

New contributions to the Hamiltonian and Lagrangian contact formalisms for dissipative mechanical systems and their symmetries

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We provide new insights into the contact Hamiltonian and Lagrangian formulations of dissipative mechanical systems. In particular, we state a new form of the contact dynamical equations, and we review two recently presented Lagrangian formalisms, studying their equivalence. We define several kinds of symmetries for contact dynamical systems, as well as the notion of dissipation laws, prove a dissipation theorem and give a way to construct conserved quantities. Some well-known examples of dissipative systems are discussed.

Keywords: Contact manifold; dissipative system; Lagrangian formalism; Hamiltonian formalism; symmetry; dissipation law.

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1. Introduction

Many theories in modern physics can be formulated using the tools of differential geometry. For instance, the natural geometric framework for autonomous

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Hamiltonian mechanical systems is symplectic geometry [1, 2, 31], whereas its nonautonomous counterpart can be nicely described using cosymplectic geometry [11, 12, 17]. Geometric approaches to field theory make use of multisymplectic and k -symplectic geometry, among others [9, 16, 36]. All these methods are developed to model systems of variational type without dissipation or damping, both in the Lagrangian and Hamiltonian formalisms.

In recent years there has been a growing interest in studying a geometric framework to describe dissipative systems [10, 18, 33, 35]. An especially useful tool is contact geometry [3, 7, 8, 21]. Contact geometry has been used to describe several types of physical systems in thermodynamics, quantum mechanics, circuit theory, control theory, etc. (see for instance [5, 24, 29, 34]). In recent papers [19, 20] a generalization of contact geometry has been used to describe field theories with dissipation.

In this paper, we are primarily interested in mechanics. Contact geometry has been applied to give a Hamiltonian-type description of mechanical systems with dissipation [4, 6, 30, 32, 38], although in recent papers the Lagrangian formalism has been also considered [13, 14]. The model introduced in [14] is more general and it is inspired in the cosymplectic formulation of nonautonomous mechanical systems; meanwhile the other one [13] describes a more particular, but very usual, type of Lagrangian systems with dissipation.

Our aim is to enlarge these lines of research, giving new insights into the contact formulation of these systems, both in the Hamiltonian and the Lagrangian formalisms. While carefully reviewing these formulations, we present a new form of the contact Hamiltonian and Lagrangian equations. We also expand the formulation introduced in [13] and compare it with the one given in [14], and we show that they are equivalent for the Lagrangians with dissipation studied in [13]. It is interesting to point out that the first Lagrangian formulation of contact systems was introduced by Herglotz, from a variational perspective, around 1930 [27, 28] (see also [22]).

We also introduce and study symmetries and dissipated quantities in a geometrical way. We define different kinds of symmetries for contact Hamiltonian and Lagrangian systems (the so-called dynamical and contact), depending on which structure is preserved, and establish the relations among them. We have also compared the characteristics of symmetries for symplectic and contact dynamical systems, showing that there are significant differences between them. Other relevant results are the statements of the so-called “dissipation theorems”, which are analogous to the conservation theorems of conservative systems, and the association of dissipated and conserved quantities to these symmetries.

Some well-known physical examples are discussed using this geometric framework. In particular, we propose contact Lagrangian functions that describe the damped harmonic oscillator, the motion in a gravitational field with friction, and the parachute equation.

The paper is organized as follows. Section 2 is devoted to review several preliminary concepts on contact geometry and contact Hamiltonian systems, including a new setting of the dynamical equations. In Sec. 3, we study the Lagrangian formalism for dissipative systems in detail, and we compare the formulations given in [13] and [14]. In Sec. 4, we analyze the concepts of symmetry and dissipation law for dissipative systems, prove a dissipation theorem and show how to obtain conserved quantities. Finally, Sec. 5 is devoted to the examples.

Throughout the paper all the manifolds and mappings are assumed to be smooth. Sum over crossed repeated indices is understood.

2. Preliminaries

In this section, we present some geometric structures that will be used to describe the Lagrangian formalism of dissipative dynamical systems (see, for instance, [4, 6, 21, 30] for details).

2.1. Contact geometry and contact Hamiltonian systems

Definition 1. Let M be a $(2n + 1)$ -dimensional manifold. A *contact form* in M is a differential 1-form $\eta \in \Omega^1(M)$ such that $\eta \wedge (d\eta)^{\wedge n}$ is a volume form in M . Then, (M, η) is said to be a *contact manifold*.

As a consequence of the condition that $\eta \wedge (d\eta)^{\wedge n}$ is a volume form we have a decomposition of TM , induced by η , in the form $TM = \ker d\eta \oplus \ker \eta \equiv \mathcal{D}^R \oplus \mathcal{D}^C$.

Proposition 2. Let (M, η) be a contact manifold. Then there exists a unique vector field $\mathcal{R} \in \mathfrak{X}(M)$, which is called *Reeb vector field*, such that

$$\begin{cases} i(\mathcal{R})d\eta = 0, \\ i(\mathcal{R})\eta = 1. \end{cases} \quad (1)$$

This vector field generates the distribution $\mathcal{D}^R$, which is called the *Reeb distribution*.

Proposition 3. On a contact manifold there exist local coordinates $(x^I; s)$ such that

$$\mathcal{R} = \frac{\partial}{\partial s}, \quad \eta = ds - f_I(x)dx^I,$$

where $f_I(x)$ are functions depending only on the x^I . (These are the so-called adapted coordinates of the contact structure.)

Proof. Let (x^I, s) , $I = 1, \dots, n$, local coordinates in M rectifying the vector field $\mathcal{R}$, that is $\mathcal{R} = \partial/\partial s$ in the domain of these coordinates.

Then $\eta = a ds - f_I(x, s)dx^I$. The conditions defining $\mathcal{R}$ imply that $a = 1$ and $\partial f_I/\partial s = 0$, hence the result is proved. $\square$

Nevertheless, one can go further and prove the existence of Darboux-type coordinates.

Theorem 4 (Darboux theorem for contact manifolds). *Let (M, η) be a contact manifold. Then around each point $p \in M$ there exist a chart $(\mathcal{U}; q^i, p_i, s)$ with $1 \leq i \leq n$ such that*

$$\eta|_{\mathcal{U}} = ds - p_i dq^i.$$

These are the so-called Darboux or canonical coordinates of the contact manifold (M, η) .

In Darboux coordinates, the Reeb vector field is $\mathcal{R}|_{\mathcal{U}} = \frac{\partial}{\partial s}$.

Example 5 (Canonical contact structure). The manifold $M = T^*Q \times \mathbb{R}$ has a canonical contact structure defined by the 1-form $\eta = ds - \theta$, where s is the Cartesian coordinate of $\mathbb{R}$, and θ is the pull-back of the canonical 1-form of T^*Q with respect to the projection $M \rightarrow T^*Q$. Using coordinates q^i on Q and natural coordinates (q^i, p_i) on T^*Q , the local expressions of the contact form is $\eta = ds - p_i dq^i$; from which $d\eta = dq^i \wedge dp_i$, and the Reeb vector field is $\mathcal{R} = \frac{\partial}{\partial s}$.

Example 6 (Contactification of a symplectic manifold). Let (P, ω) be a symplectic manifold such that $\omega = -d\theta$, and consider $M = P \times \mathbb{R}$. Denoting by s the Cartesian coordinate of $\mathbb{R}$, and representing also by θ the pull-back of θ to the product, we consider the 1-form $\eta = ds - \theta$ on M . Then (M, η) is a contact manifold which is called the *contactified* of P . Note that Example 5, is just the contactification of T^*Q with its canonical symplectic structure.

Finally, given a contact manifold (M, η) , we have the following $C^\infty(M)$ -module isomorphism

$$\begin{aligned} \flat : \mathfrak{X}(M) &\rightarrow \Omega^1(M), \\ X &\mapsto i(X)d\eta + (i(X)\eta)\eta. \end{aligned}$$

Theorem 7. *If (M, η) is a contact manifold, for every $\mathcal{H} \in C^\infty(M)$, there exists a unique vector field $X_{\mathcal{H}} \in \mathfrak{X}(M)$ such that*

$$\begin{cases} i(X_{\mathcal{H}})d\eta = d\mathcal{H} - (L_{\mathcal{R}}\mathcal{H})\eta, \\ i(X_{\mathcal{H}})\eta = -\mathcal{H}. \end{cases} \quad (2)$$

Then, the integral curves of $X_{\mathcal{H}}$, $\mathbf{c} : I \subset \mathbb{R} \rightarrow M$, are the solutions to the equations

$$\begin{cases} i(\mathbf{c}')d\eta = (d\mathcal{H} - (L_{\mathcal{R}}\mathcal{H})\eta) \circ \mathbf{c}, \\ i(\mathbf{c}')\eta = -\mathcal{H} \circ \mathbf{c}, \end{cases} \quad (3)$$

where $\mathbf{c}' : I \subset \mathbb{R} \rightarrow TM$ is the canonical lift of the curve $\mathbf{c}$ to the tangent bundle TM .

Definition 8. The vector field $X_{\mathcal{H}}$ is the *contact Hamiltonian vector field* associated to $\mathcal{H}$ and Eqs. (2) and (3) are the *contact Hamiltonian equations* for this vector field and its integral curves, respectively.

The triple $(M, \eta, \mathcal{H})$ is a *contact Hamiltonian system*.

As a consequence of the definition of $X_{\mathcal{H}}$ we have the following relation, which expresses the *dissipation of the Hamiltonian*

$$L_{X_{\mathcal{H}}} \mathcal{H} = -(L_{\mathcal{R}} \mathcal{H}) \mathcal{H}. \quad (4)$$

The proof is immediate

$$L_{X_{\mathcal{H}}} \mathcal{H} = -L_{X_{\mathcal{H}}} i(X_{\mathcal{H}}) \eta = -i(X_{\mathcal{H}}) L_{X_{\mathcal{H}}} \eta = i(X_{\mathcal{H}}) ((L_{\mathcal{R}} \mathcal{H}) \eta) = -(L_{\mathcal{R}} \mathcal{H}) \mathcal{H}.$$

Proposition 9. Let $(M, \eta, \mathcal{H})$ be a contact Hamiltonian system. The following assertions are equivalent:

- (1) $X_{\mathcal{H}}$ is a contact Hamiltonian vector field (i.e. a solution to Eqs. (2)).
- (2) $X_{\mathcal{H}}$ is a solution to the equations

$$\begin{cases} L_{X_{\mathcal{H}}} \eta = -(L_{\mathcal{R}} \mathcal{H}) \eta, \\ i(X_{\mathcal{H}}) \eta = -\mathcal{H}, \end{cases}$$

- (3) $X_{\mathcal{H}}$ satisfies that

$$\flat(X_{\mathcal{H}}) = d\mathcal{H} - (L_{\mathcal{R}} \mathcal{H} + \mathcal{H}) \eta.$$

In Sec. 2.2, we give another way of stating the equations for a contact Hamiltonian vector field.

Taking Darboux coordinates (q^i, p_i, s) , the contact Hamiltonian vector field is

$$X_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i \frac{\partial \mathcal{H}}{\partial s} \right) \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H} \right) \frac{\partial}{\partial s};$$

and its integral curves $\gamma(t) = (q^i(t), p_i(t), s(t))$ are solutions to the dissipative Hamilton equations (3) which read as

$$\begin{cases} \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \\ \dot{p}_i = - \left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i \frac{\partial \mathcal{H}}{\partial s} \right), \\ \dot{s} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}. \end{cases} \quad (5)$$

Remark 10. The case in which some of the conditions stated in Definition 1 do not hold has been recently analyzed in [14], where the definitions and properties of *precontact structures* and *precontact manifolds* are introduced and studied. In particular, for this kind of manifolds, Reeb vector fields are not uniquely determined. In these cases $(M, \eta, \mathcal{H})$ is called a *precontact Hamiltonian system*.

2.2. Equivalent form of the contact Hamiltonian equations

In this section, we present a new way of writing Eqs. (2) without making use of the Reeb vector field $\mathcal{R}$. This can be useful in the case of singular systems because, as we have pointed out, in a precontact manifold we do not have a uniquely determined Reeb vector field.

Proposition 11. *Let $(M, \eta, \mathcal{H})$ be a contact Hamiltonian system and consider the open set $U = \{p \in M; \mathcal{H}(p) \neq 0\}$. Let Ω be the 2-form defined by $\Omega = -\mathcal{H}d\eta + d\mathcal{H} \wedge \eta$ on U . A vector field $X \in \mathfrak{X}(U)$ is the contact Hamiltonian vector field if, and only if,*

$$\begin{cases} i(X)\Omega = 0, \\ i(X)\eta = -\mathcal{H}. \end{cases} \quad (6)$$

Proof. Suppose that X satisfies Eq. (6). Then,

$$0 = i(X)\Omega = -\mathcal{H}i(X)d\eta + (i(X)d\mathcal{H})\eta + \mathcal{H}d\mathcal{H},$$

and hence

$$\mathcal{H}i(X)d\eta = (i(X)d\mathcal{H})\eta + \mathcal{H}d\mathcal{H}. \quad (7)$$

Contracting with the Reeb vector field,

$$0 = \mathcal{H}i(\mathcal{R})i(X)d\eta = (i(X)d\mathcal{H})i(\mathcal{R})\eta + \mathcal{H}i(\mathcal{R})d\mathcal{H},$$

and $i(X)d\mathcal{H} = -\mathcal{H}i(\mathcal{R})d\mathcal{H}$. Using this in Eq. (7), we get

$$\mathcal{H}i(X)d\eta = \mathcal{H}(d\mathcal{H} - (i(\mathcal{R})d\mathcal{H})\eta) = \mathcal{H}(d\mathcal{H} - (\mathcal{L}_{\mathcal{R}}\mathcal{H})\eta),$$

and hence $i(X)d\eta = d\mathcal{H} - (\mathcal{L}_{\mathcal{R}}\mathcal{H})\eta$.

Now, suppose that X satisfies Eqs. (2). Then

$$\begin{aligned} i(X)\Omega &= i(X)(-\mathcal{H}d\eta + d\mathcal{H} \wedge \eta) = -\mathcal{H}i(X)d\eta + (i(X)d\mathcal{H})\eta + \mathcal{H}d\mathcal{H} \\ &= \mathcal{H}(\mathcal{L}_{\mathcal{R}}\mathcal{H})\eta + (\mathcal{L}_X\mathcal{H})\eta = (\mathcal{H}(\mathcal{L}_{\mathcal{R}}\mathcal{H}) + (\mathcal{L}_X\mathcal{H}))\eta = 0, \end{aligned}$$

hence $i(X)\Omega = 0$, using the dissipation of the Hamiltonian $\mathcal{H}$ (4). $\square$

Remark 12. Let $p \in M$ and suppose that $\mathcal{H}(p) = 0$. Then the second equation in both groups (2) and (6) gives $i(X_p)\eta = 0$, hence $X_p \in \ker \eta_p$. The remaining equation in (2) is $i(X_p)d\eta = d\mathcal{H}(p) - (\mathcal{L}_{\mathcal{R}_p}\mathcal{H})\eta_p$, and the corresponding one in (6) is $i(X_p)\Omega = (\mathcal{L}_{X_p}\mathcal{H})\eta_p = 0$. These two equations are not equivalent. In fact the first one implies the second, using the dissipation of the Hamiltonian, but not on the contrary.

Finally, bearing in mind Theorem 7, we can state as follows.

Proposition 13. *On the open set $U = \{p \in M; \mathcal{H} \neq 0\}$, a path $\mathbf{c}: I \subset \mathbb{R} \rightarrow M$ is an integral curve of the contact Hamiltonian vector field $X_{\mathcal{H}}$ if, and only if, it is a*

solution to

$$\begin{cases} i(\mathbf{c}')\Omega = 0, \\ i(\mathbf{c}')\eta = -\mathcal{H} \circ \mathbf{c}. \end{cases}$$

3. Lagrangian Formalism for Dissipative Dynamical Systems

3.1. Lagrangian phase space and geometric structures

If Q is an n -dimensional manifold, consider the product manifold $TQ \times \mathbb{R}$ with canonical projections

$$s: TQ \times \mathbb{R} \rightarrow \mathbb{R}, \quad \tau_1: TQ \times \mathbb{R} \rightarrow TQ, \quad \tau_0: TQ \times \mathbb{R} \rightarrow Q \times \mathbb{R}.$$

Note that τ_1 and τ_0 are the projection maps of two vector bundle structures. We will usually have the second one in mind; indeed, with this structure $TQ \times \mathbb{R}$ is the pull-back of the tangent bundle TQ with respect the map $Q \times \mathbb{R} \rightarrow Q$. Natural coordinates in $TQ \times \mathbb{R}$ will be denoted (q^i, v^i, s) .

In order to develop a contact Lagrangian formalism we need to extend the usual geometric structures of Lagrangian mechanics to the Lagrangian phase space. Note that, as a product manifold, we can write $T(TQ \times \mathbb{R}) = (T(TQ)) \times \mathbb{R} \oplus (TQ \times T\mathbb{R})$, so any operation that can act on tangent vectors to TQ can act on tangent vectors to $TQ \times \mathbb{R}$. In particular, the vertical endomorphism of $T(TQ)$ yields a *vertical endomorphism* $\mathcal{J}: T(TQ \times \mathbb{R}) \rightarrow T(TQ \times \mathbb{R})$. Similarly, the Liouville vector field on TQ yields a *Liouville vector field* $\Delta \in \mathfrak{X}(TQ \times \mathbb{R})$; indeed, this is the Liouville vector field of the vector bundle structure defined by τ_0 .

In natural coordinates, the local expressions of these objects are

$$\mathcal{J} = \frac{\partial}{\partial v^i} \otimes dq^i, \quad \Delta = v^i \frac{\partial}{\partial v^i}.$$

Definition 14. Let $\mathbf{c}: \mathbb{R} \rightarrow Q \times \mathbb{R}$ be a path, with $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2)$. The *prolongation* of $\mathbf{c}$ to $TQ \times \mathbb{R}$ is the path

$$\tilde{\mathbf{c}} = (\mathbf{c}'_1, \mathbf{c}_2): \mathbb{R} \rightarrow TQ \times \mathbb{R},$$

where $\mathbf{c}'_1$ is the velocity of $\mathbf{c}_1$. The path $\tilde{\mathbf{c}}$ is said to be *holonomic*.

A vector field $\Gamma \in \mathfrak{X}(TQ \times \mathbb{R})$ is said to satisfy the *second-order condition* (for short: is a SODE) when all of its integral curves are holonomic.

This definition can be equivalently expressed in terms of the canonical structures defined above.

Proposition 15. A vector field $\Gamma \in \mathfrak{X}(TQ \times \mathbb{R})$ is a SODE if, and only if, $\mathcal{J} \circ \Gamma = \Delta$.

In coordinates, if $\mathbf{c}(t) = (c^i(t), s(t))$, then

$$\tilde{\mathbf{c}}(t) = \left(c^i(t), \frac{dc^i}{dt}(t), s(t) \right).$$

The local expression of a SODE is

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i} + g \frac{\partial}{\partial s}. \quad (8)$$

So, in coordinates a SODE defines a system of differential equations of the form

$$\frac{d^2 q^i}{dt^2} = f^i(q, \dot{q}, s), \quad \frac{ds}{dt} = g(q, \dot{q}, s).$$

3.2. Contact Lagrangian systems

A general setting of the Lagrangian formalism for dissipative mechanical systems has been recently developed by León and Lainz-Valcázar [14]. Next, we review this formulation and give new additional features.

Definition 16. A *Lagrangian function* is a function $\mathcal{L}: \mathrm{T}Q \times \mathbb{R} \rightarrow \mathbb{R}$.

The *Lagrangian energy* associated with $\mathcal{L}$ is the function $E_{\mathcal{L}} := \Delta(\mathcal{L}) - \mathcal{L} \in C^\infty(\mathrm{T}Q \times \mathbb{R})$.

The *Cartan forms* associated with $\mathcal{L}$ are defined as

$$\theta_{\mathcal{L}} = {}^t\mathcal{J} \circ d\mathcal{L} \in \Omega^1(\mathrm{T}Q \times \mathbb{R}), \quad \omega_{\mathcal{L}} = -d\theta_{\mathcal{L}} \in \Omega^2(\mathrm{T}Q \times \mathbb{R}). \quad (9)$$

The *contact Lagrangian form* is

$$\eta_{\mathcal{L}} = ds - \theta_{\mathcal{L}} \in \Omega^1(\mathrm{T}Q \times \mathbb{R});$$

it satisfies that $d\eta_{\mathcal{L}} = \omega_{\mathcal{L}}$.

The couple $(\mathrm{T}Q \times \mathbb{R}, \mathcal{L})$ is a *contact Lagrangian system*.

Taking natural coordinates (q^i, v^i, s) in $\mathrm{T}Q \times \mathbb{R}$, the form $\eta_{\mathcal{L}}$ is written as

$$\eta_{\mathcal{L}} = ds - \frac{\partial \mathcal{L}}{\partial v^i} dq^i, \quad (10)$$

and consequently

$$d\eta_{\mathcal{L}} = -\frac{\partial^2 \mathcal{L}}{\partial s \partial v^i} ds \wedge dq^i - \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i} dq^j \wedge dq^i - \frac{\partial^2 \mathcal{L}}{\partial v^j \partial v^i} dv^j \wedge dq^i.$$

The next structure to be defined is the Legendre map. Before, remember that, given a (not necessarily linear) bundle map $f: E \rightarrow F$ between two vector bundles over a manifold B , the fiber derivative of f is the map $\mathcal{F}f: E \rightarrow \mathrm{Hom}(E, F) \approx F \otimes E^*$ obtained by restricting f to the fibers, $f_b: E_b \rightarrow F_b$, and computing the usual derivative of a map between two vector spaces: $\mathcal{F}f(e_b) = Df_b(e_b)$ — see [25] for a detailed account. This applies in particular when the second vector bundle is trivial of rank 1, that is, for a function $f: E \rightarrow \mathbb{R}$; then $\mathcal{F}f: E \rightarrow E^*$. This map also has a fiber derivative $\mathcal{F}^2 f: E \rightarrow E^* \otimes E^*$, which can be called the fiber Hessian of f ; for every $e_b \in E$, $\mathcal{F}^2 f(e_b)$ can be considered as a symmetric bilinear form on E_b . It is easy to check that $\mathcal{F}f$ is a local diffeomorphism at a point $e \in E$ if and only if the Hessian $\mathcal{F}^2 f(e)$ is non-degenerate.

Definition 17. Given a Lagrangian $\mathcal{L}: \text{T}Q \times \mathbb{R} \rightarrow \mathbb{R}$, its *Legendre map* is the fiber derivative of $\mathcal{L}$, considered as a function on the vector bundle $\tau_0: \text{T}Q \times \mathbb{R} \rightarrow Q \times \mathbb{R}$; that is, the map $\mathcal{FL}: \text{T}Q \times \mathbb{R} \rightarrow \text{T}^*Q \times \mathbb{R}$ given by

$$\mathcal{FL}(v_q, s) = (\mathcal{FL}(\cdot, s)(v_q), s),$$

where $\mathcal{L}(\cdot, s)$ is the Lagrangian with s freezed.

Remark 18. The Cartan forms can also be defined as

$$\theta_{\mathcal{L}} = \mathcal{FL}^*(\pi_1^*\theta), \quad \omega_{\mathcal{L}} = \mathcal{FL}^*(\pi_1^*\omega).$$

Proposition 19. For a Lagrangian function $\mathcal{L}$ the following conditions are equivalent:

- (1) The Legendre map $\mathcal{FL}$ is a local diffeomorphism.
- (2) The fiber Hessian $\mathcal{F}^2\mathcal{L}: \text{T}Q \times \mathbb{R} \rightarrow (\text{T}^*Q \times \mathbb{R}) \otimes (\text{T}^*Q \times \mathbb{R})$ of $\mathcal{L}$ is everywhere nondegenerate. (The tensor product is of vector bundles over $Q \times \mathbb{R}$.)
- (3) $(\text{T}Q \times \mathbb{R}, \eta_{\mathcal{L}})$ is a contact manifold.

The proof of this result can be easily done using natural coordinates, where

$$\mathcal{FL}: (q^i, v^i, s) \rightarrow \left(q^i, \frac{\partial \mathcal{L}}{\partial v^i}, s \right),$$

$$\mathcal{F}^2\mathcal{L}(q^i, v^i, s) = (q^i, W_{ij}, s), \quad \text{with } W_{ij} = \left(\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \right).$$

Then the conditions in the proposition mean that the matrix $W = (W_{ij})$ is everywhere nonsingular.

Definition 20. A Lagrangian function $\mathcal{L}$ is said to be *regular* if the equivalent conditions in Proposition 19 hold. Otherwise $\mathcal{L}$ is called a *singular* Lagrangian. In particular, $\mathcal{L}$ is said to be *hyperregular* if $\mathcal{FL}$ is a global diffeomorphism.

Remark 21. As a result of the preceding definitions and results, every *regular* contact Lagrangian system has associated the contact Hamiltonian system $(\text{T}Q \times \mathbb{R}, \eta_{\mathcal{L}}, E_{\mathcal{L}})$.

Given a regular contact Lagrangian system $(\text{T}Q \times \mathbb{R}, \mathcal{L})$, from (1) we have that the *Reeb vector field* $\mathcal{R}_{\mathcal{L}} \in \mathfrak{X}(\text{T}Q \times \mathbb{R})$ for this system is uniquely determined by the relations

$$\begin{cases} i(\mathcal{R}_{\mathcal{L}})d\eta_{\mathcal{L}} = 0, \\ i(\mathcal{R}_{\mathcal{L}})\eta_{\mathcal{L}} = 1. \end{cases} \quad (11)$$

Its local expression is

$$\mathcal{R}_{\mathcal{L}} = \frac{\partial}{\partial s} - W^{ji} \frac{\partial^2 \mathcal{L}}{\partial s \partial v^j} \frac{\partial}{\partial v^i}, \quad (12)$$

where (W^{ij}) is the inverse of the Hessian matrix, namely $W^{ij}W_{jk} = \delta_k^i$.

Note that the Reeb vector field does not appear in the simplest form $\partial/\partial s$. This is due to the fact that the natural coordinates in $TQ \times \mathbb{R}$ are *not* adapted coordinates for $\eta_{\mathcal{L}}$.

3.3. The contact Euler–Lagrange equations

Definition 22. Let $(TQ \times \mathbb{R}, \mathcal{L})$ be a regular contact Lagrangian system.

The *contact Euler–Lagrange equations* for a holonomic curve $\tilde{\mathbf{c}}: I \subset \mathbb{R} \rightarrow TQ \times \mathbb{R}$ are

$$\begin{cases} i(\tilde{\mathbf{c}}')d\eta_{\mathcal{L}} = (dE_{\mathcal{L}} - (L_{\mathcal{R}_{\mathcal{L}}}E_{\mathcal{L}})\eta_{\mathcal{L}}) \circ \tilde{\mathbf{c}}, \\ i(\tilde{\mathbf{c}}')\eta_{\mathcal{L}} = -E_{\mathcal{L}} \circ \tilde{\mathbf{c}}, \end{cases} \quad (13)$$

where $\tilde{\mathbf{c}}': I \subset \mathbb{R} \rightarrow T(TQ \times \mathbb{R})$ denotes the canonical lifting of $\tilde{\mathbf{c}}$ to $T(TQ \times \mathbb{R})$.

The *contact Lagrangian equations* for a vector field $X_{\mathcal{L}} \in \mathfrak{X}(TQ \times \mathbb{R})$ are

$$\begin{cases} i(X_{\mathcal{L}})d\eta_{\mathcal{L}} = dE_{\mathcal{L}} - (L_{\mathcal{R}_{\mathcal{L}}}E_{\mathcal{L}})\eta_{\mathcal{L}}, \\ i(X_{\mathcal{L}})\eta_{\mathcal{L}} = -E_{\mathcal{L}}. \end{cases} \quad (14)$$

A vector field which is a solution to these equations is called a *contact Lagrangian vector field* (it is a contact Hamiltonian vector field for the function $E_{\mathcal{L}}$).

Remark 23. Now, taking into account Propositions 11 and 13, in the open set $U = \{p \in M; E_{\mathcal{L}}(p) \neq 0\}$, the above equations can be stated equivalently as

$$\begin{cases} i(\tilde{\mathbf{c}}')\Omega_{\mathcal{L}} = 0, \\ i(\tilde{\mathbf{c}}')\eta_{\mathcal{L}} = -E_{\mathcal{L}} \circ \tilde{\mathbf{c}}, \end{cases} \quad (15)$$

and

$$\begin{cases} i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0, \\ i(X_{\mathcal{L}})\eta_{\mathcal{L}} = -E_{\mathcal{L}}, \end{cases} \quad (16)$$

where $\Omega_{\mathcal{L}} = -E_{\mathcal{L}}d\eta_{\mathcal{L}} + dE_{\mathcal{L}} \wedge \eta_{\mathcal{L}}$.

In natural coordinates, for a holonomic curve $\tilde{\mathbf{c}}(t) = (q^i(t), \dot{q}^i(t), s(t))$, Eq. (13) are

$$\dot{s} = \mathcal{L}, \quad (17)$$

$$\begin{aligned} & \frac{\partial^2 \mathcal{L}}{\partial v^j \partial v^i} \ddot{q}^j + \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i} \dot{q}^j + \frac{\partial^2 \mathcal{L}}{\partial s \partial v^i} \dot{s} - \frac{\partial \mathcal{L}}{\partial q^i} \\ &= \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial v^i} \right) - \frac{\partial \mathcal{L}}{\partial q^i} = \frac{\partial \mathcal{L}}{\partial s} \frac{\partial \mathcal{L}}{\partial v^i}, \end{aligned} \quad (18)$$

(which coincide with the so-called *generalized Euler–Lagrange equations* stated in [27]); meanwhile, for a vector field $X_{\mathcal{L}} = f^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial v^i} + g \frac{\partial}{\partial s}$,

Eq. (14) are

$$(f^j - v^j) \frac{\partial^2 \mathcal{L}}{\partial v^j \partial s} = 0, \quad (19)$$

$$(f^j - v^j) \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} = 0, \quad (20)$$

$$(f^j - v^j) \frac{\partial^2 \mathcal{L}}{\partial q^i \partial v^j} + \frac{\partial \mathcal{L}}{\partial q^i} - \frac{\partial^2 \mathcal{L}}{\partial s \partial v^i} g - \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i} f^j - \frac{\partial^2 \mathcal{L}}{\partial v^j \partial v^i} F^j + \frac{\partial \mathcal{L}}{\partial s} \frac{\partial \mathcal{L}}{\partial v^i} = 0, \quad (21)$$

$$\mathcal{L} + \frac{\partial \mathcal{L}}{\partial v^i} (f^i - v^i) - g = 0. \quad (22)$$

For these computations it is useful to use the relation

$$L_{\mathcal{R}_\mathcal{L}} E_\mathcal{L} = -\frac{\partial \mathcal{L}}{\partial s}, \quad (23)$$

which is easily proved in coordinates.

Proposition 24. *If $\mathcal{L}$ is a regular Lagrangian, then $X_\mathcal{L}$ is a SODE and the Eqs. (22) and (21) become*

$$g = \mathcal{L}, \quad (24)$$

$$\frac{\partial^2 \mathcal{L}}{\partial v^j \partial v^i} F^j + \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i} v^j + \frac{\partial^2 \mathcal{L}}{\partial s \partial v^i} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial q^i} = \frac{\partial \mathcal{L}}{\partial s} \frac{\partial \mathcal{L}}{\partial v^i}, \quad (25)$$

which, for the integral curves of $X_\mathcal{L}$, are the Euler–Lagrange equations (17) and (18).

This SODE $X_\mathcal{L} \equiv \Gamma_\mathcal{L}$ is called the Euler–Lagrange vector field associated with the Lagrangian function $\mathcal{L}$.

Proof. It follows from the coordinate expressions. If $\mathcal{L}$ is a regular Lagrangian, Eqs. (20) lead to $v^i = f^i$, which are the SODE condition for the vector field $X_\mathcal{L}$. Then, (19) holds identically, and (22) and (21) give Eqs. (24) and (25) or, equivalently, for the integral curves of $X_\mathcal{L}$, the Euler–Lagrange equations (17) and (18). $\square$

In this way, the local expression of this Euler–Lagrange vector field is

$$\Gamma_\mathcal{L} = \mathcal{L} \frac{\partial}{\partial s} + v^i \frac{\partial}{\partial q^i} + W^{ik} \left(\frac{\partial \mathcal{L}}{\partial q^k} - \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^k} v^j - \mathcal{L} \frac{\partial^2 \mathcal{L}}{\partial s \partial v^k} + \frac{\partial \mathcal{L}}{\partial s} \frac{\partial \mathcal{L}}{\partial v^k} \right) \frac{\partial}{\partial v^i}. \quad (26)$$

Remark 25. It is interesting to point out how, in the Lagrangian formalism of dissipative systems, the expression in coordinates (17) of the second Lagrangian

equation (14) relates the variation of the “dissipation coordinate” s to the Lagrangian function and, from here, we can identify this coordinate with the Lagrangian action, $s = \int \mathcal{L} dt$.

Remark 26. If $\mathcal{L}$ is singular, although $(TQ \times \mathbb{R}, \eta_{\mathcal{L}})$ is not a contact manifold, but a pre-contact one, and hence the Reeb vector field is not uniquely defined, it can be proved that the Lagrangian equations (14) are independent on the Reeb vector field used [14]. Alternatively, Proposition 11 holds also in this case and, hence, the Reeb-independent equations (16) can be used instead. In any case, solutions to the Lagrangian equations are not necessarily SODE and, in order to obtain the Euler–Lagrange equations (25) (or (18)), the condition $\mathcal{J}(X_{\mathcal{L}}) = \Delta$ must be added to the above Lagrangian equations. Furthermore, these equations are not necessarily compatible everywhere on $TQ \times \mathbb{R}$ and a suitable *constraint algorithm* must be implemented in order to find a *final constraint submanifold* $S_f \hookrightarrow TQ \times \mathbb{R}$ (if it exists) where there are SODE vector fields $X_{\mathcal{L}} \in \mathfrak{X}(TQ \times \mathbb{R})$, tangent to S_f , which are (not necessarily unique) solutions to the above equations on S_f . All these problems have been studied in detail in [14].

3.4. The canonical Hamiltonian formalism for contact Lagrangian systems

In the (hyper)regular case, we have a diffeomorphism between $(TQ \times \mathbb{R}, \eta_{\mathcal{L}})$ and $(T^*Q \times \mathbb{R}, \eta)$, where $\mathcal{FL}^*\eta = \eta_{\mathcal{L}}$. Furthermore, there exists (maybe locally) a function $\mathcal{H} \in C^\infty(T^*Q \times \mathbb{R})$ such that $\mathcal{FL}^*\mathcal{H} = E_{\mathcal{L}}$; then we have the contact Hamiltonian system $(T^*Q \times \mathbb{R}, \eta, \mathcal{H})$, for which $\mathcal{FL}_*\mathcal{R}_{\mathcal{L}} = \mathcal{R}$. Then, if $X_{\mathcal{H}} \in \mathfrak{X}(T^*Q \times \mathbb{R})$ is the contact Hamiltonian vector field associated with $\mathcal{H}$, we have that $\mathcal{FL}_*\Gamma_{\mathcal{L}} = X_{\mathcal{H}}$.

For singular Lagrangians, following [23] we define as follows.

Definition 27. A singular Lagrangian $\mathcal{L}$ is said to be *almost-regular* if $\mathcal{P} := \mathcal{FL}(TQ)$ is a closed submanifold of $T^*Q \times \mathbb{R}$, the Legendre map $\mathcal{FL}$ is a submersion onto its image, and the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(v_q, s))$, for every $(v_q, s) \in TQ \times \mathbb{R}$, are connected submanifolds of $TQ \times \mathbb{R}$.

In these cases, we have $(\mathcal{P}, \eta_{\mathcal{P}})$, where $\eta_{\mathcal{P}} = j_{\mathcal{P}}^*\eta \in \Omega^1(\mathcal{P})$, and $j_{\mathcal{P}}: \mathcal{P} \hookrightarrow T^*Q \times \mathbb{R}$ is the natural embedding. Furthermore, the Lagrangian energy function $E_{\mathcal{L}}$ is $\mathcal{FL}$ -projectable; i.e. there is a unique $\mathcal{H}_{\mathcal{P}} \in C^\infty(\mathcal{P})$ such that $E_{\mathcal{L}} = \mathcal{FL}_o^*\mathcal{H}_{\mathcal{P}}$, where $\mathcal{FL}_o: TQ \times \mathbb{R} \rightarrow \mathcal{P}$ is the restriction of $\mathcal{FL}$ to $\mathcal{P}$, defined by $\mathcal{FL} = j_{\mathcal{P}} \circ \mathcal{FL}_o$. Then, there exists a Hamiltonian formalism associated with the original Lagrangian system, which is developed on the submanifold $\mathcal{P}$, and the contact Hamiltonian equations for $X_{\mathcal{H}_{\mathcal{P}}} \in \mathfrak{X}(\mathcal{P})$ are (2) adapted to this situation or, equivalently,

$$\begin{cases} i(X_{\mathcal{H}_{\mathcal{P}}})\Omega_{\mathcal{P}} = 0, \\ i(X_{\mathcal{H}_{\mathcal{P}}})\eta_{\mathcal{P}} = -\mathcal{H}_{\mathcal{P}}, \end{cases} \quad (27)$$

where $\Omega_{\mathcal{P}} = -\mathcal{H}_{\mathcal{P}} d\eta_{\mathcal{P}} + d\mathcal{H}_{\mathcal{P}} \wedge \eta_{\mathcal{P}}$. As in the Lagrangian formalism, these equations are not necessarily compatible everywhere on $\mathcal{P}$ and a suitable *constraint algorithm* must be implemented in order to find a *final constraint submanifold* $P_f \hookrightarrow \mathcal{P}$ (if it exists) where there are vector fields $X_{\mathcal{H}_{\mathcal{P}}} \in \mathfrak{X}(\mathcal{P})$, tangent to P_f , which are (not necessarily unique) solutions to (27) on P_f . (See [14] for a detailed analysis on all these topics.)

3.5. Lagrangians with holonomic dissipation term

In a recent paper by Ciaglia *et al.* [13] a Lagrangian description for some systems with dissipation was given using a modification of Lagrangian formalism inspired by the contact Hamiltonian formalism. In this section, we will see that this description coincides with the general formalism studied in Sec. 3.2 when applied to a particular class of contact Lagrangians.

Definition 28. A *Lagrangian with holonomic dissipation term* in $TQ \times \mathbb{R}$ is a function $\mathcal{L} = L + \phi \in C^\infty(TQ \times \mathbb{R})$, where $L = \tau_1^* L_o$, for a Lagrangian function $L_o \in C^\infty(TQ)$, and $\phi = \tau_0^* \phi_o$, for $\phi_o \in C^\infty(Q \times \mathbb{R})$.

In coordinates, $\mathcal{L}(q^i, v^i, s) = L(q^i, v^i) + \phi(q^i, s)$. Observe that this implies that the momenta defined by the Legendre transformation are independent of the coordinate s . In addition, for these Lagrangians the conditions $\frac{\partial^2 \mathcal{L}}{\partial v^i \partial s} = 0$ hold. This motivates the name given in the definition.

Remark 29. The Lagrangian formalism presented in [13] is a little less general than the one we examine here, since there only the case $\phi = \phi(s)$ is considered.

Proposition 30. Let $\mathcal{L} = L + \phi$ be a Lagrangian with holonomic dissipation term. Then its Cartan 1-form, contact form, energy and Reeb vector field as a contact Lagrangian can be computed as

$$\theta_{\mathcal{L}} = \theta_L, \quad \eta_{\mathcal{L}} = ds - \theta_L, \quad E_{\mathcal{L}} = E_L - \phi, \quad \mathcal{R}_{\mathcal{L}} = \frac{\partial}{\partial s},$$

where θ_L is the Cartan 1-form of L considered (via pull-back) as a 1-form on $TQ \times \mathbb{R}$, and E_L is the energy of L as a function in the same way on $TQ \times \mathbb{R}$.

The Legendre map of $\mathcal{L}$, $\mathcal{FL}: TQ \times \mathbb{R} \rightarrow T^*Q \times \mathbb{R}$, can be expressed as $\mathcal{FL} = FL \times \text{Id}_{\mathbb{R}}$, where FL is the Legendre map of L . The Hessians are related by $\mathcal{F}^2 \mathcal{L}(v_q, s) = \mathcal{F}^2 L(v_q)$. Moreover, $\mathcal{L}$ is regular if, and only if, L is regular.

Proof. It is immediate in coordinates. In particular, the assertion about the Legendre map is a consequence of the fact that $\partial \mathcal{L} / \partial v^i = \partial L / \partial v^i$. In a similar way the relation between the Hessians can be expressed in coordinates as $\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} = \frac{\partial^2 L}{\partial v^i \partial v^j}$. This shows that $\mathcal{L}$ is regular if, and only if, L is regular. $\square$

Obviously $\mathcal{L}$ is hyperregular if, and only if, L also is. This means that the Legendre map $\mathcal{FL}$ is a diffeomorphism, and the canonical Hamiltonian formalism

for the Lagrangian with nonholonomic dissipation term can be formulated as in Sec. 3.4.

For the contact Lagrangian system $(TQ \times \mathbb{R}, \eta_{\mathcal{L}}, E_{\mathcal{L}})$ the dynamical equations for vector fields are

$$\begin{cases} i(X_{\mathcal{L}})d\eta_{\mathcal{L}} = dE_{\mathcal{L}} - (L_{\mathcal{R}_{\mathcal{L}}}E_{\mathcal{L}})\eta_{\mathcal{L}}, \\ i(X_{\mathcal{L}})\eta_{\mathcal{L}} = -E_{\mathcal{L}}. \end{cases}$$

In coordinates, writing $X_{\mathcal{L}} = g \frac{\partial}{\partial s} + f^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial v^i}$, the second Lagrangian equation for $X_{\mathcal{L}}$ reads, in coordinates,

$$\mathcal{L} + \frac{\partial \mathcal{L}}{\partial v^i} (f^i - v^i) - g = 0, \quad (28)$$

and this is Eq. (22) for $\mathcal{L} = L + \phi$. The first Lagrangian equation gives

$$(f^i - v^i) \frac{\partial^2 L}{\partial v^j \partial v^i} = 0, \quad (29)$$

and

$$\left(\frac{\partial^2 L}{\partial q^i \partial v^j} - \frac{\partial^2 L}{\partial q^j \partial v^i} \right) f^j + \frac{\partial^2 L}{\partial q^i \partial v^j} v^j - \frac{\partial^2 L}{\partial v^j \partial v^i} F^j = -\frac{\partial L}{\partial q^i} - \frac{\partial \phi}{\partial q^i} - \frac{\partial \phi}{\partial s} \frac{\partial L}{\partial v^i}, \quad (30)$$

which is Eq. (21) for $\mathcal{L}$. Observe that Eq. (19) are identities, since $\frac{\partial^2 \mathcal{L}}{\partial v^j \partial s} = 0$.

Finally, as in Proposition 24, if $\mathcal{L}$ is a regular Lagrangian then Eq. (29) implies that $f^i = v^i$, that is, $X_{\mathcal{L}}$ is a SODE, and the equations of motion become

$$g = \mathcal{L}, \quad (31)$$

$$\frac{\partial^2 L}{\partial v^j \partial v^i} \ddot{q}^j + \frac{\partial^2 L}{\partial q^j \partial v^i} \dot{q}^j - \frac{\partial L}{\partial q^i} = \frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = \frac{\partial \phi}{\partial q^i} + \frac{\partial \phi}{\partial s} \frac{\partial L}{\partial v^i}. \quad (32)$$

These are the expression in coordinates of the contact Euler–Lagrange equations.

4. Symmetries of Contact Hamiltonian and Lagrangian Systems

4.1. Symmetries for contact Hamiltonian systems

For a dynamical system, there are different concepts of symmetry which depend on which structure they preserve. Thus, one can consider the transformations that preserve the geometric structures of the system, or those that preserve its solutions [26, 37]. Next, we discuss these subjects for contact systems. (See also [15] for another complementary approach on these topics.)

Let $(M, \eta, \mathcal{H})$ be a contact Hamiltonian system with Reeb vector field $\mathcal{R}$, and $X_{\mathcal{H}}$ the contact Hamiltonian vector field for this system; that is, the solution to the Hamilton equations (2).

Definition 31. A *dynamical symmetry* of a contact Hamiltonian system is a diffeomorphism $\Phi: M \rightarrow M$ such that $\Phi_*X_{\mathcal{H}} = X_{\mathcal{H}}$ (it maps solutions into solutions).

An *infinitesimal dynamical symmetry* of a contact Hamiltonian system is a vector field $Y \in \mathfrak{X}(M)$ whose local flow is a dynamical symmetry; that is, $L_Y X_{\mathcal{H}} = [Y, X_{\mathcal{H}}] = 0$.

There are other kinds of symmetries that let the geometric structures invariant. They are the following.

Definition 32. A *contact symmetry* of a contact Hamiltonian system is a diffeomorphism $\Phi: M \rightarrow M$ such that

$$\Phi^*\eta = \eta, \quad \Phi_*\mathcal{H} = \mathcal{H}.$$

An *infinitesimal contact symmetry* of a contact Hamiltonian system is a vector field $Y \in \mathfrak{X}(M)$ whose local flow is a contact symmetry; that is,

$$L_Y\eta = 0, \quad L_Y\mathcal{H} = 0.$$

Furthermore we have the following.

Proposition 33. *Every (infinitesimal) contact symmetry preserves the Reeb vector field; that is, $\Phi^*\mathcal{R} = \mathcal{R}$ (or $[Y, \mathcal{R}] = 0$).*

Proof. We obtain that

$$i(\Phi_*^{-1}\mathcal{R})(\Phi^*d\eta) = \Phi^*(i(\mathcal{R})d\eta) = 0,$$

$$i(\Phi_*^{-1}\mathcal{R})(\Phi^*\eta) = \Phi^*(i(\mathcal{R})\eta) = 1,$$

and, as $\Phi^*\eta = \eta$ and the Reeb vector fields are unique, from these equalities we get $\Phi_*^{-1}\mathcal{R} = \mathcal{R}$. The proof for the infinitesimal case is immediate from the definition. $\square$

Finally, as a consequence of all of this we obtain the relation among contact symmetries and dynamical symmetries as follows.

Proposition 34. *(Infinitesimal) contact symmetries are (infinitesimal) dynamical symmetries.*

Proof. If $X_{\mathcal{H}}$ is the contact Lagrangian vector field, then

$$\begin{aligned} i(\Phi_*X_{\mathcal{H}})d\eta &= i(\Phi_*X_{\mathcal{H}})(\Phi^*d\eta) = \Phi^*(i(X_{\mathcal{H}})d\eta) \\ &= \Phi^*(d\mathcal{H} - (L_{\mathcal{R}}\mathcal{H})\eta) = d\mathcal{H} - (L_{\mathcal{R}}\mathcal{H})\eta, \\ i(\Phi_*X_{\mathcal{H}})\eta &= i(\Phi_*X_{\mathcal{H}})(\Phi^*\eta) = \Phi^*(i(X_{\mathcal{H}})\eta) = \Phi^*(-\mathcal{H}) = -\mathcal{H}. \end{aligned}$$

The proof for the infinitesimal case is immediate from the definition. $\square$

4.2. Dissipated and conserved quantities of contact Hamiltonian systems

Associated with symmetries of contact Hamiltonian systems are the concepts of *dissipated* and *conserved quantities* as follows.

Definition 35. A *dissipated quantity* of a contact Hamiltonian system is a function $F \in C^\infty(M)$ satisfying that

$$L_{X_{\mathcal{H}}}F = -(L_{\mathcal{R}}\mathcal{H})F. \quad (33)$$

For contact Hamiltonian systems, symmetries are associated with dissipated quantities as follows.

Theorem 36 (Dissipation theorem). *Let Y be a vector field on M . If Y is an infinitesimal dynamical symmetry ($[Y, X_H] = 0$), then the function $F = -i(Y)\eta$ is a dissipated quantity.*

Proof. This is a consequence of

$$\begin{aligned} L_{X_{\mathcal{H}}}F &= -L_{X_{\mathcal{H}}}i(Y)\eta = -i(Y)L_{X_{\mathcal{H}}}\eta - i(L_{X_H}Y)\eta \\ &= (L_{\mathcal{R}}\mathcal{H})i(Y)\eta + i([Y, X_H])\eta = -(L_{\mathcal{R}}\mathcal{H})F + i([Y, X_H])\eta \\ &= -(L_{\mathcal{R}}\mathcal{H})F, \end{aligned}$$

where we have applied the second assertion of Proposition 9. □

Remark 37. The last equality shows that, indeed, $[Y, X_{\mathcal{H}}] \in \text{Ker } \eta$ is a necessary and sufficient condition for F to be a dissipated quantity. This has been noted in [15] while this paper was under review. Nevertheless, it should be noted that such transformations are not dynamical symmetries in the sense of Definition 31, since they do not transform solutions into solutions, in general.

In particular, as it was established in (4), the Hamiltonian vector field $X_{\mathcal{H}}$ is trivially a symmetry and its dissipated quantity is the energy, $F = -i(X_{\mathcal{H}})\eta = \mathcal{H}$; that is shown in the following:

Theorem 38 (Energy dissipation theorem). $L_{X_{\mathcal{H}}}\mathcal{H} = -(L_{\mathcal{R}}\mathcal{H})\mathcal{H}$.

Remark 39. Observe that these are “non-conservation theorems”. As we are dealing with dissipative systems, dynamical symmetries are not associated with conserved quantities, but with dissipative quantities, and then these theorems account for the non-conservation of these quantities associated with the symmetries. In particular, the energy is not a conserved quantity, as it was early commented in Sec. 2.

Definition 40. A *conserved quantity* is a function $G: M \rightarrow \mathbb{R}$ satisfying that

$$L_{X_{\mathcal{H}}}G = 0.$$

Every dissipated quantity changes with the same rate $(-\mathcal{R}(\mathcal{H}))$, which suggests that the quotient of two dissipated quantities should be a conserved quantity. Indeed:

- Proposition 41.** (1) *If F_1 and F_2 are dissipated quantities and $F_2 \neq 0$, then F_1/F_2 is a conserved quantity.*
 (2) *If F is a dissipated quantity and G is a conserved quantity, then FG is a dissipated quantity.*

Proof. In fact, we have

$$\begin{aligned} L_{X_{\mathcal{H}}}(F_1/F_2) &= F_2^{-1}L_{X_{\mathcal{H}}}F_1 - F_1F_2^{-2}L_{X_{\mathcal{H}}}F_2 \\ &= -F_2^{-1}(L_{\mathcal{R}\mathcal{H}})F_1 + F_1F_2^{-2}(L_{\mathcal{R}\mathcal{H}})F_2 = 0, \\ L_{X_{\mathcal{H}}}(FG) &= GL_{X_{\mathcal{H}}}F + FL_{X_{\mathcal{H}}}G = -(L_{\mathcal{R}\mathcal{H}})FG. \end{aligned}$$

□

Remark 42. If $\mathcal{H} \neq 0$, it is possible to assign a conserved quantity to an infinitesimal dynamical symmetry Y . Indeed, from Theorem 36 and Proposition 41, the function $-i(Y)\eta/\mathcal{H}$ is a conserved quantity.

Finally, contact symmetries can be used to generate new dissipated quantities from a given dissipated quantity. In fact, as a straightforward consequence of definitions 35 and 31 we obtain the following

Proposition 43. *If $\Phi: M \rightarrow M$ is a contact symmetry and $F: M \rightarrow \mathbb{R}$ is a dissipated quantity, then so is Φ^*F .*

Proof. In fact, we have

$$L_{X_{\mathcal{H}}}(\Phi^*F) = \Phi^*L_{\Phi_*X_{\mathcal{H}}}F = \Phi^*L_{X_{\mathcal{H}}}F = \Phi^*(-L_{\mathcal{R}\mathcal{H}})F = -(L_{\mathcal{R}\mathcal{H}})(\Phi^*F).$$

The proof for the infinitesimal case is immediate from the definition. □

4.3. Symmetries for contact canonical Hamiltonian systems

Let $M = T^*Q \times \mathbb{R}$ be canonical contact structure with the contact form $\eta = ds - p_i dq^i$ (see Example 5).

Remember that, if $\varphi: Q \rightarrow Q$ is a diffeomorphism, we can construct the diffeomorphism $\Phi := (T^*\varphi, \text{Id}_{\mathbb{R}}): T^*Q \times \mathbb{R} \rightarrow T^*Q \times \mathbb{R}$, where $T^*\varphi: T^*Q \rightarrow T^*Q$ is the canonical lifting of φ to T^*Q . Then Φ is said to be the *canonical lifting* of φ to $T^*Q \times \mathbb{R}$. Any transformation Φ of this kind is called a *natural transformation* of $T^*Q \times \mathbb{R}$.

In the same way, given a vector field $Z \in \mathfrak{X}(Q)$ we can define its *complete lifting* to $T^*Q \times \mathbb{R}$ as the vector field $Y \in \mathfrak{X}(T^*Q \times \mathbb{R})$ whose local flow is the canonical lifting of the local flow of Z to $T^*Q \times \mathbb{R}$; that is, the vector field $Y = Z^{C*}$, where Z^{C*} denotes the complete lifting of Z to T^*Q , identified in a natural way as a

vector field in $T^*Q \times \mathbb{R}$. Any infinitesimal transformation Y of this kind is called a *natural infinitesimal transformation* of $T^*Q \times \mathbb{R}$.

It is well known that the canonical forms θ_o and $\omega_o = -d\theta_o$ in T^*Q are invariant under the action of canonical liftings of diffeomorphisms and vector fields from Q to T^*Q . Then, taking into account the definition of the contact form η in $T^*Q \times \mathbb{R}$, as an immediate consequence of the above considerations we have:

Proposition 44. *If $\Phi \in \text{Diff}(T^*Q \times \mathbb{R})$ (respectively, $Y \in \mathfrak{X}(T^*Q \times \mathbb{R})$) is a canonical lifting to $T^*Q \times \mathbb{R}$ of a diffeomorphism $\varphi \in \text{Diff}(Q)$ (respectively, of $Z \in \mathfrak{X}(Q)$), then*

- (1) $\Phi^*\eta = \eta$ (respectively, $L_Y\eta = 0$).
- (2) *If, in addition, $\Phi^*\mathcal{H} = \mathcal{H}$ (respectively, $L_Y\mathcal{H} = 0$), then it is a (infinitesimal) contact symmetry.*

In particular, we have the following.

Theorem 45 (Momentum dissipation theorem). *If $\frac{\partial \mathcal{H}}{\partial q^i} = 0$, then $\frac{\partial}{\partial q^i}$ is an infinitesimal contact symmetry, and its associated dissipated quantity is the corresponding momentum p_i ; that is, $L_{X_{\mathcal{H}}}p_i = -(L_{\mathcal{R}}\mathcal{H})p_i$.*

Proof. A simple computation in local coordinates shows that $L(\frac{\partial}{\partial q^i})\eta = 0$ and $L(\frac{\partial}{\partial q^i})\mathcal{H} = 0$. Therefore, it is a contact symmetry and, in particular, a dynamical symmetry. The other results are a consequence of the dissipation theorem. $\square$

4.4. Symmetries for contact Lagrangian systems

Consider a regular contact Lagrangian system $(TQ \times \mathbb{R}, \mathcal{L})$, with Reeb vector field $\mathcal{R}_{\mathcal{L}}$, and let $X_{\mathcal{L}}$ be the contact Euler–Lagrange vector field for this system; that is, the solution to the Lagrangian equations (14).

All we have said about symmetries and dissipated quantities for contact Hamiltonian systems holds when it is applied to the contact system $(TQ \times \mathbb{R}, \eta_{\mathcal{L}}, E_{\mathcal{L}})$. Thus we have the same definitions for dynamical and contact symmetries and the dissipation theorem states that $-i(Y)\eta_{\mathcal{L}}$ is a dissipated quantity, for every infinitesimal dynamical symmetry Y . In particular, the energy dissipation theorem says that

$$L_{X_{\mathcal{L}}}E_{\mathcal{L}} = -(L_{\mathcal{R}_{\mathcal{L}}}E_{\mathcal{L}})E_{\mathcal{L}}.$$

In the same way as for $T^*Q \times \mathbb{R}$, if $\varphi: Q \rightarrow Q$ is a diffeomorphism, we can construct the diffeomorphism $\Phi := (T\varphi, \text{Id}_{\mathbb{R}}): TQ \times \mathbb{R} \rightarrow TQ \times \mathbb{R}$, where $T\varphi: TQ \rightarrow TQ$ is the canonical lifting of φ to TQ . Then Φ is said to be the *canonical lifting* of φ to $TQ \times \mathbb{R}$. Any transformation Φ of this kind is called a *natural transformation* of $TQ \times \mathbb{R}$.

Moreover, given a vector field $Z \in \mathfrak{X}(TQ)$ we can define its *complete lifting* to $TQ \times \mathbb{R}$ as the vector field $Y \in \mathfrak{X}(TQ \times \mathbb{R})$ whose local flow is the canonical lifting of the local flow of Z to $TQ \times \mathbb{R}$; that is, the vector field $Y = Z^C$, where

Z^C denotes the complete lifting of Z to TQ , identified in a natural way as a vector field in $TQ \times \mathbb{R}$. Any infinitesimal transformation Y of this kind is called a *natural infinitesimal transformation* of $TQ \times \mathbb{R}$.

It is well known that the vertical endomorphism J and the Liouville vector field Δ_o in TQ are invariant under the action of canonical liftings of diffeomorphisms and vector fields from Q to TQ . Then, taking into account the definitions of the canonical endomorphism $\mathcal{J}$ and the Liouville vector field Δ in $TQ \times \mathbb{R}$, it can be proved that canonical liftings of diffeomorphisms and vector fields from Q to $TQ \times \mathbb{R}$ preserve these canonical structures as well as the Reeb vector field $\mathcal{R}$.

Therefore, as an immediate consequence, we obtain a relation between Lagrangian-preserving natural transformations and contact symmetries:

Proposition 46. *If $\Phi \in \text{Diff}(TQ \times \mathbb{R})$ (respectively, $Y \in \mathfrak{X}(TQ \times \mathbb{R})$) is a canonical lifting to $TQ \times \mathbb{R}$ of a diffeomorphism $\varphi \in \text{Diff}(Q)$ (respectively, of a vector field $Z \in \mathfrak{X}(Q)$) that leaves the Lagrangian invariant, then it is a (infinitesimal) contact symmetry, i.e.*

$$\Phi^* \eta_{\mathcal{L}} = \eta_{\mathcal{L}}, \quad \Phi^* E_{\mathcal{L}} = E_{\mathcal{L}} \quad (\text{respectively, } L_Y \eta_{\mathcal{L}} = 0, L_Y E_{\mathcal{L}} = 0).$$

As a consequence, it is a (infinitesimal) dynamical symmetry.

As a consequence we have a result similar to the momentum dissipation Theorem 45: if $\frac{\partial \mathcal{L}}{\partial q^i} = 0$, then $\frac{\partial}{\partial q^i}$ is an infinitesimal contact symmetry and its associated dissipated quantity is the momentum $\frac{\partial \mathcal{L}}{\partial v^i}$; that is,

$$L_{X_{\mathcal{L}}} \left(\frac{\partial \mathcal{L}}{\partial v^i} \right) = -(L_{\mathcal{R}_{\mathcal{L}}} E_{\mathcal{L}}) \frac{\partial \mathcal{L}}{\partial v^i} = \frac{\partial \mathcal{L}}{\partial s} \frac{\partial \mathcal{L}}{\partial v^i}.$$

Remark 47. A similar problem is considered in [22], where the used dissipation factor is $\frac{\partial \mathcal{L}}{\partial s}$ which, as we have seen in Eq. (23), is the same that we have obtained.

4.5. Symmetries of a contactified system

The dissipation theorem (Theorem 36) yields a dissipated quantity from a dynamical symmetry Y with no additional hypotheses. This is in contrast to Noether symmetries, where the generator of the symmetry has to satisfy some additional hypotheses in order to yield a conserved quantity. We want to understand this different behavior.

So, let us consider a Hamiltonian system (P, ω, H_o) on an exact symplectic manifold P , with symplectic form $\omega = -d\theta$, and Hamiltonian function $H_o : P \rightarrow \mathbb{R}$. Its associated Hamiltonian vector field $X_o \in \mathfrak{X}(P)$ is defined by $i_{X_o} \omega = dH_o$.

The contactified of (P, ω) is the contact manifold (M, η) , where $M = P \times \mathbb{R}$ is endowed with the contact form $\eta = ds - \theta$; here s is the Cartesian coordinate of $\mathbb{R}$, and we use θ for the pull-back of the 1-form to the product (see Example 6).

The pull-back of $H_\circ$ to M defines a contact Hamiltonian function $H = H_\circ$ on M . The corresponding contact Hamiltonian vector field can be written $X = X_\circ + \ell \frac{\partial}{\partial s}$, where $X_\circ$ is the Hamiltonian vector field of $H_\circ$ as a vector field on the product manifold, and $\ell = \langle \theta, X_\circ \rangle - H_\circ$.

Now, let $Y_\circ \in \mathfrak{X}(P)$ be a vector field, and construct $Y = Y_\circ + b \frac{\partial}{\partial s}$, with $b: P \rightarrow \mathbb{R}$ a function.

Lemma 48. *The vector field Y is a dynamical symmetry of X ($L_Y X = 0$) if, and only if, $Y_\circ$ is a dynamical symmetry of $X_\circ$ ($L_{Y_\circ} X_\circ = 0$) and $L_Y \ell = L_X b$.*

Proof. The proof is a consequence of a direct calculation

$$[Y, X] = [Y_\circ, X_\circ] + (L_Y \ell - L_X b) \frac{\partial}{\partial s}. \quad \square$$

Now let us consider the quantity

$$G = -i(Y)\eta = -i(Y_\circ)\eta - b.$$

We have

$$\begin{aligned} L_X G &= -i(Y)L_X \eta - i([X, Y])\eta \\ &= (L_{\mathcal{R}} H)i(Y)\eta - i(Y)(L_X \eta + (L_{\mathcal{R}} H)\eta) + i([Y, X])\eta \\ &= (L_{\mathcal{R}} H)i(Y)\eta - i(Y)(L_X \eta + (L_{\mathcal{R}} H)\eta) + i([Y_\circ, X_\circ])\eta + (L_Y \ell - L_X b). \end{aligned}$$

Let us analyze the vanishing of these summands: the first one is zero because $H = H_\circ$ does not depend on s , the second one also is because X is the contact Hamiltonian vector field, and, according to the lemma, the third and forth ones vanish if $L_Y X = 0$. So, we conclude that, if Y is a dynamical symmetry of X , G is a conserved quantity for the contact dynamics X (conserved rather than dissipated, since H does not depend on s). Now, note that

$$L_X G = L_{X_\circ} i(Y_\circ)\theta - L_X b.$$

So, we arrive at the following interpretation: when $Y_\circ$ is a dynamical symmetry of $X_\circ$, the function $G_\circ = i(Y)\theta$ is not necessarily a conserved quantity of $X_\circ$ (the hypothesis of Noether's theorem are not necessarily satisfied). However, in the contactified Hamiltonian system, with Y a dynamical symmetry projectable to $Y_\circ$,

$$L_X G_\circ = L_X(G + b) = L_X b = L_Y \ell,$$

and the time-derivative of $G_\circ$ is compensated by the time-derivative of b .

When is $G_\circ$ conserved by $X_\circ$? When $L_{Y_\circ} \ell = 0$. This happens, for instance, when $Y_\circ$ is an exact Noether symmetry, i.e. when $L_{Y_\circ} \theta = 0$ and $L_{Y_\circ} H_\circ = 0$, since

$$L_{Y_\circ} \ell = L_{Y_\circ} (i(X_\circ)\theta - H_\circ) = 0.$$

5. Examples

5.1. The damped harmonic oscillator

The Lagrangian description of the one-dimensional harmonic oscillator is given by $Q = \mathbb{R}$ and $L = \frac{1}{2}mv^2 - \frac{1}{2}m\omega^2q^2$. The Euler–Lagrange equation gives

$$\ddot{q} + \omega^2q = 0.$$

Taking $M = \mathbb{R} \times \mathbb{R}$ and the contact form $\eta = ds - v dq$, we can introduce the Lagrangian $\mathcal{L} = L + \phi$, with holonomic dissipation term $\phi = -\gamma s$, and thus $E_{\mathcal{L}} = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2q^2 + \gamma s$. We have as dynamical equation

$$\ddot{q} = -\omega^2q - \gamma\dot{q}$$

corresponding to the dynamical equation of the damped harmonic oscillator. Therefore, the contact Lagrangian vector field is

$$\Gamma_{\mathcal{L}} = \mathcal{L} \frac{\partial}{\partial s} + v \frac{\partial}{\partial q} - (\omega^2q + \gamma v) \frac{\partial}{\partial v}.$$

The dissipation of the energy is given by Theorem 38,

$$L_{\Gamma_{\mathcal{L}}} E_{\mathcal{L}} = -\gamma E_{\mathcal{L}}.$$

5.2. Motion in a constant gravitational field with friction

Consider the motion of a particle in a vertical plane under the action of constant gravity; then $Q = \mathbb{R}^2$ with coordinates (x, y) . This motion is described by the Lagrangian $L = \frac{1}{2}mv^2 - mgy$, where $v^2 = v_x^2 + v_y^2$ in TQ with coordinates (x, y, v_x, v_y) .

To introduce air friction, we consider the Lagrangian with holonomic dissipation term $\mathcal{L} = L - \gamma s$ in $M = TQ \times \mathbb{R}$, with coordinates (x, y, v_y, v_x, s) . We have

$$\theta_{\mathcal{L}} = mv_x dx + mv_y dy,$$

$$\eta_{\mathcal{L}} = ds - \theta_{\mathcal{L}} = ds - mv_x dx - mv_y dy,$$

$$d\eta_{\mathcal{L}} = m dx \wedge dv_x + m dy \wedge dv_y,$$

$$E_{\mathcal{L}} = \frac{1}{2}mv^2 + mgy + \gamma s,$$

$$\mathcal{R}_{\mathcal{L}} = \frac{\partial}{\partial s}.$$

The dynamical equations are

$$i(X)d\eta_{\mathcal{L}} = dE_{\mathcal{L}} - \mathcal{R}_{\mathcal{L}}(E_{\mathcal{L}})\eta_{\mathcal{L}}, \quad (34)$$

$$i(X)\eta = -E_{\mathcal{L}}. \quad (35)$$

Writing $X \in \mathfrak{X}(\text{T}Q \times \mathbb{R})$ as

$$X = a \frac{\partial}{\partial s} + b \frac{\partial}{\partial x} + c \frac{\partial}{\partial y} + d \frac{\partial}{\partial v_x} + e \frac{\partial}{\partial v_y},$$

and using that

$$dE_{\mathcal{L}} = mv_x dv_x + mv_y dv_y + mg dy + \gamma ds$$

and that

$$\mathcal{R}_{\mathcal{L}}(E_{\mathcal{L}})\eta_{\mathcal{L}} = \gamma ds - \gamma mv_x dx - \gamma mv_y dy,$$

from Eq. (35) we get that $a = bv_x + cv_y - E_{\mathcal{L}}$, while Eq. (34) gives the following conditions:

$$b = v_x, \quad c = v_y, \quad d = -\gamma v_x, \quad e = -\gamma v_y - g.$$

The first two conditions imply that $a = \mathcal{L}$. Summing up, the contact Lagrangian vector field is

$$\Gamma_{\mathcal{L}} = \mathcal{L} \frac{\partial}{\partial s} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} - \gamma v_x \frac{\partial}{\partial v_x} - (g + \gamma v_y) \frac{\partial}{\partial v_y}.$$

Hence, we get the following system of differential equations:

$$\begin{cases} \ddot{x} + \gamma \dot{x} = 0, \\ \ddot{y} + \gamma \dot{y} + g = 0, \\ \dot{s} = \mathcal{L}. \end{cases}$$

As in the previous example, the energy dissipation is given by Theorem 38

$$L_{\Gamma_{\mathcal{L}}} E_{\mathcal{L}} = -\gamma E_{\mathcal{L}}.$$

We also have that $\frac{\partial \mathcal{L}}{\partial x} = 0$, thus it is immediate to check that $\frac{\partial}{\partial x}$ is a contact symmetry. The associated dissipated quantity is its corresponding momentum: $p^x = \frac{\partial \mathcal{L}}{\partial v_x} = mv_x$. The dissipation of this quantity is given by Theorem 45

$$L_X p^x = -\gamma p^x.$$

Finally, as an application of Proposition 41 we have the following conserved quantity:

$$\frac{E_{\mathcal{L}}}{p^x} = \frac{\frac{1}{2}mv^2 + mgy + \gamma s}{mv_x}.$$

5.3. Parachute equation

Consider the vertical motion of a particle falling in a fluid under the action of constant gravity. If the friction is modeled by the drag equation, the force is

proportional to the square of the velocity. This motion can be described as a contact dynamical system in $M = \mathbb{T}\mathbb{R} \times \mathbb{R}$, with coordinates (y, v, s) , by the Lagrangian function

$$\mathcal{L} = \frac{1}{2}mv^2 - \frac{mg}{2\gamma}(e^{2\gamma y} - 1) + 2\gamma vs,$$

where γ is a friction coefficient depending on the density of the air, the shape of the object, etc. Note that this is *not* a Lagrangian with holonomic dissipation term. We have

$$\begin{aligned}\theta_{\mathcal{L}} &= (mv + 2\gamma s)dy, \\ \eta_{\mathcal{L}} &= ds - \theta_{\mathcal{L}} = ds - (mv + 2\gamma s)dy, \\ d\eta_{\mathcal{L}} &= m\,dy \wedge dv + 2\gamma\,dy \wedge ds, \\ E_{\mathcal{L}} &= \frac{1}{2}mv^2 + \frac{mg}{2\gamma}(e^{2\gamma y} - 1), \\ \mathcal{R}_{\mathcal{L}} &= \frac{\partial}{\partial s} - \frac{2\gamma}{m}\frac{\partial}{\partial v}.\end{aligned}$$

Now, we want to write the contact dynamical equations (14). Writing the vector field as

$$X = a\frac{\partial}{\partial s} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial v} \in \mathfrak{X}(\mathbb{T}\mathbb{R} \times \mathbb{R}),$$

and using

$$\begin{aligned}dE_{\mathcal{L}} &= mv\,dv + mg\,e^{2\gamma y}dy, \\ \mathcal{R}_{\mathcal{L}}(E_{\mathcal{L}})\eta_{\mathcal{L}} &= -2\gamma v(ds - (mv + 2\gamma s)dy),\end{aligned}$$

we derive the following conditions:

$$\begin{cases} a = (mv + 2\gamma)b - E_{\mathcal{L}}, \\ b = v, \\ c = -g\,e^{2\gamma y} - \frac{2\gamma}{m}a + 2\gamma v^2 + \frac{4\gamma^2}{m}vs. \end{cases}$$

The first two imply that $a = \mathcal{L}$. Using this, the third one becomes $c = -g + \gamma v^2$. Summing up we have that the contact Lagrangian vector field is

$$\Gamma_{\mathcal{L}} = v\frac{\partial}{\partial y} + (\gamma v^2 - g)\frac{\partial}{\partial v} + \mathcal{L}\frac{\partial}{\partial s}.$$

Hence, we get the following system of differential equations

$$\begin{cases} \ddot{y} - \gamma\dot{y}^2 + g = 0, \\ \dot{s} = \mathcal{L}. \end{cases} \quad (36)$$

As in the previous example, the energy dissipation is given by Theorem 38,

$$L_{\Gamma_{\mathcal{L}}}E_{\mathcal{L}} = 2\gamma v E_{\mathcal{L}}.$$

Remark 49. Equation (36) describes an object falling ($\dot{y} < 0$). To describe an object ascending it suffices to change γ for $-\gamma$ in the Lagrangian.

6. Conclusions and Outlook

We have studied the geometric formulation of the Lagrangian and Hamiltonian formalisms of dissipative mechanical systems using contact geometry. While contact Hamiltonian systems are well known, contact Lagrangian systems have been introduced very recently [13, 14]. Within this study, we have given a new expression of the dynamical equations of a contact Hamiltonian system without the use of the Reeb vector field. This setting would be especially useful in the case of singular contact systems, in which we have a pre-contact structure and then Reeb vector fields are not uniquely determined.

We have reviewed the Lagrangian formalism of contact systems, which takes place in the bundle $TQ \times \mathbb{R}$. The most general framework for this formalism is established in [14], while in [13] a different formalism is presented and applied to a particular type of Lagrangians with a dissipation term, which is very common in physical applications. Both formulations have been compared, studying their equivalence.

We have analyzed the concept of symmetry for dissipative systems introducing different kinds of them: for contact Hamiltonian systems in general, those preserving the dynamics and those preserving the contact structure and the Hamiltonian function, and, for contact Lagrangian systems, the natural symmetries preserving the Lagrangian function. The relation among them has been established.

We have introduced the notion of dissipated quantity and, as the most relevant result, we have proved a *dissipation theorem*, which establishes a relationship between infinitesimal symmetries and dissipated quantities. This is analogous to the Noether theorem for conservative systems. We have also proved that the quotient of two dissipated quantities is a conserved quantity.

Finally, we have also compared the notion of symmetries for symplectic dynamical systems and dissipative contactified systems, showing that the requirements to be a dynamical symmetry are substantially different.

As some examples we have discussed several common models in physics. We have proposed contact Lagrangian systems which lead to the standard dynamical equations for the damped harmonic oscillator, the motion in a gravitational field with friction, and the parachute motion.

Further research on these topics could include the study of the nonautonomous case in the Hamiltonian and Lagrangian formalisms, the characterization of equivalent Lagrangians, or to pursue the analysis of the geometric structures in the

singular case. On the other hand, it would be interesting to develop a geometric framework for Lagrangian field theories with dissipation, expanding the results recently obtained in [19].

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References

- [1] R. Abraham and J. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, 1978).
- [2] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, 1978).
- [3] A. Banyaga and D. F. Houenou, *A Brief Introduction to Symplectic and Contact Manifolds* (World Scientific, 2016), <https://doi.org/10.1142/9667>.
- [4] A. Bravetti, Contact Hamiltonian dynamics: The concept and its use, *Entropy* **10**(19) (2017) 535, <https://doi.org/10.3390/e19100535>.
- [5] A. Bravetti, Contact geometry and thermodynamics, *Int. J. Geom. Methods Mod. Phys.* **16**(suppl1) (2019) 1940003, <https://doi.org/10.1142/S0219887819400036>.
- [6] A. Bravetti, H. Cruz and D. Tapias, Contact Hamiltonian mechanics, *Ann. Phys.* **376** (2017) 17–39, <https://doi.org/10.1016/j.aop.2016.11.003>.
- [7] A. J. Bruce, K. Grabowska and J. Grabowski, Remarks on contact and Jacobi Geometry, *Symm. Integ. Geom. Meth. Appl. (SIGMA)* **13** (2017) 059, <https://doi.org/10.3842/SIGMA.2017.059>.
- [8] B. Cappelletti-Montano, A. De Nicola and I. Yudin, A survey on cosymplectic geometry, *Rev. Math. Phys.* **25**(10) (2013) 1343002, <https://doi.org/10.1142/S0129055X13430022>.
- [9] J. Cariñena, M. Crampin and L. A. Ibort, On the multisymplectic formalism for first order field theories, *Differ. Geom. Appl.* **1**(4) (1991) 345–374, [https://doi.org/10.1016/0926-2245\(91\)90013-Y](https://doi.org/10.1016/0926-2245(91)90013-Y).
- [10] J. Cariñena and P. Guha, Nonstandard Hamiltonian structures of the Liénard equation and contact geometry, *Int. J. Geom. Methods Mod. Phys.* **16**(suppl 1) (2019) 1940001, <https://doi.org/10.1142/S0219887819400012>.
- [11] D. Chinea, M. de León and J. C. Marrero, Locally conformal cosymplectic manifolds and time-dependent Hamiltonian systems, *Comment. Math. Univ. Carolin.* **32**(2) (1991) 383–387.
- [12] D. Chinea, M. de León and J. C. Marrero, The constraint algorithm for time-dependent Lagrangians, *J. Math. Phys.* **35**(7) (1994) 3410–3447, <https://doi.org/10.1063/1.530476>.
- [13] F. Ciaglia, H. Cruz and G. Marmo, Contact manifolds and dissipation, classical and quantum, *Ann. Phys.* **398** (2018) 159–179, <https://doi.org/10.1016/j.aop.2018.09.012>.

- [14] M. de León and M. Lainz-Valcázar, Singular Lagrangians and precontact Hamiltonian Systems, *Int. J. Geom. Methods Mod. Phys.* **16**(10) (2019) 1950158, <https://doi.org/10.1142/S0219887819501585>.
- [15] M. de León and M. Lainz-Valcázar, Infinitesimal symmetries in contact Hamiltonian systems, *J. Geom. Phys.* **153** (2020) 103651, <https://doi.org/10.1016/j.geomphys.2020.103651>.
- [16] M. de León, M. Salgado and S. Vilariño, *Methods of Differential Geometry in Classical Field Theories: k-Symplectic and k-Cosymplectic Approaches* (World Scientific, 2016).
- [17] M. de León and C. Sardón, Cosymplectic and contact structures to resolve time-dependent and dissipative Hamiltonian systems, *J. Phys. A: Math. Theor.* **50**(25) (2017) 255205, <https://doi.org/10.1088/1751-8121/aa711d>.
- [18] C. R. Galley, Classical mechanics of nonconservative systems, *Phys. Rev. Lett.* **110**(17) (2013) 174301, <https://doi.org/10.1103/PhysRevLett.110.174301>.
- [19] J. Gaset, X. Gràcia, M. Muñoz-Lecanda, X. Rivas and N. Román-Roy, A contact geometry framework for field theories with dissipation, *Ann. Phys.* **414** (2020) 168092, <https://doi.org/10.1016/j.aop.2020.168092>.
- [20] J. Gaset, X. Gràcia, M. Muñoz-Lecanda, X. Rivas and N. Román-Roy, A k -contact Lagrangian formulation for nonconservative field theories, arXiv: 2002.10458 [math-ph].
- [21] H. Geiges, *An Introduction to Contact Topology* (Cambridge University Press, 2008).
- [22] B. Georgieva and R. Guenther, First Noether-type theorem for the generalized variational principle of Herglotz, *Topol. Methods Nonlinear Anal.* **20**(2) (2002) 261–273.
- [23] M. J. Gotay and J. M. Nester, Presymplectic Lagrangian systems I: The constraint algorithm and the equivalence theorem, *Ann. Inst. Henri Poincaré* **30**(2) (1979) 129–142.
- [24] S. Goto, Contact geometric descriptions of vector fields on dually flat spaces and their applications in electric circuit models and nonequilibrium statistical mechanics, *J. Math. Phys.* **57**(10) (2016) 102702, <https://doi.org/10.1063/1.4964751>.
- [25] X. Gràcia, Fibre derivatives: Some applications to singular Lagrangians, *Rep. Math. Phys.* **45**(1) (2000) 67–84, [https://doi.org/10.1016/S0034-4877\(00\)88872-2](https://doi.org/10.1016/S0034-4877(00)88872-2).
- [26] X. Gràcia and J. M. Pons, Symmetries and infinitesimal symmetries of singular differential equations, *J. Phys. A Math. Gen.* **35**(24) (2002) 5059–5077, <https://doi.org/10.1088/0305-4470/35/24/306>.
- [27] G. Herglotz, *Berührungstransformationen*, Lectures at the University of Gottingen (1930).
- [28] G. Herglotz, *Vorlesungen über die Mechanik der Kontinua*, Teubner-Archiv zur Mathematik, Vol. 3 (Teubner, Leipzig, 1985).
- [29] A. L. Kholodenko, *Applications of Contact Geometry and Topology in Physics* (World Scientific, 2013).
- [30] M. Lainz Valcázar and M. de León, Contact Hamiltonian systems, *J. Math. Phys.* **60**(10) (2019) 102902, <https://doi.org/10.1063/1.5096475>.
- [31] P. Libermann and C. Marle, *Symplectic Geometry and Analytical Mechanics* (D. Reidel, 1987).
- [32] Q. Liu, P. J. Torres and C. Wang, Contact Hamiltonian dynamics: Variational principles, invariants, completeness and periodic behavior, *Ann. Phys.* **395** (2018) 26–44, <https://doi.org/10.1016/j.aop.2018.04.035>.
- [33] N. E. Martínez-Pérez and C. Ramírez, On the Lagrangian description of dissipative systems, *J. Math. Phys.* **59**(3) (2018) 032904, <https://doi.org/10.1063/1.5004796>.
- [34] H. Ramírez, B. Maschke and D. Sbarbaro, Partial stabilization of input-output contact systems on a Legendre submanifold, *IEEE Trans. Automat. Control* **62**(3) (2017)

1431–1437, <https://doi.org/10.1109/TAC.2016.2572403>.

- [35] M. Razavy, *Classical and Quantum Dissipative Systems* (Imperial College Press, 2006).
- [36] A. M. Rey, N. Román-Roy, M. Salgado and S. Vilariño, On the k -symplectic, k -cosymplectic and multisymplectic formalisms of classical field theories, *J. Geom. Mech.* **3**(1) (2011) 113–137, <https://doi.org/10.3934/jgm.2011.3.113>.
- [37] N. Román-Roy, A summary on symmetries and conserved quantities of autonomous Hamiltonian systems, to appear in *J. Geom. Mech.* (2020), <https://doi.org/10.3934/jgm.2020009>.
- [38] M. Visinescu, Contact Hamiltonian systems and complete integrability, *AIP Conf. Proc.* **1916**(1) (2017) 020002.