^i_\alpha, s^\alpha) = \left(q^i, \frac{\partial \mathcal{L}}{\partial v^i_\alpha}, s^\alpha\right),$$

$$\mathcal{F}^2\mathcal{L}(q^i, v^i_\alpha, s^\alpha) = (q^i, W^{\alpha\beta}_{ij}, s^\alpha), \text{ with } W^{\alpha\beta}_{ij} = \left(\frac{\partial^2 \mathcal{L}}{\partial v^i_\alpha \partial v^j_\beta}\right).$$

Then the conditions in the proposition mean that the matrix $W = (W^{\alpha\beta}_{ij})$ is everywhere nonsingular. □

DEFINITION 6. A Lagrangian function $\mathcal{L}$ is said to be *regular* if the equivalent conditions in Proposition 2 hold. Otherwise $\mathcal{L}$ is called a *singular* Lagrangian. In particular, $\mathcal{L}$ is said to be *hyperregular* if $\mathcal{F}\mathcal{L}$ is a global diffeomorphism.

Given a regular k-contact Lagrangian system $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$, from (4) we have that the *Reeb vector fields* $(\mathcal{R}_\mathcal{L})_\alpha \in \mathfrak{X}(\oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ for this system are the unique solution to

$$i((\mathcal{R}_\mathcal{L})_\alpha) d\eta^\beta_\mathcal{L} = 0, \quad i((\mathcal{R}_\mathcal{L})_\alpha)\eta^\beta_\mathcal{L} = \delta^\beta_\alpha.$$

If $\mathcal{L}$ is regular, there exists the inverse $W^{ij}_{\alpha\beta}$ of the Hessian matrix, namely $W^{ij}_{\alpha\beta} \dfrac{\partial^2 \mathcal{L}}{\partial v^j_\beta \partial v^k_\gamma} = \delta^i_k \delta^\gamma_\alpha$, and then a simple calculation in coordinates leads to

$$(\mathcal{R}_\mathcal{L})_\alpha = \frac{\partial}{\partial s^\alpha} - W^{ji}_{\gamma\beta} \frac{\partial^2 \mathcal{L}}{\partial s^\alpha \partial v^j_\gamma} \frac{\partial}{\partial v^i_\beta}.$$

3.2. The k-contact Euler–Lagrange equations

As a result of the preceding definitions and results, every *regular* k-contact Lagrangian system has associated the k-contact Hamiltonian system $(\oplus^k \mathrm{T}Q \times \mathbb{R}, \eta^\alpha_\mathcal{L}, E_\mathcal{L})$.

DEFINITION 7. Let $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$ be a k-contact Lagrangian system. The *k-contact Euler–Lagrange equations* for a holonomic maps $\boldsymbol{\sigma}\colon \mathbb{R}^k \to \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ are

$$\begin{cases} i(\boldsymbol{\sigma}'_\alpha) d\eta^\alpha_\mathcal{L} = \Big(dE_\mathcal{L} - (\mathscr{L}_{(\mathcal{R}_\mathcal{L})_\alpha} E_\mathcal{L})\eta^\alpha_\mathcal{L}\Big) \circ \boldsymbol{\sigma}, \\ i(\boldsymbol{\sigma}'_\alpha)\eta^\alpha_\mathcal{L} = -E_\mathcal{L} \circ \boldsymbol{\sigma}. \end{cases} \tag{13}$$

The *k-contact Lagrangian equations* for a k-vector field $\mathbf{X}_{\mathcal{L}} = ((X_{\mathcal{L}})_\alpha)$ in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ are

$$\begin{cases} i((X_{\mathcal{L}})_\alpha) d\eta^\alpha_{\mathcal{L}} = dE_{\mathcal{L}} - (\mathscr{L}_{(\mathcal{R}_{\mathcal{L}})_\alpha} E_{\mathcal{L}})\eta^\alpha_{\mathcal{L}}, \\ i((X_{\mathcal{L}})_\alpha)\eta^\alpha_{\mathcal{L}} = -E_{\mathcal{L}}. \end{cases} \tag{14}$$

A k-vector field which is solution to these equations is called a *Lagrangian k-vector field*.

A first relevant result is as follows.

PROPOSITION 3. *Let* $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$ *be a k-contact regular Lagrangian system. Then, the k-contact Euler–Lagrange equations* (14) *admit solutions. They are not unique if* $k > 1$.

Proof: The proof is the same as that of Proposition 4.3 in [15]. □

In a natural chart of coordinates of $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$, Eqs. (13) read

$$\frac{\partial}{\partial t^\alpha}\left(\frac{\partial \mathcal{L}}{\partial v^i_\alpha} \circ \boldsymbol{\sigma}\right) = \left(\frac{\partial \mathcal{L}}{\partial q^i} + \frac{\partial \mathcal{L}}{\partial s^\alpha}\frac{\partial \mathcal{L}}{\partial v^i_\alpha}\right) \circ \boldsymbol{\sigma}, \qquad \frac{\partial s^\alpha}{\partial t^\alpha} = \mathcal{L} \circ \boldsymbol{\sigma}, \tag{15}$$

meanwhile, for a k-vector field $\mathbf{X}_{\mathcal{L}} = ((X_{\mathcal{L}})_\alpha)$ with $(X_{\mathcal{L}})_\alpha = (X_{\mathcal{L}})^i_\alpha \dfrac{\partial}{\partial q^i} + (X_{\mathcal{L}})^i_{\alpha\beta}\dfrac{\partial}{\partial v^i_\beta} + (X_{\mathcal{L}})^\beta_\alpha \dfrac{\partial}{\partial s^\beta}$, the Lagrangian equations (14) are

$$0 = \left((X_{\mathcal{L}})^j_\alpha - v^j_\alpha\right)\frac{\partial^2 \mathcal{L}}{\partial v^j_\alpha \partial s^\beta}, \tag{16}$$

$$0 = \left((X_{\mathcal{L}})^j_\alpha - v^j_\alpha\right)\frac{\partial^2 \mathcal{L}}{\partial v^i_\beta \partial v^j_\alpha}, \tag{17}$$

$$\begin{aligned} 0 = &\left((X_{\mathcal{L}})^j_\alpha - v^j_\alpha\right)\frac{\partial^2 \mathcal{L}}{\partial q^i \partial v^j_\alpha} + \frac{\partial \mathcal{L}}{\partial q^i} - \frac{\partial^2 \mathcal{L}}{\partial s^\beta \partial v^i_\alpha}(X_{\mathcal{L}})^\beta_\alpha \\ &- \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i_\alpha}(X_{\mathcal{L}})^j_\alpha - \frac{\partial^2 \mathcal{L}}{\partial v^j_\beta \partial v^i_\alpha}(X_{\mathcal{L}})^j_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial s^\alpha}\frac{\partial \mathcal{L}}{\partial v^i_\alpha}, \end{aligned} \tag{18}$$

$$0 = \mathcal{L} + \frac{\partial \mathcal{L}}{\partial v^i_\alpha}\left((X_{\mathcal{L}})^i_\alpha - v^i_\alpha\right) - (X_{\mathcal{L}})^\alpha_\alpha. \tag{19}$$

If $\mathcal{L}$ is a regular Lagrangian, equations (17) lead to $v^i_\alpha = (X_{\mathcal{L}})^i_\alpha$, which are the SOPDE condition for the k-vector field $\mathbf{X}$. Then, (16) holds identically, and (19) and (18) give

$$(X_{\mathcal{L}})^\alpha_\alpha = \mathcal{L},$$

$$-\frac{\partial \mathcal{L}}{\partial q^i} + \frac{\partial^2 \mathcal{L}}{\partial s^\beta \partial v^i_\alpha}(X_{\mathcal{L}})^\beta_\alpha + \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i_\alpha} v^j_\alpha + \frac{\partial^2 \mathcal{L}}{\partial v^j_\beta \partial v^i_\alpha}(X_{\mathcal{L}})^j_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial s^\alpha}\frac{\partial \mathcal{L}}{\partial v^i_\alpha}.$$

Notice that, if this SOPDE $\mathbf{X}_{\mathcal{L}}$ is integrable, these last equations are the Euler–Lagrange equations (15) for its integral maps. In this way, we have proved the following result.

PROPOSITION 4. *If $\mathcal{L}$ is a regular Lagrangian, then the corresponding Lagrangian k-vector fields $\mathbf{X}_{\mathcal{L}}$ (solutions to the k-contact Lagrangian equations* (14)*) are* SOPDE*'s and if, in addition, $\mathbf{X}_{\mathcal{L}}$ is integrable, then its integral maps are solutions to the k-contact Euler–Lagrange field equations* (13).

This SOPDE $\mathbf{X}_{\mathcal{L}} \equiv \mathbf{\Gamma}_{\mathcal{L}}$ *is called the* Euler–Lagrange k-vector field *associated with the Lagrangian function $\mathcal{L}$.*

REMARK 4. It is interesting to point out how, in the Lagrangian formalism of dissipative field theories, the second equation in (15) relates the variation of the "dissipation coordinates" s^α to the Lagrangian function.

REMARK 5. If $\mathcal{L}$ is not regular then $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \eta^\alpha_{\mathcal{L}}, E_{\mathcal{L}})$ is a k-precontact system and, in general, Eqs. (13) and (14) have no solutions everywhere in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ but, in the most favourable situations, they do in a submanifold of $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$ which is obtained by applying a suitable constraint algorithm. Nevertheless, solutions to Eqs. (14) are not necessarily SOPDE (unless it is required as the additional condition $J^\alpha(X_\alpha) = \Delta$) and, as a consequence, if they are integrable, their integral maps are not necessarily holonomic.

REMARK 6. Observe that the particular case $k = 1$ gives the Lagrangian formalism for mechanical systems with dissipation [9, 16].

3.3. k-contact canonical Hamiltonian formalism

In the regular or the hyper-regular cases we have that $\mathcal{FL}$ is a (local) diffeomorphism between $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \eta^\alpha_{\mathcal{L}})$ and $(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, \eta^\alpha)$, where $\mathcal{FL}^*\eta^\alpha = \eta^\alpha_{\mathcal{L}}$. Furthermore, there exists (maybe locally) a function $\mathcal{H} \in \mathscr{C}^\infty(\oplus^k \mathrm{T}^*Q \times \mathbb{R})$ such that $\mathcal{H} = E_{\mathcal{L}} \circ \mathcal{FL}^{-1}$; then we have the k-contact Hamiltonian system $(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, \eta^\alpha, \mathcal{H})$, for which $\mathcal{FL}_*(\mathcal{R}_{\mathcal{L}})_\alpha = \mathcal{R}_\alpha$. Therefore, if $\mathbf{\Gamma}_{\mathcal{L}}$ is an Euler–Lagrange k-vector field associated with $\mathcal{L}$ in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$, then $\mathcal{FL}_*\mathbf{\Gamma}_{\mathcal{L}} = \mathbf{X}_{\mathcal{H}}$ is a contact Hamiltonian k-vector field associated with $\mathcal{H}$ in $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$, and conversely.

For singular Lagrangians, following [19] we define such an object.

DEFINITION 8. A singular Lagrangian $\mathcal{L}$ is *almost-regular* if:

1. $\mathcal{P} := \mathcal{FL}(\oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ is a closed submanifold of $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$.
2. $\mathcal{FL}$ is a submersion onto its image.
3. The fibres $\mathcal{FL}^{-1}(p)$, for every $p \in \mathcal{P}$, are connected submanifolds of $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$.

If $\mathcal{L}$ is almost-regular and $\jmath_0 \colon \mathcal{P} \hookrightarrow \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ is the natural embedding, denoting by $\mathcal{FL}_0 \colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathcal{P}$ the restriction of $F\mathcal{L}$ given by $\jmath_0 \circ \mathcal{FL}_0 = \mathcal{FL}$;

then there exists $\mathcal{H}_0 \in \mathscr{C}^\infty(\mathcal{P})$ such that $(\mathcal{FL}_0)^*\mathcal{H}_0 = E_{\mathcal{L}}$. Furthermore, we can define $\eta_0^\alpha = \jmath_0^*\eta^\alpha$, and then, the triple $(\mathcal{P}, \eta_0^\alpha, \mathcal{H}_0)$ is the *k-precontact Hamiltonian system associated with* $\mathcal{L}$, and the corresponding Hamiltonian fields equations are (8) or (9) (in $\mathcal{P}$). In general, these equations have no solutions everywhere in $\mathcal{P}$ but, in the most favourable situations, they do in a submanifold $P_f \hookrightarrow \mathcal{P}$, which is obtained applying a suitable constraint algorithm, and where there are Hamiltonian k-vector fields in $\mathcal{P}$, tangent to P_f.

4. Symmetries and dissipated quantities in the Lagrangian formalism

As in [15], we introduce different concepts of symmetry of the system, depending on which structure is preserved, putting the emphasis on the transformations that leave the geometric structures invariant, or on the transformations that preserve the solutions of the system (see, for instance [22, 32]). In this way, the following definitions and properties are adapted from those stated for generic k-contact Hamiltonian systems to the case of a k-contact regular Lagrangian system $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$; that is, for the system $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \eta_{\mathcal{L}}^\alpha, E_{\mathcal{L}})$. The proofs of the results for the general case are given in [15].

4.1. Symmetries

DEFINITION 9. Let $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$ be a k-contact regular Lagrangian system.

- A *Lagrangian dynamical symmetry* is a diffeomorphism $\Phi\colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ such that, for every solution $\boldsymbol{\sigma}$ to the k-contact Euler–Lagrange equations (13), $\Phi \circ \boldsymbol{\sigma}$ is also a solution.
- An *infinitesimal Lagrangian dynamical symmetry* is a vector field $Y \in \mathfrak{X}(\oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ whose local flow is made of local symmetries.

The following results give characterizations of symmetries in terms of k-vector fields.

LEMMA 1. *Let $\Phi\colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ be a diffeomorphism and $\mathbf{X} = (X_1, \dots, X_k)$ a k-vector field in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$. If ψ is an integral map of $\mathbf{X}$, then $\Phi \circ \psi$ is an integral map of $\Phi_*\mathbf{X} = (\Phi_* X_\alpha)$. In particular, if $\mathbf{X}$ is integrable then $\Phi_*\mathbf{X}$ is also integrable.*

PROPOSITION 5. *If $\Phi\colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ is a Lagrangian dynamical symmetry then, for every integrable k-vector field $\mathbf{X}$ solution to the k-contact Lagrangian equations* (14)*, $\Phi_*\mathbf{X}$ is another solution.*

On the other side, if Φ transforms every k-vector field $\mathbf{X}_{\mathcal{L}}$ solution to the k-contact Lagrangian equations (14) *into another solution, then for every integral map ψ of $\mathbf{X}_{\mathcal{L}}$, we have that $\Phi \circ \psi$ is a solution to the k-contact Euler–Lagrange equations* (13).

Among the most relevant symmetries are those that leave the geometric structures invariant.

DEFINITION 10. A *Lagrangian k-contact symmetry* is a diffeomorphism $\Phi\colon \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ such that

$$\Phi^* \eta^\alpha_{\mathcal{L}} = \eta^\alpha_{\mathcal{L}}, \quad \Phi^* E_{\mathcal{L}} = E_{\mathcal{L}}.$$

An *infinitesimal Lagrangian k-contact symmetry* is a vector field $Y \in \mathfrak{X}(\oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ whose local flow is a Lagrangian k-contact symmetry; that is,

$$\mathscr{L}(Y)\eta^\alpha_{\mathcal{L}} = 0, \quad \mathscr{L}(Y)E_{\mathcal{L}} = 0.$$

PROPOSITION 6. *Every (infinitesimal) Lagrangian k-contact symmetry preserves the Reeb vector fields, that is* $\Phi_*(\mathcal{R}_{\mathcal{L}})_\alpha = (\mathcal{R}_{\mathcal{L}})_\alpha$ *(or* $[Y, (\mathcal{R}_{\mathcal{L}})_\alpha] = 0$*).*

And, as a consequence of these results, we obtain the relation between these kinds of symmetries.

PROPOSITION 7. *(Infinitesimal) Lagrangian k-contact symmetries are (infinitesimal) Lagrangian dynamical symmetries.*

4.2. Dissipation laws

DEFINITION 11. A map $F\colon M \to \mathbb{R}^k$, $F = (F^1, \dots, F^k)$, is said to satisfy:

1. The *dissipation law for maps* if, for every map $\boldsymbol{\sigma}$ solution to the k-contact Euler–Lagrange equations (13), the divergence of $F \circ \boldsymbol{\sigma} = (F^\alpha \circ \boldsymbol{\sigma})\colon \mathbb{R}^k \to \mathbb{R}^k$, which is defined as usual by $\operatorname{div}(F \circ \boldsymbol{\sigma}) = \partial(F^\alpha \circ \boldsymbol{\sigma})/\partial t^\alpha$, satisfies that
$$\operatorname{div}(F \circ \boldsymbol{\sigma}) = -\left[(\mathscr{L}_{(\mathcal{R}_{\mathcal{L}})_\alpha} E_{\mathcal{L}})F^\alpha\right] \circ \boldsymbol{\sigma}. \tag{20}$$
2. The *dissipation law for k-vector fields* if, for every k-vector field $\mathbf{X}_{\mathcal{L}}$ solution to the k-contact Lagrangian equations (14), the following equation holds.
$$\mathscr{L}_{(X_{\mathcal{L}})_\alpha} F^\alpha = -(\mathscr{L}_{(\mathcal{R}_{\mathcal{L}})_\alpha} E_{\mathcal{L}})F^\alpha. \tag{21}$$

Both concepts are partially related by the following property.

PROPOSITION 8. *If* $F = (F^\alpha)$ *satisfies the dissipation law for maps then, for every integrable k-vector field* $\mathbf{X}_{\mathcal{L}} = ((X_{\mathcal{L}})_\alpha)$ *which is a solution to the k-contact Lagrangian equations* (14)*, we have that Eq.* (21) *holds for* $\mathbf{X}_{\mathcal{L}}$*.*

On the other side, if (21) *holds for a k-vector field* $\mathbf{X}$*, then* (20) *holds for every integral map* ψ *of* $\mathbf{X}$*.*

PROPOSITION 9. *If Y is an infinitesimal dynamical symmetry then, for every solution* $\mathbf{X}_{\mathcal{L}} = ((X_{\mathcal{L}})_\alpha)$ *to the k-contact Lagrangian equations* (14)*, we have that*

$$i([Y, (X_{\mathcal{L}})_\alpha])\eta^\alpha_{\mathcal{L}} = 0, \quad i([Y, (X_{\mathcal{L}})_\alpha])d\eta^\alpha_{\mathcal{L}} = 0.$$

Finally, we have the following fundamental result which associates dissipated quantities with symmetries.

THEOREM 3. (Dissipation theorem). *If Y is an infinitesimal dynamical symmetry, then* $F^\alpha = -i(Y)\eta^\alpha_{\mathcal{L}}$ *satisfies the dissipation law for k-vector fields* (21).

4.3. Symmetries of the Lagrangian function

Consider a k-contact regular Lagrangian system $(\oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$.

First, remember that, if $\varphi\colon Q \to Q$ is a diffeomorphism, we can construct the diffeomorphism $\Phi := (\mathrm{T}^k\varphi, \mathrm{Id}_{\mathbb{R}^k})\colon \oplus^k \mathrm{T}Q\times\mathbb{R}^k \longrightarrow \oplus^k \mathrm{T}Q\times\mathbb{R}^k$, where $\mathrm{T}^k\varphi\colon \oplus^k \mathrm{T}Q \to \oplus^k \mathrm{T}Q$ denotes the canonical lifting of φ to $\oplus^k \mathrm{T}Q$. Then Φ is said to be the *canonical lifting* of φ to $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$. Any transformation Φ of this kind is called a *natural transformation* of $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$.

Moreover, given a vector field $Z \in \mathfrak{X}(Q)$ we can define its *complete lifting* to $\oplus^k \mathrm{T}Q\times\mathbb{R}^k$ as the vector field $Y \in \mathfrak{X}(\oplus^k \mathrm{T}Q\times\mathbb{R}^k)$ whose local flow is the canonical lifting of the local flow of Z to $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$; that is, the vector field $Y = Z^C$, where Z^C denotes the complete lifting of Z to $\oplus^k \mathrm{T}Q$, identified in a natural way as a vector field in $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$. Any infinitesimal transformation Y of this kind is called a *natural infinitesimal transformation* of $\oplus^k \mathrm{T}Q \times \mathbb{R}^k$.

It is well known that the canonical k-tangent structure (J^α) and the Liouville vector field Δ in $\oplus^k \mathrm{T}Q$ are invariant under the action of canonical liftings of diffeomorphisms and vector fields from Q to $\oplus^k \mathrm{T}Q$. Then, taking into account the definitions of the canonical k-tangent structure (J^α) and the Liouville vector field Δ in $\otimes^k \mathrm{T}Q \otimes \mathbb{R}^k$, it can be proved that canonical liftings of diffeomorphisms and vector fields from Q to $\otimes^k \mathrm{T}Q \otimes \mathbb{R}^k$ preserve these canonical structures as well as the Reeb vector fields $(\mathcal{R}_\mathcal{L})_\alpha$.

Therefore, as an immediate consequence, we obtain a relationship between Lagrangian-preserving natural transformations and contact symmetries.

PROPOSITION 10. *If* $\Phi \in \mathrm{Diff}(\otimes^k \mathrm{T}Q \otimes \mathbb{R}^k)$ *(resp.* $Y \in \mathfrak{X}(\otimes^k \mathrm{T}Q \otimes \mathbb{R}^k)$*) is a canonical lifting to* $\otimes^k \mathrm{T}Q \otimes \mathbb{R}^k$ *of a diffeomorphism* $\varphi \in \mathrm{Diff}(Q)$ *(resp. of a vector field* $Z \in \mathfrak{X}(Q)$*) that leaves the Lagrangian* $\mathcal{L}$ *invariant, then it is an (infinitesimal) contact symmetry, i.e.*

$$\Phi^*\eta^\alpha_\mathcal{L} = \eta^\alpha_\mathcal{L}\,, \quad \Phi^* E_\mathcal{L} = E_\mathcal{L} \qquad (\text{resp. } \mathscr{L}_Y \eta^\alpha_\mathcal{L} = 0\,, \ \ \mathscr{L}_Y E_\mathcal{L} = 0\,)\,.$$

As a consequence, it is an (infinitesimal) Lagrangian dynamical symmetry.

As an immediate consequence we have the following *momentum dissipation theorem*.

PROPOSITION 11. *If* $\dfrac{\partial \mathcal{L}}{\partial q^i} = 0$*, then* $\dfrac{\partial}{\partial q^i}$ *is an infinitesimal contact symmetry and its associated dissipation law is given by the "momenta"* $\left(\dfrac{\partial \mathcal{L}}{\partial v^i_\alpha}\right)$*; that is, for every k-vector field* $\mathbf{X}_\mathcal{L} = ((X_\mathcal{L})_\alpha)$ *solution to the k-contact Lagrangian equations* (14)*, then*

$$\mathscr{L}_{(X_\mathcal{L})_\alpha}\left(\frac{\partial \mathcal{L}}{\partial v^i_\alpha}\right) = -(\mathscr{L}_{(\mathcal{R}_\mathcal{L})_\alpha} E_\mathcal{L})\frac{\partial \mathcal{L}}{\partial v^i_\alpha} = \frac{\partial \mathcal{L}}{\partial s^\alpha}\frac{\partial \mathcal{L}}{\partial v^i_\alpha}.$$

5. Examples

5.1. An inverse problem for a class of elliptic and hyperbolic equations

A generic second-order linear PDE in $\mathbb{R}^2$ is

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0\,,$$

where A, B, C, D, E, F, G are functions of (x, y), with $A > 0$. If $B^2 - AC > 0$, the equation is said to be hyperbolic, if $B^2 - AC < 0$, is elliptic, and if $B^2 - AC = 0$, is parabolic. In $\mathbb{R}^n$ we consider the equation

$$A^{\alpha\beta}u_{\alpha\beta} + D^\alpha u_\alpha + G(u) = 0\,, \tag{22}$$

where $1 \leq \alpha, \beta \leq n$; and now we consider the following case: $A^{\alpha\beta}$ is constant and invertible (not parabolic), D^α is constant and G is an arbitrary function in u.

In order to find a Lagrangian k-contact formulation of this kind of PDE's, consider $\oplus^n T\mathbb{R} \times \mathbb{R}^n$, with coordinates (u, u_α, s^α) and a generic Lagrangian of the form

$$L = \frac{1}{2}a^{\alpha\beta}(u)u_\alpha u_\beta + b(u)u_\alpha s^\alpha + d(u, s)\,.$$

The associated k-contact structure is given by

$$\eta^\alpha = ds^\alpha - \frac{\partial L}{\partial u_\alpha}du = ds^\alpha - (a^{\alpha\beta}u_\beta + bs^\alpha + c^\alpha)du\,.$$

The k-contact Euler–Lagrange equations associated to L are

$$a^{\alpha\beta}u_{\alpha\beta} + \left(\frac{1}{2}\frac{\partial a^{\alpha\beta}}{\partial u} - \frac{1}{2}ba^{\alpha\beta}\right)u_\alpha u_\beta - \frac{\partial d}{\partial s^\beta}a^{\beta\alpha}u_\alpha + \left(-\frac{\partial d}{\partial s^\alpha}bs^\alpha + bd - \frac{\partial d}{\partial u}\right) = 0\,. \tag{23}$$

If this equation has to match (22) then

$$a^{\alpha\beta} = A^{\alpha\beta}, \quad b = 0, \quad d = -(a^{-1})_{\alpha\beta}D^\beta s^\alpha - \overline{g},$$

where $a = (a^{\alpha\beta})$ and $\dfrac{\partial \overline{g}}{\partial u} = G$.

Damped vibrating membrane As a particular example consider the damped vibrating membrane, which is described by the PDE

$$u_{tt} - \mu^2(u_{xx} + u_{yy}) + \gamma u_t = 0\,;$$

then

$$A^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mu^2 & 0 \\ 0 & 0 & -\mu^2 \end{pmatrix}, \quad D^\alpha = \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}, \quad G = 0,$$

and therefore

$$a^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mu^2 & 0 \\ 0 & 0 & -\mu^2 \end{pmatrix}, \quad b = 0, \quad d = -\gamma s^t.$$

Then, a Lagrangian that leads to this equation is

$$L = \frac{1}{2}u_t^2 - \frac{\mu^2}{2}(u_x^2 + u_y^2) - \gamma s^t,$$

for which

$$\eta^t = ds^t - u_t du, \quad \eta^x = ds^x + \mu^2 u_x du, \quad \eta^y = ds^y + \mu^2 u_y du.$$

In this case, we have the contact symmetry $\dfrac{\partial}{\partial u}$ and the associated map $\mathbf{F} = (F^t, F^x, F^y)$ that satisfies the dissipation law for 3-vector fields is

$$F^t = -i(Y)\eta^t = u_t, \quad F^x = -i(Y)\eta^x = -\mu^2 u_x, \quad F^y = -i(Y)\eta^y = -\mu^2 u_y.$$

5.2. A vibrating string: Lorentz-like forces versus dissipation forces

Terms linear in velocities can be found in Euler–Lagrange equations of symplectic systems. However, they have a specific form, arising from the coefficients of a closed 2-form in the configuration space. The canonical example is the force of a magnetic field acting on a moving charged particle; such forces do not dissipate energy. By contrast, other forces linear in the velocities do dissipate energy; for instance, damping forces. To illustrate the difference between the equations arising from magnetic-like terms in the Lagrangian and the equations given by the k-contact formulation of a linear dissipation, we analyze the following academic example.

Consider an infinite string aligned with the z-axis, each of whose points can vibrate in a horizontal plane. So, the independent variables are $(t, z) \in \mathbb{R}^2$, and the phase space is the bundle manifold $\oplus^2 \mathrm{T}\mathbb{R}^2$ with coordinates $(x, y, x_t, x_z, y_t, y_z)$. Let us imagine that the string is nonconducting, but charged with linear density charge λ. Then, inspired by the Lagrangian formulation of the Lorentz force, we set the Lagrangian

$$L_o = \frac{1}{2}\rho(x_t^2 + y_t^2) - \frac{1}{2}\tau(x_z^2 + y_z^2) - \lambda\,(\phi - A_1 x_t - A_2 y_t)$$

depending on some fixed functions $A_1(x, y)$, $A_2(x, y)$ and $\phi(x, y)$. The resulting Euler–Lagrange equations are

$$\begin{aligned} \rho x_{tt} - \tau x_{zz} &= -\lambda\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) y_t + \lambda\frac{\partial \phi}{\partial x}, \\ \rho y_{tt} - \tau y_{zz} &= \lambda\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) x_t + \lambda\frac{\partial \phi}{\partial y}. \end{aligned} \tag{24}$$

The left-hand side is the string equation with two modes of vibration in the plane XY and in the right-hand side we have an electromagnetic-like term.

Now, consider the 2-contact phase space $\oplus^2 T\mathbb{R}^2 \times \mathbb{R}^2$, with cartesian coordinates $(x, y, x_t, x_z, y_t, y_z, s^t, s^z)$. We add a simple dissipation term to the preceding

Lagrangian,

$$L = L_o + \gamma\, s^t = \frac{1}{2}\rho(x_t^2 + y_t^2) - \frac{1}{2}\tau(x_z^2 + y_z^2) - \lambda\,(\phi - A_1 x_t - A_2 y_t) + \gamma s^t .$$

The induced 2-contact structure is

$$\eta^t = ds^t - (\rho x_t + \lambda A_1)\, dx - (\rho y_t + \lambda A_2)\, dy\,; \quad \eta^z = ds^z + \tau x_z\, dx + \tau y_z\, dy.$$

The 2-contact Euler–Lagrange equations are

$$\begin{aligned} \rho x_{tt} - \tau x_{zz} &= -\lambda \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) y_t + \lambda \frac{\partial \phi}{\partial x} + \gamma \rho x_t + \gamma \lambda A_1\,, \\ \rho y_{tt} - \tau y_{zz} &= \lambda \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) x_t + \lambda \frac{\partial \phi}{\partial y} + \gamma \rho y_t + \gamma \lambda A_2\,. \end{aligned} \tag{25}$$

Comparing Eqs. (24) and (25) we observe that the dissipation originates two new terms: a dissipation force proportional to the velocity, and an extra term proportional to (A_1, A_2). This last term comes from the nonlinearity of the 2-contact Euler–Lagrange equations with respect to the Lagrangian.

This system has the Lagrangian 2-contact symmetry

$$Y = \frac{\partial A_2}{\partial x}\frac{\partial}{\partial x} + \frac{\partial A_1}{\partial y}\frac{\partial}{\partial y}.$$

The associated map $\mathbf{F} = (F^t, F^z)$ that satisfies the dissipation law for 2-vector fields is

$$\begin{aligned} F^t &= -i(Y)\eta^t = \rho x_t \frac{\partial A_2}{\partial x} + \lambda \frac{\partial A_2}{\partial x} A_1 + \rho y_t \frac{\partial A_1}{\partial y} + \lambda \frac{\partial A_1}{\partial y} A_2, \\ F^z &= -i(Y)\eta^z = -\tau x_z \frac{\partial A_2}{\partial x} - \tau y_z \frac{\partial A_1}{\partial y}. \end{aligned}$$

6. Conclusions and outlook

In a previous paper [15] we introduced the notion of k-contact structure to describe Hamiltonian (De Donder–Weyl) covariant field theories with dissipation, bringing together contact Hamiltonian mechanics and k-symplectic field theory.

In this paper, we have developed the Lagrangian counterpart of this theory, basing on contact Lagrangian and k-contact Hamiltonian formalisms. Thus, we have obtained and analyzed the Lagrangian (Euler–Lagrange) equations of dissipative field theories. It should be pointed out that the regularity of the Lagrangian is required to obtain a k-contact structure.

We have also studied several kinds of symmetries: dynamical symmetries (those preserving solutions), k-contact symmetries (those preserving the k-contact structure and the energy) and symmetries of the Lagrangian function. We have shown how to associate a dissipation law with any dynamical symmetry.

As interesting examples, we have constructed contact Lagrangian functions for certain classes of elliptic and hyperbolic partial differential equations; in particular, we have analyzed the example of the damped vibrating membrane. Another example has shown the difference between the equations of the k-contact formulation of a linear dissipation and the equations arising from magnetic-like terms appearing in some Lagrangian functions of field theories.

Among future lines of research, the case of singular Lagrangians seems especially interesting, though it would require to define the notions of *k-precontact structure* and *k-precontact Hamiltonian system*, and to develop a constraint analysis to check the consistency of field equations.

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