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Practical Introduction to Action-Dependent Field Theories

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Action-dependent field theories are systems where the Lagrangian or Hamiltonian depends on new variables that encode the action. They model a larger class of field theories, including non-conservative behavior, while maintaining a well-defined notion of symmetries and a Noether theorem. This makes them especially suited for open systems. After a conceptual introduction, a quick presentation of a new mathematical framework is made for action-dependent field theory: *multicontact geometry*. The formalism is illustrated with a variety of action-dependent Lagrangians, some of which are regular and others singular, derived from well-known theories whose Lagrangians have been modified to incorporate action-dependent terms. Detailed computations are provided, including the constraint algorithm for the singular cases, in both the Lagrangian and Hamiltonian formalisms. These are the one-dimensional wave equation, the Klein–Gordon equation and the telegrapher equation, Maxwell's electromagnetism, Metric-affine gravity, the heat equation and Burgers' equation, the Bosonic string theory, and (2 + 1)-dimensional gravity and Chern–Simons equation.

1. Introduction

An action-dependent field theory is a system where the Lagrangian or the Hamiltonian depends on new variables that represent the flow of action. They are a natural generalization of classical field theories with an array of appealing properties that make them exceptionally interesting.

To begin with, action-dependent field theories provide terms in the field equations that cannot be generated with standard classical field theories, even considering the addition of new particles. These terms have been used to model dissipative behavior, but their vast potential goes far beyond. Moreover, the new variables related to the action create a bridge between the boundary and the bulk.

In addition, action-dependent field theories are nonconservative, which makes them especially suited for open systems. Remarkably, the conservation failure is tightly controlled. Noether's theorem relates symmetries to dissipated quantities, which are not conserved but have a very specific behavior. This is encoded by a differential 1-form (called dissipation form) that is easily computed from the Lagrangian or Hamiltonian, giving a clear control over the nonconservation of the system.

Importantly, action-dependent field theories are grounded in a rigorous geometric formulation: *multicontact geometry*. A geometric description also provides the generality to incorporate singular and higher-order theories, as well as the underlying bracket structure, a prerequisite for quantization. The benefits of such a mathematical approach are profuse, as evidenced by the successful history of symplectic geometry.

Symplectic Mechanics

Mechanics experienced a drastic change as soon as it was able to use symplectic geometry in its description. This occurred in the 50's and 60's of the last century, and made it possible to obtain Hamilton's equations as the integral curves of a vector field on a symplectic manifold, in fact, in the cotangent bundle of the configuration space of the system. This implied a liberation from coordinates and the possibility of obtaining the usual properties of mechanical systems (conservation of energy, other conserved quantities, integration, …) in a simple and elegant way. But it was not only a different way of approaching the study of mechanical systems, as this formulation served to go further in this task. For example, the definition of the momentum map made possible to create a bridge between geometry and dynamics with algebra and combinatorics through the use of polytopes,[1] obtained as images of the phase space in the dual of the Lie algebra of the Lie group of symmetries of the system.

Another subject in which symplectic geometry has been used to the benefit of mechanics is integration problems, by exploiting the symmetries of the system in the so-called *Meyer–Marsden–Weinstein Reduction Theorem*.[2] Or, for example, the development

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of the *Hamilton–Jacobi equation*,[3] which has led to a better understanding of the problem and extensions of the procedure to more general situations. It has also been possible to study integrability problems and to develop the *Liouville–Arnold Theorem*,[4] with the corresponding action-angle coordinates. Other important developments achieved through this geometric vision are the applications to numerical integration (geometric integrators), the study of *Noether's Theorem* and its generalizations developing the different types of infinitesimal symmetries and the corresponding conserved quantities,[5] or the applications to geometric quantization.[6] Another subject that has received a boost from the geometric description is systems subject to nonholonomic constraints,[7] which have been extensively studied since the 1980s in this new light. This has made it possible, for example, to obtain an appropriate Hamilton–Jacobi equation.

The above items are just some of the milestones that symplectic geometry has achieved in the study of what is now called geometric mechanics. There are countless references to these results, that is why we cite a series of books (some of them already classics) where these developments and the original references to them can be found.[3,4,8–13]

Contact Mechanics and Dissipation

Recently, interest has grown in the use of *contact geometry* in the description of mechanical systems that experience dissipation phenomena. Although this was known and used in the case of, for example, thermodynamics, it has now been used in the general context, and this geometry has been identified as a particular case of Jacobi structures. These types of systems are what in physics and in their Lagrangian version are called action-dependent Lagrangian systems. The interest in this description has led to the publication of numerous papers that extend to this scenario what had already been done in the symplectic case. For a motivated introduction see refs. [14, 15]. Contact Hamiltonian systems have been extensible studied[16–19] and the description of contact Lagrangian systems can be found in ref. [20]. For contact systems with an explicit time dependence, see refs. [21–24]. For the quantization of contact systems, we recommend ref. [25]. Of course, there is a variational description that comes back to the original papers by Herglotz and generalizes the well-known Hamilton's principle.[26,27] Generalities about contact geometry can be found in refs. [8, 10, 28–30]).

One of the key contributions of both symplectic and contact geometry is that in both cases a bracket of (observable) functions is obtained, which in the first case is Poisson, and in the second Jacobi, the difference being precisely that the Jacobi bracket does not satisfy Leibniz's identity (in other words, it is not a derivation in its arguments but a first-order differential operator). Let us not forget that the existence of a function bracket is essential for the quantization process.

Multisymplectic Field Theories

At the end of the 60's of the last century, something similar happened in the case of classical field theories. Three groups of physicist-mathematicians independently developed a formalism called *multisymplectic*, which sought to extend the symplectic of mechanics to this case.[31–33] The difficulty of this new geometry is that, while symplectic geometry is very rigid (it is always locally equivalent to the canonical of a cotangent bundle via the *Darboux Theorem*), the situation is very different in the multisymplectic case. In the last 50 years, much effort has been made to achieve progress in this direction, and despite many achievements, a theory as satisfactory as for mechanics has not yet been achieved, being still a field of research in full development. For Lagrangian formalism of classical field theories we refer to refs. [31, 32, 34–40], and for the Hamiltonian formalism, see refs. [39–50]. These are only some references, since the literature about this subject is really large. In any case, a brief historical account on the subject can be found in ref. [51].

Multicontact Field Theories

In the same way that there are mechanical systems with dissipation, the same occurs in the case of classical field theories. Echoing contact geometry, this leads to action-dependent field theory, that is, field theories where the Lagrangian or the Hamiltonian depends on new variables that represent the flow of the action. This leads to a modification of the field equations. And, again, there is a geometric structure, called *multicontact*, that allows us to obtain these field equations. This has been developed in a recent paper.[52] Let us notice that in ref. [53], Vitagliano introduces higher codimensional versions of contact manifolds that he calls "multicontact manifolds". This notion is different from ours. As in the contact case, one can derive the field equations from a variational principle, and we refer to several papers on this subject[52,54–57] (for a precise and general derivation, including higher-order, we recommend ref. [56]). We would like also to refer to an alternative (but related) approach for action-dependent field theories using the so-called *k-contact structures*.[58–61]

The multicontact geometry was derived as a mathematical generalization of other approaches, motivated by the capacity to formalize phenomenology outside the scope of the usual Hamilton's principle. So far, there are only a handful of examples developed with any detail. This paper aims to provide an easy introduction to action-dependent field theories and the main associated geometric tools. We lay out the field equations for regular systems, and provide an explicit guide for the multicontact formalism. The method is showcased with a diverse list of fully worked-out examples. We hope this paper will facilitate the work of those interested in action-dependent field theory and the new geometric objects underlying it.

Structure of the Paper

The paper is structured as follows. In Section 2 we introduce the so-called action-dependent field theories, and in Section 3, we develop the Lagrangian and Hamiltonian formalism for these theories, making use of the multicontact geometry. The rest of the paper is devoted to applying the above framework to particular field theories. In Section 4 we consider general Lagrangians that are quadratic (regular and singular) and affine, and that include terms depending on "action variables". In Section 5,

a number of particular cases are discussed: one-dimensional wave equation with first-order terms and external force, Klein–Gordon equation and the telegrapher equation, Metric-affine gravity, Maxwell's electromagnetism, and Burgers' equation from the heat equation; all of them including terms depending on "action variables". Finally, we enumerate some relevant topics for further investigation. Two additional simple examples, Bosonic string theory and $(2+1)$-dimensional gravity and Chern–Simons equation, are added in a first Appendix A. To keep the paper self-contained, we include a second Appendix B on premulticontact and multicontact structures that is not necessary to follow the rest of the paper but may be useful to the reader.

Some General Conventions and Notation

All the manifolds are real, second-countable and of class $\mathscr{C}^\infty$, and the mappings are assumed to be smooth. Sum over crossed repeated indices is understood. The signature of the space-time metric that has been taken is $(-+++)$.

A brief summary of conventional notation is the following:

- $\mathscr{C}^\infty(\mathcal{M})$: Smooth functions in a manifold $\mathcal{M}$.
- $\Omega^k(\mathcal{M})$: Module of differential forms of degree k in a manifold $\mathcal{M}$.
- $\mathsf{X}(\mathcal{M})$: Module of vector fields in a manifold $\mathcal{M}$.
- $\mathsf{X}^m(\mathcal{M})$: Module of m-multivector fields in a manifold $\mathcal{M}$.
- $\imath\,(X)\,\Omega$: Natural contraction of a vector field $X \in \mathsf{X}(\mathcal{M})$ and a k-form $\Omega \in \Omega^k(\mathcal{M})$.
- $J^1\pi$: 1-jet bundle over a bundle $\pi : E \to M$. Coordinates: (x^μ, y^a, y^a_μ), $0 \le \mu \le m-1$, $1 \le a \le n$.
- $J^{1*}\pi$: "Dual" jet bundle. Coordinates: (x^μ, y^a, p^μ_a).
- $\Lambda^{m-1}(\mathrm{T}^*\mathcal{M})$: Bundle of $(m-1)$-differential forms on a manifold $\mathcal{M}$ (multicotangent bundle).
- d: Exterior differential of differential forms.
- $\wedge$: Exterior product of differential forms or vector fields.
- $\mathrm{d}^m x \equiv \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{m-1}$: Coordinate expression of a volume form.
- $\mathrm{d}^{m-1}x_\mu = \imath\left(\frac{\partial}{\partial x^\mu}\right)\mathrm{d}^m x \equiv (-1)^\mu \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{\mu-1} \wedge \mathrm{d}x^{\mu+1} \wedge \cdots \wedge \mathrm{d}x^{m-1}$.
- $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

2. Action-Dependent Field Theories

In this preliminary section, we discuss the main features of the so-called action-dependent classical field theories.

2.1. General Aspects

An *action-dependent* first-order field theory is a system where the Lagrangian or the Hamiltonian functions depend on the action in an implicit way. Since the action is not a local operator, we consider a localized version of the action, given by an $(m-1)$-form S over space-time. It is related to the action in the following way: in classical field theories, the action of a field $\gamma(x) = (\gamma^a(x))$ is an integral over a domain in space-time, $D \subset M$,

$$\mathcal{A}(\gamma(x)) = \int_D L(\gamma(x))\mathrm{d}^m x\,. \tag{1}$$

The action $\mathcal{A}$ is not local in the sense that it depends on the domain D. The density of action is the Lagrangian density $L\,\mathrm{d}^m x$. Evaluated on a field, $L(\gamma(x))\mathrm{d}^m x$ is an m-form over space-time and, therefore, it is a closed form. Locally, we can consider a form whose differential is the Lagrangian density, which is the $(m-1)$-form represented by $S = s^\mu(x^\nu)\mathrm{d}^{m-1}x_\mu$,

$$\mathrm{d}S = \mathrm{d}\left(s^\mu \mathrm{d}^{m-1}x_\mu\right) = \frac{\partial s^\mu}{\partial x^\mu}\mathrm{d}^m x = L(\gamma(x))\mathrm{d}^m x\,. \tag{2}$$

Therefore, S plays the role of the potential of the density of action. The value of S depends on the field $\gamma(x)$; hence, the components s^μ are new fields; that is, fiber coordinates to be determined as functions of (x^μ).

To be more precise, an action-dependent field theory is a system where the Lagrangian or Hamiltonian depends on the components s^μ of a $(m-1)$ form over space-time, with the relation (2). This leads to an implicit variational principle called *Herglotz's variational principle*.[56,57] A pair $(\gamma(x), s(x)) \equiv (\gamma^a(x), s^\mu(x))$ is a solution to the Herglotz variational principle if they are critical for the action

$$\mathcal{A}(\gamma(x), s(x)) = \int_D L(x^\mu, \gamma^a, \partial_\mu \gamma^a, s^\mu)\mathrm{d}^m x\,, \tag{3}$$

under the constraint

$$\frac{\partial s^\mu}{\partial x^\mu} = L(x^\mu, \gamma^a, \partial_\mu \gamma^a, s^\mu)\,. \tag{4}$$

To solve this principle one should first solve (4) for the unknowns $(\gamma^a(x), s^\mu(x))$, if possible, or consider (4) as a nonholonomic constraint, which establishes nontrivial restrictions on the possible variations. For more details on Herglotz's variational principle for fields, see ref. [56].

The constraint (4) is crucial for the theory because it is the difference between Hamilton's and Herglotz's variational principles. Due to it, the s^μ variables generate the action; a property that differentiates them from other fields. Explicitly, as a consequence of the constraint (4) we can write the action as,

$$\mathcal{A}(\gamma(x), s(x)) = \int_D L(x^\mu, \gamma^a, \partial_\mu \gamma^a, s^\mu)\mathrm{d}^m x = \int_{\partial D} s^\mu \mathrm{d}^{m-1}x_\mu\,. \tag{5}$$

Namely, given a solution $(\gamma^a(x), s^\mu(x))$, the integral of the $(m-1)$-form $s^\mu \mathrm{d}^{m-1}x_\mu$ over the boundary of the domain is actually the value of the action on that solution. The unique character of the s^μ variables is reflected in its particular geometric description and are sometimes called "action variables". This relation suggests that the frontier could play a more relevant role in action-dependent field theories, although this is a feature that has not been explored so far.

2.2. Lagrangian Formalism

Given a Lagrangian $L(x^\mu, \gamma^a, \gamma^a_\mu, s^\mu)$, the functions $(\gamma^a(x), s^\mu(x))$ are a solution to the Herglotz variational principle for fields if, and only if,

$$\frac{\partial L}{\partial \gamma^a} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\partial L}{\partial \gamma^a_\mu} + \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial \gamma^a_\mu} = 0\,, \quad \frac{\partial s^\mu}{\partial x^\mu} = L\,. \tag{6}$$

These are the *Herglotz–Euler–Lagrange equations*[57,59] (see also Section 3.2). The first group of Equations (6) are the usual Euler–Lagrange equations with two extra terms: $\frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^a_\mu}$, which is a term proportional to the momenta, and $\frac{\partial^2 L}{\partial s^\nu y^a_\mu}\frac{\partial s^\nu}{\partial x^\mu}$, from the total derivative with respect to x^μ. Notice that the above equations are not linear on the Lagrangian, as in the standard Euler–Lagrange equations. Thus, any new term added to the Lagrangian could interact with all the other ones. The last equation of (6) is the constraint (4).

In most applications so far considered, the action-dependent Lagrangian density is the standard one of the system of interest, $L_0\,\mathrm{d}^m x$, plus a term linear on the s^μ of the form $\lambda_\mu(x^\nu)s^\mu \mathrm{d}^m x$. Therefore, $L\,\mathrm{d}^m x = (L_0 + \lambda_\mu s^\mu)\mathrm{d}^m x$. In these cases, we have that,

$$\frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^a_\mu} = \lambda_\mu \frac{\partial L}{\partial y^a_\mu}\,, \quad \frac{\partial^2 L}{\partial s^\nu y^a_\mu}\frac{\partial s^\nu}{\partial x^\mu} = 0\,, \tag{7}$$

that is, a linear term on the momenta, $\frac{\partial L}{\partial y^a_\mu}$, has been added to the equations. For Lagrangians with kinetic energy, this is generally a term linear in velocities, which is related to dissipative phenomena.

A more general dependence of the Lagrangian on the action usually leads to nonlinear terms of this kind in the field equations (for instance, see Section 5.5 for a nonlinear example).

2.3. Hamiltonian Formalism

There are two main approaches to Hamiltonian field theory: the space-time split formalism on an infinite-dimensional phase space,[62] and the covariant formalism, also known as the *De Donder–Weyl theory*.[63,64] The last one has been developed for action-dependent field theories using *k-contact* and *multicontact geometries*.[52,58–60] We now motivate the Herglotz–Hamilton–De Donder–Weyl equations from the Lagrangian Equations (6) with a simple manipulation in coordinates. The reader can find a more detailed derivation in Section 3 and in refs. [52, 58].

In the covariant approach, all the space-time coordinates are considered on the same footing. Therefore, we have momenta in each of the space-time coordinates, which are defined as:

$$p^\mu_a = \frac{\partial L}{\partial y^a_\mu}\,. \tag{8}$$

The quantities p^μ_a are usually called the *multimomenta*. We can express the velocities in terms of the multimomenta as long as the matrix $\left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu}\right)$ is regular. In this case, we say that the Lagrangian is *regular*, and we can define the Hamiltonian as:

$$\begin{aligned} &H(x^\mu, y^a, p^\mu_a, s^\mu) \\ &\quad = y^a_\mu(x^\mu, y^a, p^\mu_a, s^\mu)p^\mu_a - L(x^\mu, y^a, y^a_\mu(x^\mu, y^a, p^\mu_a, s^\mu), s^\mu)\,, \end{aligned} \tag{9}$$

and, by direct calculation, we derive that,

$$y^a_\mu = \frac{\partial H}{\partial p^\mu_a}\,. \tag{10}$$

From the above definition of H we obtain,

$$\begin{aligned} &L(x^\mu, y^a, y^a_\mu, s^\mu) \\ &\quad = y^a_\mu p^\mu_a(x^\mu, y^a, y^a_\mu, s^\mu) - H(x^\mu, y^a, p^\mu_a(x^\mu, y^a, y^a_\mu, s^\mu), s^\mu)\,. \end{aligned} \tag{11}$$

Using these transformations we can see that the Equations 6 are equivalent to

$$\frac{\partial p^\mu_a}{\partial x^\mu} = -\frac{\partial H}{\partial y^a} - p^\mu_a \frac{\partial H}{\partial s^\mu}\,, \tag{12}$$

$$\frac{\partial s^\mu}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_a}p^\mu_a - H\,. \tag{13}$$

These equations, together with (10), are the *Herglotz–Hamilton–de Donder-Weyl equations* for action-dependent field theories. One can consider these equations for an arbitrary Hamiltonian, even if it does not arise from a Lagrangian.

Most of the relevant field theories, like Einsteins' equations and Yang–Mills, are described by singular Lagrangian functions. In that case, one cannot find a one-to-one correspondence between velocities and multimomenta. Moreover, the equations are usually not compatible in the full space and constraints appear. To solve these problems, the geometric approach to field theories, in this case premulticontact geometry, gives us a method to obtain the set where solutions exist and which equations they actually satisfy. This procedure is explained in Section 3.2 for Lagrangian systems and in 3.4 for the Hamiltonian counterpart. We will see some examples in the second part of the paper.

The space-time split formalism considers the evolution of a field with respect to only one coordinate, usually time. This approach is preferred when quantizing field theories. Moreover, it is more natural to incorporate boundary conditions for the field equations, using the Cartan formalism in the space of Cauchy data.[38,51,65] Furthermore, this formulation of the problem has been useful to numerically simulate solutions of the equations.[66,67] This approach benefits from a different constraint algorithm (see ref. [68] for a complete algorithm including functional independence). Nevertheless, to our knowledge, the space-time split formalism for action-dependent field theories has not been developed.

2.4. Properties of Action-Dependent Field Theories

Nonconservation and symmetries: A relevant feature of action-dependent field theories is that they are nonconservative in the following sense: a conserved quantity in field theories is a $(m-1)$-form $\xi^\mu(x^\nu)\,\mathrm{d}^{m-1}x_\mu$ such that, on a solution, $\frac{\partial \xi^\mu}{\partial x^\mu} = 0$ (see, for instance, refs. [38, 44, 69–74]). In integral form, this translates to the conservation of flow through the boundary of a closed region. In action-dependent field theories, we have dissipated quantities, which are governed by the following relation, which holds on any solution:

$$\frac{\partial \xi^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial s^\mu}\xi^\mu\,, \tag{14}$$

in the Hamiltonian formalism. In the Lagrangian formalism it is:

$$\frac{\partial \xi^\mu}{\partial x^\mu} = \frac{\partial L}{\partial s^\mu}\xi^\mu \,. \tag{15}$$

This relation is called the *dissipation law*.[59] Despite its name, it does not always describe a decrease of an observable, but a non-conservative behavior in a particular way. The name originates from the use of contact geometry to study friction in mechanical systems. All the dissipated quantities of a system have the same dissipation law, which is given by the Hamiltonian (or the Lagrangian).

The dissipation law (14) (or (15)) has a rich behavior in field theories, far beyond dissipation. For instance, in Electromagnetism only a particular selection of covectors (λ_μ) lead to a clear dissipation of energy.[75] Meanwhile, one can construct cosmological models with an action-dependent Lagrangian where the energy-momentum tensor is conserved.[76]

The notions of symmetries and a Noether theorem exist for action-dependent field theories, that is, a way to construct a dissipated quantity from a symmetry.[58,59,77] Nevertheless, the topic is far less developed than in contact mechanical systems.[20,24]

New terms in the equations: Another relevant reason to study action-dependent field theories is their capacity to model a larger pool of PDE's. It is clear that the Herglotz–Euler–Lagrange Equations 6 contain more terms than the Euler–Lagrange equations, but the question remains if those terms could be recovered with a different Lagrangian without invoking action terms. This is a challenging question because it is closely related to the *inverse problem*, a classical problem only solved in particular cases[78] (see ref. [79] for recent advances in contact mechanics). Nevertheless, several results indicate that action-dependent Lagrangians, and the corresponding contact and multicontact geometry, are the adequate description for these systems.

As a first approach, consider the quadratic Lagrangian of a scalar field y with a linear dependence on the action (see 4.1 for a more general case)

$$L = \frac{1}{2}f^{\mu\nu}y_\mu y_\nu - V(y) - \lambda_\mu(x^\rho)s^\mu \,. \tag{16}$$

The corresponding Herglotz–Euler–Lagrange equations are

$$\frac{\partial V}{\partial y} + f^{\mu\nu}y_{\mu\nu} = -\lambda_\nu f^{\mu\nu}y_\mu \,. \tag{17}$$

Assuming $f^{\mu\nu}$ is a regular metric, the right-hand side is a generic linear term in velocities. This term can be partially recovered without the action-dependent term by considering a space-time-depending global factor:

$$L_b = b(x^\rho)\left[\frac{1}{2}f^{\mu\nu}y_\mu y_\nu - V(y)\right] . \tag{18}$$

The resulting equations are

$$b\left[\frac{\partial V}{\partial y} + f^{\mu\nu}y_{\mu\nu}\right] = -\frac{\partial b}{\partial x^\nu}f^{\mu\nu}y_\mu \,. \tag{19}$$

We recover the same equations if, and only if, $\mathrm{d}\ln b = \lambda_\mu \mathrm{d}x^\mu$, which is only possible if $\lambda_\mu \mathrm{d}x^\mu$ is a closed form. In other words, not all terms linear in velocities can be recovered by a global factor.

The problem of the particularity of action-dependent Lagrangians has been studied in more generality in contact mechanical systems. The dynamics of a contact mechanical system is of Reeb type (instead of symplectic) outside the set where the energy vanishes.[18] In some cases, is it possible to recast the contact dynamics as a subset of a symplectic system, although there are obstructions for a general result.[80,81] In summary, the dynamics of an action-dependent mechanical system are associated with contact geometry, which is, in general, qualitatively different from symplectic geometry.

There are a lot of topics to explore, but one should advance with care because our intuition about the standard classical field theories does not always translate to action-dependent field theories. To begin with, the equations are not linear in the Lagrangian. Another important case is that of equivalent Lagrangians: if we add a total derivative to an action-dependent Lagrangian, we do not obtain the same equations. To construct equivalent Lagrangians one usually needs to also perform a change of variables in the s^μ coordinates. This topic has been studied for mechanical systems in ref. [79]. A notorious example in field theories is General Relativity, where different equivalent Lagrangians lead to different dynamics when an action-dependent term is added[55,76] (see also example 5.4).

3. Lagrangian and Hamiltonian Formalisms for Action-Dependent Field Theories

This section is devoted to reviewing the main features of the multicontact formulation for action-dependent field theories introduced in ref. [52]. For this, it is necessary to make a minimal presentation of the inherent geometric structures; although our attention will be focused on obtaining and discussing the field equations. As this framework is used in the following sections to describe both regular and singular field theories, both cases are explained.

3.1. Multivector Fields

For details on multivector fields see, for instance, refs. [37, 82, 83]. Let $\mathcal{M}$ be a manifold with $\dim \mathcal{M} = n$. The m-**multivector fields** or **multivector fields of order** m on $\mathcal{M}$ are the contravariant skew-symmetric tensor fields of order m in $\mathcal{M}$. The set of m-multivector fields in $\mathcal{M}$ is denoted $\mathfrak{X}^m(\mathcal{M})$. A multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is **locally decomposable** if, for every point $\mathrm{p} \in \mathcal{M}$, there exists an open neighborhood $U_\mathrm{p} \subset \mathcal{M}$ such that

$$\mathbf{X}|_{U_\mathrm{p}} = X_0 \wedge \cdots \wedge X_{m-1} \,, \quad \text{for some } X_0, \ldots, X_{m-1} \in \mathfrak{X}(U_\mathrm{p}) \,. \tag{20}$$

The *contraction* of a locally decomposable multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ and a differentiable form $\Omega \in \Omega^k(\mathcal{M})$ is the natural contraction between contravariant and covariant tensor fields; i.e., $\imath(\mathbf{X})\,\Omega|_{U_p} = 0$, if $k < m$, and

$$\begin{aligned} \imath(\mathbf{X})\,\Omega|_{U_p} &= \imath\big(X_0 \wedge \cdots \wedge X_{m-1}\big)\Omega \\ &= \imath\big(X_{m-1}\big)\cdots\imath\big(X_0\big)\Omega \,, \quad \text{if } k \geq m \,. \end{aligned} \tag{21}$$

Let $\kappa : \mathcal{M} \to M$ be a fiber bundle with $\dim M = m$ and $\dim \mathcal{M} = N + m$, and denote (x^μ, z^a) the local coordinates on $\mathcal{M}$; where x^μ are coordinates on the base M and z^a are coordinates on the fibers (with $0 \le \mu \le m-1$ and $1 \le I \le N$). A multivector field $\mathbf{X} \in \mathsf{X}^m(\mathcal{M})$ is κ-*transverse* if $\imath(\mathbf{X})(\kappa^*\beta)|_{\mathrm{p}} \neq 0$, for every $\mathrm{p} \in \mathcal{M}$ and $\beta \in \Omega^m(\mathcal{M})$. The local expression for a locally decomposable κ-transverse multivector field $\mathbf{X} \in \mathsf{X}^m(\mathcal{M})$ is

$$\mathbf{X} = \bigwedge_{\mu=0}^{m-1} \mathbf{X}_\mu = f \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + F^a_\mu \frac{\partial}{\partial z^a} \right), \quad \text{where } f \in \mathscr{C}^\infty(\mathcal{M}). \tag{22}$$

If M is an orientable manifold with volume form $\omega \in \Omega^m(\mathcal{M})$, then the condition for $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ to be κ-transverse can be expressed as $\imath(\mathbf{X})(\kappa^*\omega) \neq 0$. This condition can be fixed by taking $\imath(\mathbf{X})(\kappa^*\omega) = 1$, which implies $f = 1$ in (22).

If $\mathbf{X} \in \mathsf{X}^m(\mathcal{M})$ is a locally decomposable and κ-transverse multivector field and $\psi : U \subset M \to \mathcal{M}$ is a local section of κ, with local expression $\psi(x^\mu) = (x^\mu, z^a(x^\nu))$; then ψ is an **integral section** of $\mathbf{X}$ if $\frac{\partial z^a}{\partial x^\mu} = F^a_\mu$. Then, $\mathbf{X}$ is **integrable** if, for $\mathrm{p} \in \mathcal{M}$, there exist $x \in M$ and an integral section, $\psi : U_x \subset M \to \mathcal{M}$, of $\mathbf{X}$ such that $\mathrm{p} = \psi(x)$.

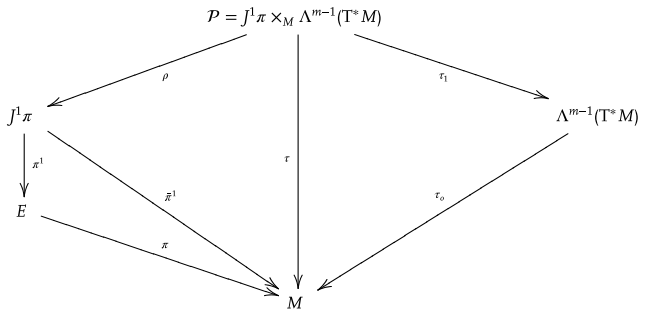
3.2. Multicontact Lagrangian Formalism

In the multisymplectic formulation of conservative classical field theories,[31,32,34–39,84] a first-order Lagrangian field theory is described by the following elements. First we have the *configuration bundle* $\pi : E \to M$, where $\dim M = m$ and $\dim E = n + m$, and M is an orientable manifold with volume form $\omega \in \Omega^m(M)$, which usually represents space-time. Then, the theory is developed on the first-order jet bundle $J^1\pi \to E \to M$, with $\dim J^1\pi = nm + n + m$. Natural coordinates in $J^1\pi$ adapted to the bundle structure are denoted by (x^μ, y^a, y^a_μ) ($\mu = 0, \dots, m-1$ and $a = 1, \dots, n$), and are such that $\omega = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{m-1} \equiv \mathrm{d}^m x$.

For the Lagrangian formulation of nonconservative (or action-dependent) first-order field theories (see ref. [52]), consider the bundle,

$$\mathcal{P} = J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M), \tag{23}$$

where $\Lambda^{m-1}(\mathrm{T}^*M)$ denotes the bundle of $(m-1)$-forms on M. This bundle can be identified with $J^1\pi \times \mathbb{R}^m$, and so we shall do henceforth. We denote by $\tau : \mathcal{P} \to M$ the bundle projection. Natural coordinates in $\mathcal{P}$ are $(x^\mu, y^a, y^a_\mu, s^\mu)$, and so $\dim \mathcal{P} = 2m + n + nm$.



Now, a **Lagrangian density** is introduced on $\mathcal{P}$ as a m-form $\mathcal{L} \in \Omega^m(\mathcal{P})$, whose expression is

$$\mathcal{L}(x^\mu, y^a, y^a_\mu, s^\mu) = L(x^\mu, y^a, y^a_\mu, s^\mu)\, \mathrm{d}^m x, \tag{24}$$

where $L \in \mathscr{C}^\infty(\mathcal{P})$ is the *Lagrangian function* associated with $\mathcal{L}$. Then, the **Lagrangian** m**-form** associated with $\mathcal{L}$ on $\mathcal{P}$ is

$$\Theta_{\mathcal{L}} = -\frac{\partial L}{\partial y^a_\mu} \mathrm{d}y^a \wedge \mathrm{d}^{m-1}x_\mu + \left(\frac{\partial L}{\partial y^a_\mu} y^a_\mu - L \right) \mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu . \tag{25}$$

The local function $E_{\mathcal{L}} = \frac{\partial L}{\partial y^a_\mu} y^a_\mu - L$ is called the **energy Lagrangian function** associated with L.

A Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$ is **regular** if the Hessian matrix

$$(W^{\mu\nu}_{ab}) = \left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu} \right) \tag{26}$$

is regular everywhere; then $\Theta_{\mathcal{L}}$ is a *variational multicontact form* on $\mathcal{P}$ and the triple $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ is called a **multicontact Lagrangian system** (see Appendix B for a quick review of multicontact and premulticontact structures and their terminology). Otherwise, L is a **singular** Lagrangian.

Remark 3.1. It is important to note that, given a singular Lagrangian, the associated Lagrangian form $\Theta_{\mathcal{L}}$ is not always a *premulticontact form* on $\mathcal{P}$. Consider, for instance, the example given by the singular Lagrangian $L = \sum_{a=1}^{n} y^a_\mu s^\mu$ which yields a structure $(\Theta_{\mathcal{L}}, \omega)$ which has no Reeb distribution associated with it, and then it is neither a multicontact, nor a premulticontact structure (see Appendix B and ref. [52] for details on these definitions and results). Nevertheless, in this paper we only consider singular Lagrangians for which $\Theta_{\mathfrak{L}}$ is a premulticontact form, and then the triple $(\mathcal{P}, \Theta_{\mathfrak{L}}, \omega)$ will be called a **premulticontact Lagrangian system**.

To set the field equations for action-dependent field theories in an intrinsic way, we need first to introduce the so-called **dissipation form** whose expression in coordinates is

$$\sigma_{\Theta_{\mathcal{L}}} = -\frac{\partial L}{\partial s^\mu} \mathrm{d}x^\mu . \tag{27}$$

Then we define the **Lagrangian** $(m+1)$**-form**

$$\begin{aligned} \overline{\Omega}_{\mathcal{L}} &= \mathrm{d}\Theta_{\mathcal{L}} + \sigma_{\Theta_{\mathcal{L}}} \wedge \Theta_{\mathcal{L}} = \mathrm{d}\Theta_{\mathcal{L}} - \frac{\partial L}{\partial s^\nu} \mathrm{d}x^\nu \wedge \Theta_{\mathcal{L}} \\ &= \mathrm{d}\left(-\frac{\partial L}{\partial y^a_\mu} \mathrm{d}y^a \wedge \mathrm{d}^{m-1}x_\mu + \left(\frac{\partial L}{\partial y^a_\mu} y^a_\mu - L \right) \mathrm{d}^m x \right) \\ &\quad - \left(\frac{\partial L}{\partial s^\mu} \frac{\partial L}{\partial y^a_\mu} \mathrm{d}y^a - \frac{\partial L}{\partial s^\mu} \mathrm{d}s^\mu \right) \wedge \mathrm{d}^m x . \end{aligned} \tag{28}$$

Now, a section $\boldsymbol{\psi} : M \to \mathcal{P}$ of the projection τ is said to be a **holonomic section** on $\mathcal{P}$ if it is locally expressed as

$$\boldsymbol{\psi}(x^\mu) = \left(x^\mu, y^a(x^\nu), \frac{\partial y^a}{\partial x^\mu}(x^\nu), s^\mu(x^\nu) \right). \tag{29}$$

Then $\boldsymbol{X} \in \mathsf{X}^m(\mathcal{P})$ is a **holonomic** m**-multivector field** on $\mathcal{P}$ if it is τ-transverse, integrable, and its integral sections are holonomic on $\mathcal{P}$. The local expression of a holonomic m-multivector field on $\mathcal{P}$ verifying the condition $\imath(\boldsymbol{X})\,\omega = 1$ is

$$\boldsymbol{X} = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + y^a_\mu \frac{\partial}{\partial y^a} + F^a_{\mu\nu} \frac{\partial}{\partial y^a_\nu} + g^\nu_\mu \frac{\partial}{\partial s^\nu} \right),$$
$$\text{where } F^a_{\mu\nu}, g^\nu_\mu \in \mathscr{C}^\infty(\mathcal{P}). \tag{30}$$

Holonomic multivector fields are also called SOPDEs because the equations of their integral sections are the solutions to a system of second-order partial differential equations of the form

$$y^a_\mu = \frac{\partial y^a}{\partial x^\mu}, \quad F^a_{\mu\nu} = \frac{\partial^2 y^a}{\partial x^\mu \partial x^\nu}. \tag{31}$$

Multivector fields on $\mathcal{P}$ which have the local expression (30), but are not integrable, are usually called **semiholonomic**.

Bearing all this in mind, for a (pre)multicontact Lagrangian system $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ the Lagrangian field equations can be derived from a variational principle which is called the *generalized Herglotz Principle* and they can be stated alternatively as:

(1) The *(pre)multicontact Lagrangian equations for holonomic sections*, $\boldsymbol{\psi} : M \to \mathcal{P}$, which are

$$\boldsymbol{\psi}^* \Theta_{\mathcal{L}} = 0, \quad \boldsymbol{\psi}^* \imath(Y) \overline{\Omega}_{\mathcal{L}} = 0, \quad \text{for every } Y \in \mathfrak{X}(\mathcal{P}). \tag{32}$$

(2) The *(pre)multicontact Lagrangian equations for τ-transverse and locally decomposable multivector fields* $\mathbf{X}_{\mathcal{L}} \in \mathsf{X}^m(\mathcal{P})$ which are

$$\imath(\mathbf{X}_{\mathcal{L}})\Theta_{\mathcal{L}} = 0, \quad \imath(\mathbf{X}_{\mathcal{L}})\overline{\Omega}_{\mathcal{L}} = 0, \tag{33}$$

and the condition of τ-transversality can be imposed by asking that $\imath(\mathbf{X}_{\mathcal{L}})\,\omega = 1$. A τ-transverse and locally decomposable m-multivector field solution to these equations is called a **Lagrangian multivector field**.

In a natural chart of coordinates of $\mathcal{P}$, a τ-transverse and locally decomposable m-multivector field $\mathbf{X}_{\mathcal{L}} \in \mathsf{X}^m(\mathcal{P})$ verifying the condition $\imath(\boldsymbol{X})_{\mathcal{L}}\,\omega = 1$ is given by

$$\mathbf{X}_{\mathcal{L}} = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + (X_{\mathcal{L}})^a_\mu \frac{\partial}{\partial y^a} + (X_{\mathcal{L}})^a_{\mu\nu} \frac{\partial}{\partial y^a_\nu} + (X_{\mathcal{L}})^\nu_\mu \frac{\partial}{\partial s^\nu} \right). \tag{34}$$

Taking into account (25) and (28), we have that Equations (33) lead to

$$0 = L + \frac{\partial L}{\partial y^a_\mu} \left((X_{\mathcal{L}})^a_\mu - y^a_\mu \right) - (X_{\mathcal{L}})^\mu_\mu, \tag{35}$$

$$0 = \left((X_{\mathcal{L}})^b_\mu - y^b_\mu \right) \frac{\partial^2 L}{\partial y^b_\mu \partial s^\nu}, \tag{36}$$

$$0 = \left((X_{\mathcal{L}})^b_\mu - y^b_\mu \right) \frac{\partial^2 L}{\partial y^b_\mu \partial x^\nu}, \tag{37}$$

$$0 = \left((X_{\mathcal{L}})^b_\mu - y^b_\mu \right) \frac{\partial^2 L}{\partial y^a_\nu \partial y^b_\mu}, \tag{38}$$

$$0 = \left((X_{\mathcal{L}})^b_\mu - y^b_\mu \right) \frac{\partial^2 L}{\partial y^a \partial y^b_\mu} + \frac{\partial L}{\partial y^a} - \frac{\partial^2 L}{\partial x^\mu \partial y^a_\mu} - \frac{\partial^2 L}{\partial s^\nu \partial y^a_\mu} (X_{\mathcal{L}})^\nu_\mu$$
$$- \frac{\partial^2 L}{\partial y_b \partial y^a_\mu} (X_{\mathcal{L}})^b_\mu - \frac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu} (X_{\mathcal{L}})^b_{\mu\nu} + \frac{\partial L}{\partial s^\mu} \frac{\partial L}{\partial y^a_\mu}, \tag{39}$$

and a last group of equations which are identities when they are combined with the above ones.

If $X_{\mathcal{L}}$ is a semiholonomic solution to (33), then

$$y^a_\mu = (X_{\mathcal{L}})^a_\mu, \tag{40}$$

and Equations (36), (37), and (38) hold identically, and (35) and (39) give

$$(X_{\mathcal{L}})^\mu_\mu = L, \tag{41}$$

$$\frac{\partial L}{\partial y^a} - \frac{\partial^2 L}{\partial x^\mu \partial y^a_\mu} - \frac{\partial^2 L}{\partial y^b \partial y^a_\mu} y^b_\mu - \frac{\partial^2 L}{\partial s^\nu \partial y^a_\mu} (X_{\mathcal{L}})^\nu_\mu$$
$$- \frac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu} (X_{\mathcal{L}})^b_{\mu\nu} = - \frac{\partial L}{\partial s^\mu} \frac{\partial L}{\partial y^a_\mu}. \tag{42}$$

Furthermore, if these semiholonomic multivector fields $\mathbf{X}_{\mathcal{L}}$ are integrable then, for their holonomic integral sections

$$\boldsymbol{\psi}(x^\nu) = \left(x^\mu, y^a(x^\nu), \frac{\partial y^a}{\partial x^\mu}(x^\nu), s^\mu(x^\nu) \right), \tag{43}$$

Equations (41) and (42) transform into

$$\frac{\partial s^\mu}{\partial x^\mu} = L \circ \boldsymbol{\psi}, \tag{44}$$

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial y^a_\mu} \circ \boldsymbol{\psi} \right) = \left(\frac{\partial L}{\partial y^a} + \frac{\partial L}{\partial s^\mu} \frac{\partial L}{\partial y^a_\mu} \right) \circ \boldsymbol{\psi}, \tag{45}$$

which are precisely the coordinate expression of the Lagrangian Equations (32) for holonomic sections. Thus, Equations (33) and (32) are equivalent in this way. In particular, (45) are the so-called *Herglotz–Euler–Lagrange field equations*.

In the case that L is a regular Lagrangian, Equations (38) lead to the condition of semiholonomy (40), since the Hessian matrix $\left(\frac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu} \right)$ is regular. Therefore, Equations (42) always have solution for the components $(\mathbf{X}_{\mathcal{L}})^b_{\mu\nu}$ of $\mathbf{X}_{\mathcal{L}}$ (although the system is undetermined, unless $m = 1$).

Thus, we have proved that:

Theorem 3.2. *Let $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ be a multicontact Lagrangian system. If $\mathbf{X}_{\mathcal{L}}$ is a semiholonomic and integrable solution to the Equations (33) (that is, a* SOPDE*), its integral sections are the solutions to the multicontact Euler–Lagrange field equations for holonomic sections (32). These* SOPDEs *are called the* ***Euler–Lagrange multivector fields*** *associated with $\mathcal{L}$.*

In addition, if the Lagrangian system is regular then:

(1) *The multicontact Lagrangian field equations for multivector fields (53) have solutions on $\mathcal{P}$. The solutions are not unique if $m > 1$.*
(2) *The Lagrangian m-multivector fields $\mathbf{X}_{\mathcal{L}}$ solution to Equations (53) are semiholonomic.*

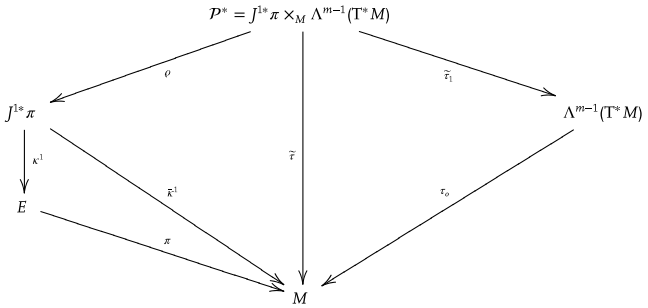
When L is not regular and assuming that $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ is a premulticontact system, in general, the Equations (42) have no solutions everywhere on $\mathcal{P}$, since the compatibility of the system (42) depends on the rank of the Hessian matrix $\left(\frac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu}\right)$. In the most favorable cases, these field equations are compatible only on a submanifold of $\mathcal{P}$ which is obtained by applying a suitable constraint algorithm. In any case, solutions to Equations (33) are not necessarily SOPDEs and, as a consequence, if they are integrable, their integral sections are not necessarily holonomic; so this requirement must be imposed as an additional condition. Hence, the final objective is to find the maximal submanifold $\mathcal{S}_f$ of $\mathcal{P}$ where there are Euler–Lagrange multivector fields $\mathbf{X}_{\mathcal{L}}$ which are solutions to the premulticontact Lagrangian field equations on $\mathcal{S}_f$ and are tangent to $\mathcal{S}_f$; that is to say, where consistent solutions exist. This procedure is explained in some of the applications developed in Sections 4 and 5.

3.3. Multicontact Hamiltonian Formalism: The (hyper)regular Case

For the Hamiltonian formulation of nonconservative (or action-dependent) first-order field theories, first consider the so-called *restricted multimomentum bundle* $J^{1*}\pi$, whose natural coordinates are (x^μ, y^a, p^μ_a) such that $\omega = \mathrm{d}^m x$, and so $\dim J^{1*}\pi = n + m + nm$. (This is the bundle where the Hamiltonian multisymplectic formalism of conservative field theories takes place[39,41–48,50].) Then, consider the manifold

$$\mathcal{P}^* = J^{1*}\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M)\,, \tag{46}$$

which has natural coordinates $(x^\mu, y^a, p^\mu_a, s^\mu)$. This bundle is identified with $J^{1*}\pi \times \mathbb{R}^m$, and so we shall do henceforth. It is a bundle over M with projection $\widetilde{\tau} : \mathcal{P}^* \to M$.



Let $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ be a Lagrangian system, with $\mathcal{L} = L\,\omega$. The **Legendre map** associated with the Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$ is the map $\mathcal{FL} : \mathcal{P} \to \mathcal{P}^*$ locally given by

$$\mathcal{FL}^* x^\nu = x^\nu\,, \quad \mathcal{FL}^* y^a = y^a\,, \quad \mathcal{FL}^* p^\nu_a = \frac{\partial L}{\partial y^a_\nu}\,, \quad \mathcal{FL}^* s^\mu = s^\mu\,. \tag{47}$$

The Lagrangian L is regular if, and only if, $\mathcal{FL}$ is a local diffeomorphism, and L is said to be **hyperregular** when $\mathcal{FL}$ is a global diffeomorphism.

Next, we consider the hyperregular case (the regular case is the same but changing $\mathcal{P}$ and $\mathcal{P}^*$ by the corresponding open sets). In this case $\mathcal{FL}(\mathcal{P}) = \mathcal{P}^*$. Then, the multicontact form $\Theta_{\mathcal{L}} \in \Omega^m(\mathcal{P})$ can be projected by $\mathcal{FL}$ to an m-form in $\mathcal{P}^*$ and then there exists the **Hamiltonian m-form** $\Theta_{\mathcal{H}} \in \Omega^m(\mathcal{P}^*)$ such that $\Theta_{\mathcal{L}} = \mathcal{FL}^*\Theta_{\mathcal{H}}$, whose local expression is

$$\Theta_{\mathcal{H}} = -p^\mu_a \mathrm{d}y^a \wedge \mathrm{d}^{m-1}x_\mu + H\,\mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu\,. \tag{48}$$

Here $H \in \mathscr{C}^\infty(\mathcal{P}^*)$ is the *Hamiltonian function*, which is obtained as $H = (\mathcal{FL}^{-1})^* E_{\mathcal{L}}$; that is, its local expression is given by

$$H = p^\mu_a (\mathcal{FL}^{-1})^* y^a_\mu - (\mathcal{FL}^{-1})^* L\,. \tag{49}$$

Then, $\Theta_{\mathcal{H}}$ is a variational multicontact form and hence $(\mathcal{P}^*, \Theta_{\mathcal{H}}, \omega)$ is called the **multicontact Hamiltonian system** associated with the multicontact Lagrangian systems $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$. In this formalism, the *dissipation form* is expressed as

$$\sigma_{\mathcal{H}} = \frac{\partial H}{\partial s^\mu}\,\mathrm{d}x^\mu\,, \tag{50}$$

and we can define the **Hamiltonian $(m+1)$-form**

$$\begin{aligned}\overline{\Omega}_{\mathcal{H}} &= \mathrm{d}\Theta_{\mathcal{H}} + \sigma_{\mathcal{H}} \wedge \Theta_{\mathcal{H}} = \mathrm{d}\Theta_{\mathcal{H}} + \frac{\partial H}{\partial s^\mu}\,\mathrm{d}x^\mu \wedge \Theta_{\mathcal{H}} \\ &= \mathrm{d}(-p^\mu_a \mathrm{d}y^a \wedge \mathrm{d}^{m-1}x_\mu + H\,\mathrm{d}^m x) \\ &\quad + \left(\frac{\partial H}{\partial s^\mu}\,p^\mu_a\,\mathrm{d}y^a - \frac{\partial H}{\partial s^\mu}\,\mathrm{d}s^\mu\right) \wedge \mathrm{d}^m x\,;\end{aligned} \tag{51}$$

which verifies that $\overline{\Omega}_{\mathcal{L}} = \mathcal{FL}^*\overline{\Omega}_{\mathcal{H}}$.

For this multicontact Hamiltonian system, the field equations can be stated alternatively as:

(1) The *multicontact Hamilton–de Donder–Weyl equations for sections* $\boldsymbol{\psi} : M \to \mathcal{P}^*$ are

$$\boldsymbol{\psi}^*\Theta_{\mathcal{H}} = 0\,, \quad \boldsymbol{\psi}^*\imath(Y)\overline{\Omega}_{\mathcal{H}} = 0\,, \quad \text{for every } Y \in \mathfrak{X}(\mathcal{P}^*)\,. \tag{52}$$

(2) The *multicontact Hamilton–de Donder–Weyl equations for $\widetilde{\tau}$-transverse and locally decomposable multivector fields* $\mathbf{X}_{\mathcal{H}} \in \mathfrak{X}^m(\mathcal{P}^*)$ are

$$\imath(\mathbf{X}_{\mathcal{H}})\Theta_{\mathcal{H}} = 0\,, \quad \imath(\mathbf{X}_{\mathcal{H}})\overline{\Omega}_{\mathcal{H}} = 0\,, \tag{53}$$

and the condition of $\widetilde{\tau}$-transversality can be imposed by asking that $\imath\left(\mathbf{X}_{\mathcal{H}}\right)\omega = 1$.

In natural coordinates, if

$$\mathbf{X}_{\mathcal{H}} = \bigwedge_{\mu=0}^{m-1}\left(\frac{\partial}{\partial x^\mu} + (X_{\mathcal{H}})^a_\mu \frac{\partial}{\partial y^a} + (X_{\mathcal{H}})^\nu_{\mu a}\frac{\partial}{\partial p^\nu_a} + (X_{\mathcal{H}})^\nu_\mu \frac{\partial}{\partial s^\nu}\right) \tag{54}$$

is the expression of the $\widetilde{\tau}$-transverse and locally decomposable multivector field $\mathbf{X}_{\mathcal{H}} \in \mathfrak{X}^m(\mathcal{P}^*)$, and they are solutions to Equations (53), taking into account the local expressions (48) and (51),

these field equations lead to

$$(X_{\mathcal{H}})^{\mu}_{\mu} = p^{\mu}_{a}\frac{\partial H}{\partial p^{\mu}_{a}} - H\,, \quad (X_{\mathcal{H}})^{a}_{\mu} = \frac{\partial H}{\partial p^{\mu}_{a}}\,,$$
$$(X_{\mathcal{H}})^{\mu}_{\mu a} = -\left(\frac{\partial H}{\partial y^{a}} + p^{\mu}_{a}\frac{\partial H}{\partial s^{\mu}}\right), \tag{55}$$

together with a last group of equations which are identities when the above ones are taken into account. If $\mathbf{X}_{\mathcal{H}}$ are integrable, then their integral sections $\boldsymbol{\psi}(x^{\nu}) = (x^{\mu}, y^{a}(x^{\nu}), p^{\mu}_{a}(x^{\nu}), s^{\mu}(x^{\nu}))$ are the solutions to the Equations (52) which read as

$$\frac{\partial s^{\mu}}{\partial x^{\mu}} = \left(p^{\mu}_{a}\frac{\partial H}{\partial p^{\mu}_{a}} - H\right)\circ\boldsymbol{\psi}\,, \quad \frac{\partial y^{a}}{\partial x^{\mu}} = \frac{\partial H}{\partial p^{\mu}_{a}}\circ\boldsymbol{\psi}\,,$$
$$\frac{\partial p^{\mu}_{a}}{\partial x^{\mu}} = -\left(\frac{\partial H}{\partial y^{a}} + p^{\mu}_{a}\frac{\partial H}{\partial s^{\mu}}\right)\circ\boldsymbol{\psi}\,. \tag{56}$$

These are the *Herglotz–Hamilton–de Donder–Weyl equations* for action-dependent field theories. These equations are compatible in $\mathcal{P}^{*}$.

As $\mathcal{FL}$ is a diffeomorphism, the solutions to the Lagrangian field equations for $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ are in one-to-one correspondence to those of the Hamilton–de Donder–Weyl field equations for $(\mathcal{P}^{*}, \Theta_{\mathcal{H}}, \omega)$.

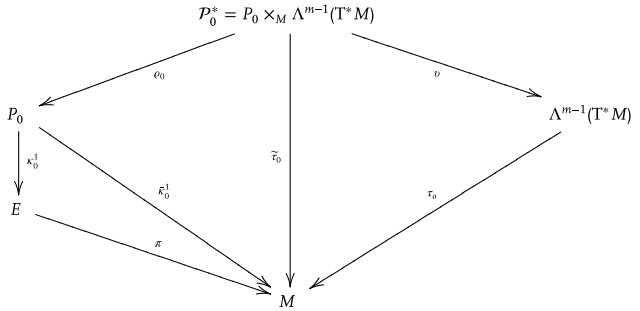
3.4. Multicontact Hamiltonian formalism: The Singular Case

See ref. [52]. For singular Lagrangians, the existence of an associated Hamiltonian formalism is not assured unless some minimal regularity conditions are assumed. For instance, in the standard multisymplectic formulation of conservative field theories, a sufficient condition is that the Lagrangian functions be *almost-regular*.[41,42,45] Similarly, in the multicontact formalism of action-dependent field theories, we say that a Lagrangian $L \in \mathscr{C}^{\infty}(\mathcal{P})$ is said to be **almost-regular** if (i) $\mathcal{P}^{*}_{0} = \mathcal{FL}(\mathcal{P})$ is a submanifold of $\mathcal{P}^{*}$, (ii) $\mathcal{FL}$ is a submersion onto its image, and (iii) the fibers $\mathcal{FL}^{-1}(\mathrm{p})$, for every $\mathrm{p} \in \mathcal{P}^{*}_{0}$, are connected submanifolds of $\mathcal{P}$. Nevertheless, even in these cases, the existence of a premulticontact structure in $\mathcal{P}^{*}_{0}$ is not assured unless additional conditions are assumed, as we will see in the examples.

Now, as a consequence of the definition of the Legendre map $\mathcal{FL}$ (see (47)), we have that

$$\mathcal{P}^{*}_{0} = P_{0} \times \Lambda^{m-1}(\mathrm{T}^{*}M) \simeq P_{0} \times \mathbb{R}^{m}\,. \tag{57}$$

It is a bundle over M with projection $\widetilde{\tau}_{0} : \mathcal{P}^{*}_{0} \to M$; which is endowed with natural coordinates denoted $(x^{\mu}, y^{a}, p^{I}, s^{\mu})$, $1 \le I \le \dim \mathcal{P}^{*}_{0} - n - 2m$, and are such that $\omega = \mathrm{d}^{m}x$.



We denote by $\mathcal{FL}_{0} : \mathcal{P} \to \mathcal{P}^{*}_{0}$ the restriction of $\mathcal{FL}$ to $\mathcal{P}^{*}_{0}$; that is, $\mathcal{FL} = \jmath_{0} \circ \mathcal{FL}_{0}$, where $\jmath_{0} : \mathcal{P}^{*}_{0} \hookrightarrow J^{1*}\pi$ denotes the canonical inclusion. Then, for almost-regular Lagrangian systems, the form $\Theta_{\mathcal{L}}$ projects to $\mathcal{P}^{*}_{0}$ by $\mathcal{FL}_{0}$, and then we have a Hamiltonian m-form $\Theta^{0}_{\mathcal{H}} \in \Omega^{m}(\mathcal{P}^{*}_{0})$ such that $\Theta_{\mathcal{L}} = \mathcal{FL}^{*}_{0}\Theta^{0}_{\mathcal{H}}$ whose local expression is

$$\Theta^{0}_{\mathcal{H}} = F^{\mu}_{a}\mathrm{d}y^{a} \wedge \mathrm{d}^{m-1}x_{\mu} + H_{0}\,\mathrm{d}^{m}x + \mathrm{d}s^{\mu} \wedge \mathrm{d}^{m-1}x_{\mu}\,, \tag{58}$$

where $H_{0} \in \mathscr{C}^{\infty}(\mathcal{P}^{*}_{0})$ is now the Hamiltonian function which is obtained as the projection of the Lagrangian energy $E_{\mathcal{L}}$ by $\mathcal{FL}_{0}$; that is, $E_{\mathcal{L}} = \mathcal{FL}_{0}{}^{*}H_{0}$. The functions $F^{\mu}_{a} \in \mathscr{C}^{\infty}(\mathcal{P}^{*}_{0})$ are determined by the condition $\Theta_{\mathcal{L}} = \mathcal{FL}^{*}_{0}\Theta^{0}_{\mathcal{H}}$. If $\Theta^{0}_{\mathcal{H}}$ is a premulticontact form, then $(\mathcal{P}^{*}_{0}, \Theta^{0}_{\mathcal{H}}, \omega)$ is said to be the **premulticontact Hamiltonian system** associated with the premulticontact Lagrangian system $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$.

Let $(x^{\mu}, y^{a}, p^{I}, s^{\mu})$ be adapted coordinates for the premulticontact manifold $(\mathcal{P}^{*}_{0}, \Theta^{0}_{\mathcal{H}}, \omega)$ (see the Appendix B). If F^{μ}_{a} are such that

$$\frac{\partial F^{\mu}_{a}}{\partial s^{\nu}} = 0\,, \tag{59}$$

then

$$\sigma_{\Theta^{0}_{\mathcal{H}}} = \frac{\partial H_{0}}{\partial s^{\mu}}\,\mathrm{d}x^{\mu}\,, \tag{60}$$

and then the corresponding Hamiltonian $(m+1)$-form $\overline{\Omega}^{0}_{\mathcal{H}} \in \Omega^{m+1}(\mathcal{P}^{*}_{0})$ is

$$\begin{aligned}\overline{\Omega}^{0}_{\mathcal{H}} &= \mathrm{d}\Theta^{0}_{\mathcal{H}} + \sigma_{\Theta^{0}_{\mathcal{H}}} \wedge \Theta^{0}_{\mathcal{H}} = \mathrm{d}\Theta^{0}_{\mathcal{H}} + \frac{\partial H_{0}}{\partial s^{\mu}}\,\mathrm{d}x^{\mu} \wedge \Theta^{0}_{\mathcal{H}}\\ &= \mathrm{d}(F^{\mu}_{a}\mathrm{d}y^{a} \wedge \mathrm{d}^{m-1}x_{\mu} + H_{0}\,\mathrm{d}^{m}x) - \left(\frac{\partial H_{0}}{\partial s^{\mu}}\,F^{\mu}_{a}\,\mathrm{d}y^{a} + \frac{\partial H_{0}}{\partial s^{\mu}}\,\mathrm{d}s^{\mu}\right) \wedge \mathrm{d}^{m}x\\ &= \frac{\partial F^{\mu}_{a}}{\partial p^{I}}\,\mathrm{d}p^{I} \wedge \mathrm{d}y^{a} \wedge \mathrm{d}^{m-1}x_{\mu} + \frac{\partial F^{\mu}_{a}}{\partial y^{b}}\,\mathrm{d}y^{b} \wedge \mathrm{d}y^{a} \wedge \mathrm{d}^{m-1}x_{\mu}\\ &\quad + \frac{\partial H_{0}}{\partial p^{I}}\,\mathrm{d}p^{I} \wedge \mathrm{d}^{m}x + \left(\frac{\partial H_{0}}{\partial y^{a}} - \frac{\partial H_{0}}{\partial s^{\mu}}\,F^{\mu}_{a} - \frac{\partial F^{\mu}_{a}}{\partial x^{\mu}}\right)\mathrm{d}y^{a} \wedge \mathrm{d}^{m}x\,.\end{aligned} \tag{61}$$

For this system the field equations for sections $\boldsymbol{\psi} : M \to \mathcal{P}^{*}_{0}$ and for $\widetilde{\tau}_{0}$-transverse, locally decomposable multivector fields $\mathbf{X}_{\mathcal{H}_{0}} \in \mathfrak{X}^{m}(\mathcal{P}^{*}_{0})$ are, respectively,

$$\boldsymbol{\psi}^{*}\Theta^{0}_{\mathcal{H}} = 0\,, \quad \boldsymbol{\psi}^{*}\imath(Y)\overline{\Omega}^{0}_{\mathcal{H}} = 0\,, \quad \text{for every } Y \in \mathfrak{X}(\mathcal{P}^{*})\,. \tag{62}$$

and

$$\imath(\mathbf{X}_{\mathcal{H}_{0}})\Theta^{0}_{\mathcal{H}} = 0\,, \quad \imath(\mathbf{X}_{\mathcal{H}_{0}})\overline{\Omega}^{0}_{\mathcal{H}} = 0\,; \tag{63}$$

where the condition of $\widetilde{\tau}_{0}$-transversality is imposed again by requiring that $\imath\left(\mathbf{X}_{\mathcal{H}_{0}}\right)\omega = 1$. In general, these equations are not compatible on $\mathcal{P}^{*}_{0}$ and a constraint algorithm must be implemented in order to find a final constraint submanifold $\mathcal{P}^{*}_{f} \hookrightarrow \mathcal{P}^{*}_{0}$ (if it exists) where there are integrable multivector fields solutions to the field equations on $\mathcal{P}^{*}_{f}$ and being tangent to $\mathcal{P}^{*}_{f}$. This algorithm is explained in some of the applications developed in Sections 4 and 5.

If $\mathbf{X}_{\mathcal{H}_0} \in \mathsf{X}^m(\mathcal{P}_0^*)$ is a $\widetilde{\tau}_0$-transverse and locally decomposable multivector field whose local expression is

$$\mathbf{X}_{\mathcal{H}_0} = \bigwedge_{\mu=0}^{m-1}\left(\frac{\partial}{\partial x^\mu} + (X_{\mathcal{H}_0})^a_\mu \frac{\partial}{\partial y^a} + (X_{\mathcal{H}_0})^I_\mu \frac{\partial}{\partial p^I} + (X_{\mathcal{H}_0})^\nu_\mu \frac{\partial}{\partial s^\nu}\right), \tag{64}$$

and it is a solution to Equations (63); taking into account the local expressions (58) and (61), these field equations lead to

$$0 = (X_{\mathcal{H}_0})^a_\mu F^\mu_a + H_0 + (X_{\mathcal{H}_0})^\mu_\mu, \tag{65}$$

$$0 = -\frac{\partial F^\mu_a}{\partial p^I}(X_{\mathcal{H}_0})^I_\mu + \left(\frac{\partial F^\mu_b}{\partial y^a} - \frac{\partial F^\mu_a}{\partial y^b}\right)(X_{\mathcal{H}_0})^b_\mu + \frac{\partial H_0}{\partial y^a} - \frac{\partial H_0}{\partial s^\mu}F^\mu_a - \frac{\partial F^\mu_a}{\partial x^\mu}, \tag{66}$$

$$0 = \frac{\partial F^\mu_a}{\partial p^I}(X_{\mathcal{H}_0})^a_\mu + \frac{\partial H_0}{\partial p^I}, \tag{67}$$

together with a last group of equations which are identities when the above ones are taken into account. If $\boldsymbol{\psi}(x^\nu) = (x^\mu, y^a(x^\nu), p^I(x^\nu), s^\mu(x^\nu))$ is an integral section of $\mathbf{X}_{\mathcal{H}_0}$, then it is a solution to the Equations (62) which read as

$$0 = (F^\mu_a \circ \boldsymbol{\psi})\frac{\partial y^a}{\partial x^\mu} + (H_0 \circ \boldsymbol{\psi}) + \frac{\partial s^\mu}{\partial x^\mu}, \tag{68}$$

$$0 = \left(\frac{\partial F^\mu_a}{\partial p^I}\circ\boldsymbol{\psi}\right)\frac{\partial p^I}{\partial x^\mu} + \left(\left(\frac{\partial F^\mu_b}{\partial y^a} - \frac{\partial F^\mu_a}{\partial y^b}\right)\circ\boldsymbol{\psi}\right)\frac{\partial y^b}{\partial x^\mu} + \left(\frac{\partial H_0}{\partial y^a} - \frac{\partial H_0}{\partial s^\mu}F^\mu_a - \frac{\partial F^\mu_a}{\partial x^\mu}\right)\circ\boldsymbol{\psi}, \tag{69}$$

$$0 = \left(\frac{\partial F^\mu_a}{\partial p^I}\circ\boldsymbol{\psi}\right)\frac{\partial y^a}{\partial x^\mu} + \left(\frac{\partial H_0}{\partial p^I}\circ\boldsymbol{\psi}\right). \tag{70}$$

In general, these equations have no solutions everywhere on $\mathcal{P}_0^*$. In the most favorable cases, the Hamiltonian field equations are compatible only on a submanifold of $\mathcal{P}_f^* \subseteq \mathcal{P}_0^*$ which is obtained by applying a suitable constraint algorithm. This algorithm is explained in some of the applications developed in Sections 4 and 5.

4. Application to General Lagrangians

As first applications of this (pre)multicontact formulation, we consider, in general, the cases of field theories described by quadratic or affine Lagrangians.

4.1. Quadratic (regular and singular) Lagrangians

Many field theories in physics are described by quadratic-type Lagrangians,[85,86]

$$L(x^\mu, y^a, y^a_\mu) = \frac{1}{2}f^{\mu\nu}_{ab}(x^\alpha, y^c)\, y^a_\mu\, y^b_\nu - V(x^\alpha, y^c). \tag{71}$$

They are usually called *mechanical-type Lagrangians*, or *modified kinetic energy Lagrangians* when the functions $f^{\mu\nu}_{ab}$ are not constant. We will assume, without lost of generality, that $f^{\mu\nu}_{ab} = f^{\nu\mu}_{ba}$.

We study a modification of these Lagrangians including a term that depends on "action variables".

4.1.1. Lagrangian Formalism

As in Section 3.2, consider the bundle $\mathcal{P} \equiv J^1\pi \times \mathbb{R}^m$ with coordinates $(x^\mu, y^a, y^a_\mu, s^\mu)$. A *quadratic Lagrangian* in $\mathcal{P}$ has the form,

$$L(x^\mu, y^a, y^a_\mu, s^\mu) = \frac{1}{2}f^{\mu\nu}_{ab}(x^\alpha, y^c, s^\alpha)\, y^a_\mu\, y^b_\nu - V(x^\alpha, y^c, s^\alpha) \in \mathscr{C}^\infty(\mathcal{P}), \tag{72}$$

where we have included a dependence on the additional variables s^μ to the functions $f^{\mu\nu}_{ab}$ and V. This Lagrangian is either regular or singular, depending on the regularity of the quadratic form whose matrix is $\left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu}\right) = f^{\mu\nu}_{ab}$. The energy Lagrangian function is

$$E_{\mathcal{L}} = \frac{\partial L}{\partial y^a_\mu}\, y^a_\mu - L = \frac{1}{2}f^{\mu\nu}_{ab}\, y^a_\mu\, y^b_\nu + V \in \mathscr{C}^\infty(\mathcal{P}), \tag{73}$$

the Lagrangian m-form (25) is

$$\Theta_{\mathcal{L}} = -f^{\mu\nu}_{ab}\, y^b_\nu\, \mathrm{d}y^a \wedge \mathrm{d}^{m-1}x_\mu + E_{\mathcal{L}}\, \mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu \in \Omega^m(\mathcal{P}). \tag{74}$$

and

$$\sigma_{\Theta_{\mathcal{L}}} = -\left(\frac{1}{2}\frac{\partial f^{\mu\nu}_{ab}}{\partial s^\rho}\, y^a_\mu\, y^b_\nu - \frac{\partial V}{\partial s^\rho}\right)\mathrm{d}x^\rho. \tag{75}$$

Given a semiholonomic m-multivector field,

$$\mathbf{X}_{\mathcal{L}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1}\left(\frac{\partial}{\partial x^\mu} + y^a_\mu\frac{\partial}{\partial y^a} + (X_{\mathcal{L}})^a_{\mu\nu}\frac{\partial}{\partial y^a_\nu} + (X_{\mathcal{L}})^\nu_\mu\frac{\partial}{\partial s^\nu}\right) \in \mathsf{X}^m(\mathcal{P}), \tag{76}$$

the Lagrangian Equations (41) and (42) take the form

$$(X_{\mathcal{L}})^\mu_\mu = L, \tag{77}$$

$$\frac{1}{2}\frac{\partial f^{\mu\nu}_{bc}}{\partial y^a}\, y^b_\mu\, y^c_\nu - \frac{\partial V}{\partial y^a} - \frac{\partial f^{\mu\nu}_{ab}}{\partial x^\mu}\, y^b_\nu - \frac{\partial f^{\mu\nu}_{ac}}{\partial y^b}\, y^c_\nu\, y^b_\mu - \frac{\partial f^{\mu\alpha}_{ab}}{\partial s^\nu}\, y^b_\alpha (X_{\mathcal{L}})^\nu_\mu$$
$$- f^{\mu\nu}_{ab}(X_{\mathcal{L}})^b_{\mu\nu} = \left(\frac{\partial V}{\partial s^\mu} - \frac{1}{2}\frac{\partial f^{\alpha\beta}_{dc}}{\partial s^\mu}\, y^d_\alpha\, y^c_\beta\right) f^{\mu\nu}_{ab}\, y^b_\nu. \tag{78}$$

For the integral (holonomic) sections $\boldsymbol{\psi}(x^\nu) = \left(x^\mu, y^a(x^\nu), \frac{\partial y^a}{\partial x^\mu}(x^\nu), s^\mu(x^\nu)\right)$ of $\mathbf{X}_{\mathcal{L}}$, Equations (77) and (78) read,

$$\frac{\partial s^\mu}{\partial x^\mu} = L, \tag{79}$$

$$\frac{1}{2}\frac{\partial f^{\mu\nu}_{bc}}{\partial y^a}\frac{\partial y^b}{\partial x^\mu}\frac{\partial y^c}{\partial x^\nu} - \frac{\partial V}{\partial y^a} - \frac{\partial f^{\mu\nu}_{ab}}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu} - \frac{\partial f^{\mu\nu}_{ac}}{\partial y^b}\frac{\partial y^c}{\partial x^\nu}\frac{\partial y^b}{\partial x^\mu} - \frac{\partial f^{\mu\alpha}_{ab}}{\partial s^\nu}\frac{\partial y^b}{\partial x^\alpha}\frac{\partial s^\nu}{\partial x^\mu}$$
$$- f^{\mu\nu}_{ab}\frac{\partial^2 y^b}{\partial x^\mu \partial x^\nu} = \left(\frac{\partial V}{\partial s^\mu} - \frac{1}{2}\frac{\partial f^{\alpha\beta}_{dc}}{\partial s^\mu}\frac{\partial y^d}{\partial x^\alpha}\frac{\partial y^c}{\partial x^\beta}\right) f^{\mu\nu}_{ab}\frac{\partial y^b}{\partial x^\nu} \tag{80}$$

which are the Equations (44) and (45) for this Lagrangian.

When the Lagrangian is regular, $\Theta_{\mathcal{L}}$ is a multicontact form and these equations are compatible. In the singular case, if $\Theta_{\mathcal{L}}$ is a premulticontact form, Equations (78) could be incompatible; then compatibility constraints appear, and they define a submanifold of $\mathcal{P}$ where the equations have solutions. Therefore, a constraint algorithm must be implemented as usual in order to find the final constraint submanifold $S_f \subseteq \mathcal{P}$, if it exists, where there are consistent solutions.

As a particular situation, we can assume that $\frac{\partial f_{ab}^{\mu\nu}}{\partial s^a} = 0$ and that any coordinate has a velocity present, that is, for any a there exist μ, ν and b such that $f_{ab}^{\mu\nu} \neq 0$. Then, the Equations (78) reduce to

$$\frac{1}{2}\frac{\partial f_{bc}^{\mu\nu}}{\partial y^a} y_\mu^b y_\nu^c - \frac{\partial V}{\partial y^a} - \frac{\partial f_{ab}^{\mu\nu}}{\partial x^\mu} y_\nu^b - \frac{\partial f_{ac}^{\mu\nu}}{\partial y^b} y_\nu^c y_\mu^b - f_{ab}^{\mu\nu} (X_{\mathcal{L}})_{\mu\nu}^b = \frac{\partial V}{\partial s^\mu} f_{ab}^{\mu\nu} y_\nu^b \,. \tag{81}$$

which, as mentioned above, can be compatible or not depending on the regularity of L; namely, on the quadratic form $f_{ab}^{\mu\nu}$. For the integral (holonomic) sections of $\mathbf{X}_{\mathcal{L}}$, these equations lead to

$$\frac{1}{2}\frac{\partial f_{bc}^{\mu\nu}}{\partial y^a}\frac{\partial y^b}{\partial x^\mu}\frac{\partial y^c}{\partial x^\nu} - \frac{\partial V}{\partial y^a} - \frac{\partial f_{ab}^{\mu\nu}}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu} - \frac{\partial f_{ac}^{\mu\nu}}{\partial y^b}\frac{\partial y^c}{\partial x^\nu}\frac{\partial y^b}{\partial x^\mu} - f_{ab}^{\mu\nu}\frac{\partial^2 y^b}{\partial x^\mu \partial x^\nu} = \frac{\partial V}{\partial s^\mu} f_{ab}^{\mu\nu}\frac{\partial y^b}{\partial x^\nu} \,. \tag{82}$$

4.1.2. Hamiltonian Formalism

The Legendre map generated by the Lagrangian (72) gives the multimomenta

$$p_a^\mu = f_{ab}^{\mu\nu}\, y_\nu^b \,. \tag{83}$$

In the hyperregular case this map is a diffeomorphism and these conditions give us the Hamiltonian function

$$H(x^\mu, y^a, p_a^\mu, s^\mu) = \frac{1}{2} f_{\mu\nu}^{ab}(x^\mu, y^a, s^\mu)\, p_a^\mu\, p_b^\nu + V(x^\mu, y^a, s^\mu) \in \mathscr{C}^\infty(\mathcal{P}^*), \tag{84}$$

where $(f_{\mu\nu}^{ab}) = (f_{ab}^{\mu\nu})^{-1}$ in the sense $f_{\mu\nu}^{ab} f_{bc}^{\nu\rho} = \delta_c^a\, \delta_\mu^\rho$. The Hamiltonian premulticontact m-form has the local expression (48), and

$$\sigma_{\mathcal{H}} = \left(\frac{1}{2}\frac{\partial f_{\mu\nu}^{ab}}{\partial s^\rho} p_a^\mu\, p_b^\nu + \frac{\partial V}{\partial s^\rho}\right) \mathrm{d}x^\rho \,. \tag{85}$$

For a multivector field $\mathbf{X}_{\mathcal{H}} \in \mathfrak{X}^m(\mathcal{P}^*)$ whose local expression is (54), the Hamilton–De Donder–Weyl Equations (55) are

$$(X_{\mathcal{H}})_\mu^\mu = f_{\mu\nu}^{ab}\, p_a^\mu\, p_b^\nu - V \,, \quad (X_{\mathcal{H}})_\mu^a = f_{\mu\nu}^{ab}\, p_b^\nu \,, \tag{86}$$

$$(X_{\mathcal{H}})_{\mu a}^\mu = -\left(\frac{1}{2}\frac{\partial f_{\nu\rho}^{bc}}{\partial y^a} p_b^\nu\, p_c^\rho + \frac{\partial V}{\partial y^a} + \frac{1}{2}\frac{\partial f_{\nu\rho}^{bc}}{\partial s^\mu} p_a^\mu\, p_b^\nu\, p_c^\rho + \frac{\partial V}{\partial s^\mu} p_a^\mu\right). \tag{87}$$

The integral sections $\boldsymbol{\psi}(x^\nu) = (x^\mu, y^a(x^\nu), p_a^\mu(x^\nu), s^\mu(x^\nu))$ of $\mathbf{X}_{\mathcal{H}}$ are the solutions to the Equations (56) which in this case are

$$\frac{\partial s^\mu}{\partial x^\mu} = f_{\mu\nu}^{ab}\, p_a^\mu\, p_b^\nu - V \,, \quad \frac{\partial y^a}{\partial x^\mu} = f_{\mu\nu}^{ab}\, p_b^\nu \,, \tag{88}$$

$$\frac{\partial p_a^\mu}{\partial x^\mu} = -\left(\frac{1}{2}\frac{\partial f_{\nu\rho}^{bc}}{\partial y^a} p_b^\nu\, p_c^\rho + \frac{\partial V}{\partial y^a} + \frac{1}{2}\frac{\partial f_{\nu\rho}^{bc}}{\partial s^\mu} p_a^\mu\, p_b^\nu\, p_c^\rho + \frac{\partial V}{\partial s^\mu} p_a^\mu\right). \tag{89}$$

The singular case corresponds to the situation in which the matrix $(f_{ab}^{\mu\nu})$ is not regular. If the rank of this matrix is constant everywhere on $\mathcal{P}$, then the Lagrangian is almost-regular. Therefore, the Legendre map is not a diffeomorphism but a submersion onto its image, and the Equations (83) yield constraints defining the submanifold $\mathcal{P}_0^* \subset \mathcal{P}^*$. Then, we have the Hamiltonian function $H_0 \in \mathscr{C}^\infty(\mathcal{P}_0^*)$ and the Hamiltonian premulticontact m-form $\Theta_{\mathcal{H}}^0 \in \Omega^m(\mathcal{P}_0^*)$ given by (48), and we are in a situation like the one described generically in Section 3.4.

4.2. Affine Lagrangians

Some relevant classical field theories in physics such as the *Einstein–Palatini* (or *metric-affine*) approach to gravitation,[87–89] or Dirac fermion fields[85] (among others), are described by affine Lagrangians. In natural coordinates, their general expression is

$$L(x^\mu, y^b, y_\mu^b) = f_a^\alpha(x^\mu, y^b)\, y_\alpha^a - V(x^\mu, y^b) \,. \tag{90}$$

Next, we study a modification of these Lagrangians which include a dependence on "action variables".

4.2.1. Lagrangian Formalism

As above, consider the bundle $\mathcal{P} \equiv J^1\pi \times \mathbb{R}^m$ with coordinates $(x^\mu, y^a, y_\mu^a, s^\mu)$. An *affine Lagrangian* in $\mathcal{P}$ is a function of the form,

$$L(x^\mu, y^b, y_\mu^b, s^\mu) = f_a^\alpha(x^\mu, y^b, s^\mu)\, y_\alpha^a - V(x^\mu, y^b, s^\mu) \in \mathscr{C}^\infty(\mathcal{P}) \,, \tag{91}$$

which is a singular Lagrangian since $\frac{\partial^2 L}{\partial y_\nu^b \partial y_\mu^a} = 0$.

Note that, as pointed out in Remark 3.1, not every Lagrangian function of the form (91) yields a premulticontact structure. A particular case giving premulticontact structures are those affine Lagrangians such that $\frac{\partial f_a^\alpha}{\partial s^\mu} = 0$, and there exist functions $J_\mu^b \in C^\infty(\mathcal{P})$ such that

$$\frac{\partial V}{\partial s^\mu} f_a^\mu + \frac{\partial V}{\partial y^a} + \frac{\partial f_a^\mu}{\partial x^\mu} = J_\mu^b \left(\frac{\partial f_a^\mu}{\partial y^b} - \frac{\partial f_b^\mu}{\partial y^a}\right). \tag{92}$$

A relevant example of an affine Lagrangian satisfying these conditions is the metric-affine Lagrangian for General Relativity (see Section 5.4). Hereafter, we assume that all the affine Lagrangians are of this kind.

The Lagrangian energy function is

$$E_{\mathcal{L}} = \frac{\partial L}{\partial y_\mu^a} y_\mu^a - L = V \in \mathscr{C}^\infty(\mathcal{P}) \,, \tag{93}$$

the Lagrangian m-form (25) is

$$\Theta_{\mathcal{L}} = -f_a^\mu \, \mathrm{d}y^a \wedge \mathrm{d}^{m-1}x_\mu + V \, \mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu \in \Omega^m(\mathcal{P}) \,, \tag{94}$$

and

$$\sigma_{\Theta_{\mathcal{L}}} = -\left(\frac{\partial f_a^\alpha}{\partial s^\mu} y_\alpha^a - \frac{\partial V}{\partial s^\mu} \right) \mathrm{d}x^\mu \,. \tag{95}$$

For a semiholonomic m-multivector field such as

$$\mathbf{X}_{\mathcal{L}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + y_\mu^a \frac{\partial}{\partial y^a} + (X_{\mathcal{L}})_{\mu\nu}^a \frac{\partial}{\partial y_\nu^a} + (X_{\mathcal{L}})_\mu^\nu \frac{\partial}{\partial s^\nu} \right) \in \mathsf{X}^m(\mathcal{P}) \,, \tag{96}$$

the Lagrangian Equations (41) and (42) read

$$(X_{\mathcal{L}})_\mu^\mu = L \,, \tag{97}$$

$$\frac{\partial f_b^\mu}{\partial y^a} y_\mu^b - \frac{\partial V}{\partial y^a} - \frac{\partial f_a^\mu}{\partial x^\mu} - \frac{\partial f_a^\mu}{\partial y^b} y_\mu^b = \frac{\partial V}{\partial s^\mu} f_a^\mu \,. \tag{98}$$

For the integral (holonomic) sections $\boldsymbol{\psi}(x^\nu) = \left(x^\mu, y^a(x^\nu), \frac{\partial y^a}{\partial x^\mu}(x^\nu), s^\mu(x^\nu) \right)$ of $\mathbf{X}_{\mathcal{L}}$, Equations (97) and (98) lead to

$$\frac{\partial s^\mu}{\partial x^\mu} = L \,, \tag{99}$$

$$\frac{\partial f_b^\mu}{\partial y^a} \frac{\partial y^b}{\partial x^\mu} - \frac{\partial V}{\partial y^b} - \frac{\partial f_a^\mu}{\partial x^\mu} - \frac{\partial f_a^\mu}{\partial y^b} \frac{\partial y^b}{\partial x^\mu} = \frac{\partial V}{\partial s^\mu} f_a^\mu \,, \tag{100}$$

which are the Equations (44) and (45) for this system.

Due to the singularity of affine Lagrangians, Equations (98) are constraints, which we denote by ζ_a. They define the compatibility submanifold $\mathcal{S}_1 \subset \mathcal{P}$. Next, we impose the tangency of the multivector fields (96) which are solutions to the Lagrangian equations on $\mathcal{S}_1$. These conditions are

$$X_\alpha(\zeta_a)|_{\mathcal{S}_1} = 0 \,, \tag{101}$$

and originate a system of equations giving relations among the component functions $(X_{\mathcal{L}})_a^\nu$ and $(X_{\mathcal{L}})_\mu^\nu$, and/or new constraints. In the case where new constraints arise, the constraint algorithm continues as usual until obtaining the final constraint submanifold $\mathcal{S}_f \subseteq \mathcal{P}$.

4.2.2. Hamiltonian Formalism

The Legendre map generated by the Lagrangian (91) gives the relations

$$p_a^\mu = f_a^\mu \,, \tag{102}$$

which are *primary constraints* defining the submanifold $\mathcal{P}_0^* \subset \mathcal{P}^*$; thus $\mathcal{P}_0^*$ is diffeomorphic to E and local coordinates in $\mathcal{P}_0^*$ are (x^μ, y^a, s^μ). The Lagrangian is almost-regular and, as stated above, we assume the conditions $\frac{\partial f_a^\mu}{\partial s^\nu} = 0$ and (92).

The Hamiltonian function is

$$H_0 = V \in \mathscr{C}^\infty(\mathcal{P}_0^*) \,. \tag{103}$$

The Hamiltonian premulticontact m-form (58) is

$$\Theta_{\mathcal{H}}^0 = -f_a^\mu \, \mathrm{d}y^a \wedge \mathrm{d}^{m-1}x_\mu + V \, \mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu \in \Omega^m(\mathcal{P}_0^*), \tag{104}$$

and (x^μ, y^a, s^μ) are adapted coordinates for the premulticontact structure $(\Theta_{\mathcal{H}}^0, \omega)$. Furthermore,

$$\sigma_{\mathcal{H}^0} = \frac{\partial V}{\partial s^\mu} \, \mathrm{d}x^\mu \,. \tag{105}$$

For a $\widetilde{\tau}_0$-transverse, locally decomposable multivector field, such as

$$\mathbf{X}_{\mathcal{H}_0} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + (X_{\mathcal{H}_0})_\mu^a \frac{\partial}{\partial y^a} + (X_{\mathcal{H}_0})_\mu^\nu \frac{\partial}{\partial s^\nu} \right) \in \mathsf{X}^m(\mathcal{P}_0^*), \tag{106}$$

the Hamiltonian Equations (65) and (66) lead to

$$0 = -(X_{\mathcal{H}_0})_\mu^a f_a^\mu + V + (X_{\mathcal{H}_0})_\mu^\mu \,, \tag{107}$$

$$0 = \left(\frac{\partial f_a^\mu}{\partial y^b} - \frac{\partial f_b^\mu}{\partial y^a} \right) (X_{\mathcal{H}_0})_\mu^b + \frac{\partial V}{\partial y^a} + \frac{\partial V}{\partial s^\mu} f_a^\mu + \frac{\partial f_a^\mu}{\partial x^\mu} \,, \tag{108}$$

and Equations (67) vanish identically. For the integral sections $\boldsymbol{\psi}(x^\nu) = \left(x^\mu, y^a(x^\nu), s^\mu(x^\nu) \right)$ of $\mathbf{X}_{\mathcal{H}_0}$, Equations (107) and (108) read,

$$0 = -\frac{\partial y^a}{\partial x^\mu} f_a^\mu + V + \frac{\partial s^\mu}{\partial x^\mu} \,, \tag{109}$$

$$0 = \left(\frac{\partial f_a^\mu}{\partial y^b} - \frac{\partial f_b^\mu}{\partial y^a} \right) \frac{\partial y^b}{\partial x^\mu} + \frac{\partial V}{\partial y^a} + \frac{\partial V}{\partial s^\mu} f_a^\mu + \frac{\partial f_a^\mu}{\partial x^\mu} \,. \tag{110}$$

The compatibility of Equations (108) depends on the rank of the matrix $\left(\frac{\partial f_a^\mu}{\partial y^b} - \frac{\partial f_b^\mu}{\partial y^a} \right)$. If this system is incompatible, then compatibility constraints could appear, defining a submanifold of $\mathcal{P}_0^*$ where the equations have solutions, and therefore the constraint algorithm must be implemented as usual until finding the final constraint submanifold $\mathcal{P}_f^* \subseteq \mathcal{P}_0^*$ where consistent solutions exist.

5. Application to Particular Theories

In this section, we study some specific classical field theories in physics described by quadratic and affine Lagrangians. We will keep a generic expression for the Lagrangian's coefficients. Due to this generality, not all possible values will lead to stable or physically sound systems. Some particular instances with an interesting interpretation will be considered.

5.1. One-Dimensional Wave Equation with First-Order Terms and External Force

We study a one-dimensional wave equation, which can be considered an infinitely long vibrating string. Consider the coordinates (t, x) for the time and the space. Denote by y the separation of a point in the string from its equilibrium point, and hence y_t and y_x denote the multivelocities associated with the two independent variables. The Lagrangian function for this system is

$$L(y, y_t, y_x) = \frac{1}{2}\rho y_t^2 - \frac{1}{2}\tau y_x^2 , \tag{111}$$

where ρ is the linear mass density of the string and τ is the tension of the string. We assume that these quantities are constant. The Euler–Lagrange equation for this Lagrangian density is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} , \tag{112}$$

where $c^2 = \dfrac{\tau}{\rho}$ is the square of the propagation speed of the wave. This last equation is the one-dimensional wave equation.

5.1.1. Lagrangian Formalism

This is a particular case of a hyperregular quadratic Lagrangian.

To model a vibrating string with linear damping, we can modify the Lagrangian function (111) so that it becomes a multicontact Lagrangian.[52] The fiber bundle $\tau : \mathcal{P} \to \mathbb{R}^2$ has adapted coordinates $(t, x, y, y_t, y_x, s^t, s^x)$, and the volume form of the base space $\mathbb{R}^2$ reads $\omega = \mathrm{d}t \wedge \mathrm{d}x$. The modified Lagrangian function L reads

$$L(t, x, y, y_t, y_x, s^t, s^x) = \frac{1}{2}\rho y_t^2 - \frac{1}{2}\tau y_x^2 + a s^t - b\,\frac{\rho}{\tau}\, s^x + \rho y f(t, x) , \tag{113}$$

where $a, b \in \mathbb{R}$ are constants, and $f(t, x)$ represents an external force acting on the string. It is a regular Lagrangian. The Lagrangian energy associated with the Lagrangian (113) is

$$E_{\mathcal{L}} = \frac{1}{2}\rho y_t^2 - \frac{1}{2}\tau y_x^2 - a s^t + b\,\frac{\rho}{\tau}\, s^x - \rho y f(t, x) . \tag{114}$$

The Lagrangian multicontact two-form (25) is now

$$\Theta_{\mathcal{L}} = -\rho y_t\, \mathrm{d}y \wedge \mathrm{d}x - \tau y_x\, \mathrm{d}y \wedge \mathrm{d}t + E_{\mathcal{L}}\, \mathrm{d}t \wedge \mathrm{d}x + \mathrm{d}s^t \wedge \mathrm{d}x - \mathrm{d}s^x \wedge \mathrm{d}t . \tag{115}$$

and

$$\sigma_{\Theta_{\mathcal{L}}} = -\frac{\partial L}{\partial s^t}\mathrm{d}t - \frac{\partial L}{\partial s^x}\mathrm{d}x = -a\mathrm{d}t + b\frac{\rho}{\tau}\mathrm{d}x . \tag{116}$$

The Lagrangian Equation (79) for a holonomic section

$$\boldsymbol{\psi}(t, x) = \left(t, x, y(t, x), \frac{\partial y}{\partial t}(t, x), \frac{\partial y}{\partial x}(t, x), s^t(t, x), s^x(t, x)\right)$$

read

$$\frac{\partial s^t}{\partial t} + \frac{\partial s^x}{\partial x} = L , \tag{117}$$

$$\frac{\partial^2 y}{\partial t^2} - \frac{\tau}{\rho}\frac{\partial^2 y}{\partial x^2} = a\frac{\partial y}{\partial t} + b\frac{\partial y}{\partial x} + f(t, x) . \tag{118}$$

which are the Equations (44) and (45) for this Lagrangian. In particular, Equation (118) is the one-dimensional wave equation with an external force $f(t, x)$ and two first-order terms accounting for the nonconservativity.

5.1.2. Hamiltonian Formalism

The fiber bundle $\widetilde{\tau} : \mathcal{P}^* \to \mathbb{R}^2$ has adapted coordinates $(t, x, y, p^t, p^x, s^t, s^x)$. The Legendre map $\mathcal{FL} : \mathcal{P} \to \mathcal{P}^*$ associated with the Lagrangian (113) is given by

$$\mathcal{FL}(t, x, y, y_t, y_x, s^t, s^x) = (t, x, y, p^t, p^x, s^t, s^x) , \tag{119}$$

where $p^t = \rho y_t$ and $p^x = -\tau y_x$. The Legendre map is a global diffeomorphism since the Lagrangian function is hyperregular. The Hamiltonian function is

$$H = \frac{1}{2\rho}(p^t)^2 - \frac{1}{2\tau}(p^x)^2 - a s^t + b\,\frac{\rho}{\tau}\, s^x - \rho y f(t, x) . \tag{120}$$

The Hamiltonian m-form (48) is

$$\Theta_{\mathcal{H}} = -p^t\, \mathrm{d}y \wedge \mathrm{d}x + p^x\, \mathrm{d}y \wedge \mathrm{d}t + \mathrm{d}s^t \wedge \mathrm{d}x - \mathrm{d}s^x \wedge \mathrm{d}t \tag{121}$$

$$+ \left(\frac{1}{2\rho}(p^t)^2 - \frac{1}{2\tau}(p^x)^2 - a s^t + b\,\frac{\rho}{\tau}\, s^x - \rho y f(t, x)\right) \mathrm{d}t \wedge \mathrm{d}x , \tag{122}$$

and now

$$\sigma_{\Theta_{\mathcal{H}}} = \frac{\partial H}{\partial s^t}\mathrm{d}t + \frac{\partial H}{\partial s^x}\mathrm{d}x = -a\mathrm{d}t + b\frac{\rho}{\tau}\mathrm{d}x . \tag{123}$$

For a section $\boldsymbol{\psi}(t, x) = (t, x, y(t, x), p^t(t, x), p^x(t, x), s^t(t, x), s^x(t, x))$, the Herglotz–Hamilton–De Donder–Weyl Equations (56) read

$$\frac{\partial s^t}{\partial t} + \frac{\partial s^x}{\partial x} = \frac{1}{2\rho}(p^t)^2 - \frac{1}{2\tau}(p^x)^2 + a s^t - b\frac{\rho}{\tau}s^x + \rho y f(t, x) , \tag{124}$$

$$\frac{\partial y}{\partial t} = \frac{1}{\rho}p^t , \quad \frac{\partial y}{\partial x} = -\frac{1}{\tau}p^x , \tag{125}$$

$$\frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} = \rho f(t, x) + a p^t - b\frac{\rho}{\tau}p^x . \tag{126}$$

Combining the last three equations above, we obtain Equation (118). Thus, the Lagrangian and Hamiltonian formalisms are equivalent.

5.2. Klein–Gordon Equation: The Telegrapher Equation

The *Klein–Gordon equation*,

$$(\Box + m^2)y \equiv \partial_\mu \partial^\mu y + m^2 y \equiv \frac{\partial^2 y}{\partial x_\mu \partial x^\mu} + m^2 y = 0 , \tag{127}$$

is one of the most important equations in classical and quantum field theory [57,90, p. 108], In it, $\gamma(x^\mu)$ is a scalar field in the Minkowski space-time $\mathbb{R}^4$, we denote $\partial^\mu \equiv g^{\mu\nu}\partial_\nu$, $m \geq 0$ is a constant parameter, and $\Box$ denotes de D'Alembert operator in $\mathbb{R}^4$.

To develop the multisymplectic formulation of this theory,[74,91] we use space-time coordinates (x^μ), $\mu = 0, \dots, 3$, we denote γ the field variable, and γ_μ the corresponding multivelocities. Then, the multivelocity phase space $J^1\pi$ has natural coordinates $(x^\mu, \gamma, \gamma_\mu)$, and hence, the Lagrangian function for the Klein–Gordon equation is

$$L_0(x^\mu, \gamma, \gamma_\mu) = \frac{1}{2}\gamma_\mu \gamma^\mu - \frac{1}{2}m^2\gamma^2 , \tag{128}$$

(where $\gamma^\mu = g^{\mu\nu}\gamma_\nu$), which, evaluated on sections $\phi(x) = \left(x^\mu, \gamma(x), \frac{\partial\gamma}{\partial x^\mu}(x)\right)$, gives $L_0(\phi) = \frac{1}{2}\partial_\mu\gamma\,\partial^\mu\gamma - \frac{1}{2}m^2\gamma^2$. This Lagrangian function can be slightly modified to include a more generic potential, $L_0 = \frac{1}{2}\gamma_\mu\gamma^\mu - V(\gamma)$; however, we stick to the simplest case. Note that, in any case, it is a quadratic hyperregular Lagrangian (see Section 4.1).

5.2.1. Lagrangian Formalism

Consider now the bundle $\tau : \mathcal{P} \to \mathbb{R}^4$, with adapted coordinates $(x^\mu, \gamma, \gamma_\mu, s^\mu)$, and the volume form $\omega = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \equiv \mathrm{d}^4x$ on $\mathbb{R}^4$. The contactified Lagrangian $L \in \mathscr{C}^\infty(\mathcal{P})$ we propose is given by

$$L(x^\mu, \gamma, \gamma_\mu, s^\mu) = L_0(x^\mu, \gamma, \gamma_\mu) + \gamma_\mu s^\mu = \frac{1}{2}\gamma_\mu\gamma^\mu - \frac{1}{2}m^2\gamma^2 + \gamma_\mu s^\mu , \tag{129}$$

where $\gamma \equiv (\gamma_\mu) \in \mathbb{R}^4$ is a constant vector. It is a regular Lagrangian. Its associated Lagrangian energy is

$$E_{\mathcal{L}} = \frac{1}{2}\gamma_\mu\gamma^\mu + \frac{1}{2}m^2\gamma^2 - \gamma_\mu s^\mu . \tag{130}$$

The Lagrangian multicontact 4-form (25) is now

$$\Theta_{\mathcal{L}} = \gamma^\mu \mathrm{d}\gamma \wedge \mathrm{d}x_\mu + E_{\mathcal{L}}\mathrm{d}^4x + \mathrm{d}s^\mu \wedge \mathrm{d}x_\mu . \tag{131}$$

Now, $\sigma_{\Theta_{\mathcal{L}}} = -\gamma_\mu \mathrm{d}x^\mu$. The Equations (80) for holonomic sections $\boldsymbol{\psi}(x^\nu) = \left(x^\mu, \gamma(x^\nu), \frac{\partial\gamma}{\partial x^\mu}(x^\nu), s^\mu(x^\nu)\right)$ are

$$\frac{\partial s^\mu}{\partial x^\mu} = L , \tag{132}$$

$$m^2\gamma + g^{\mu\nu}\frac{\partial^2\gamma}{\partial x^\mu \partial x^\nu} = g^{\mu\nu}\gamma_\nu\frac{\partial\gamma}{\partial x_\mu} . \tag{133}$$

and they are the Equations (44) and (45) for this Lagrangian. Denoting $\gamma^\mu = g^{\mu\nu}\gamma_\nu$, we can write Equation (133) as

$$\frac{\partial^2\gamma}{\partial x_\mu \partial x^\mu} + m^2\gamma = \gamma^\mu\frac{\partial\gamma}{\partial x^\mu} , \tag{134}$$

which is the Klein–Gordon equation with additional first-order terms.

Remark 5.1. For simplicity, in this example, we have considered the Minkowski metric and γ_μ constants. However, a similar procedure can be performed for a generic metric $g_{\mu\nu} = g_{\mu\nu}(x^\nu)$ and functions $\gamma_\mu = \gamma_\mu(x^\nu)$, thus obtaining the equation

$$\frac{\partial^2\gamma}{\partial x_\mu \partial x^\mu} + m^2\gamma + \frac{\partial g_{\mu\nu}}{\partial x^\mu}\frac{\partial\gamma}{\partial x^\nu} = \gamma^\mu\frac{\partial\gamma}{\partial x^\mu} . \tag{135}$$

Telegrapher's equation: An interesting application of this modified Klein–Gordon Equation (134) is that we can derive from it the so-called *telegrapher's equation* (see refs. [92, p. 306] and [93 p. 653]) which describes the current and voltage on a uniform electrical transmission line:

$$\begin{cases} \dfrac{\partial V}{\partial x} = -L\dfrac{\partial I}{\partial t} - RI , \\ \dfrac{\partial I}{\partial x} = -C\dfrac{\partial V}{\partial t} - GV , \end{cases} \tag{136}$$

where V is the voltage, I is the current, R is the resistance, L is the inductance, C is the capacitance, and G is the conductance. This system can be decoupled, obtaining the system

$$\begin{cases} \dfrac{\partial^2 V}{\partial x^2} = LC\dfrac{\partial^2 V}{\partial t^2} + (LG + RC)\dfrac{\partial V}{\partial t} + RGV , \\ \dfrac{\partial^2 I}{\partial x^2} = LC\dfrac{\partial^2 I}{\partial t^2} + (LG + RC)\dfrac{\partial I}{\partial t} + RGI . \end{cases} \tag{137}$$

Note that the two equations in the system above are identical, and also known as telegrapher's equations. Both of them can be written as

$$\Box\gamma + \gamma\frac{\partial\gamma}{\partial t} + m^2\gamma = 0 , \tag{138}$$

where $\Box$ is the d'Alembert operator in 1+1 dimensions, and γ and m^2 are adequate constants. Written this way, we can see the telegrapher equation as a modified Klein–Gordon equation. More precisely, taking $\gamma_\mu = (-\gamma, 0, 0, 0)$ in (134), we obtain the telegrapher Equation (138).

5.2.2. Hamiltonian Formalism

The adapted coordinates of the fiber bundle $\widetilde{\tau} : \mathcal{P}^* \to \mathbb{R}^2$ are $(x^\mu, \gamma, p^\mu, s^\mu)$. The Legendre map $\mathcal{FL} : \mathcal{P} \to \mathcal{P}^*$ associated with the Lagrangian (129) is

$$\mathcal{FL}(x^\mu, \gamma, \gamma_\mu, s^\mu) = (x^\mu, \gamma, p^\mu, s^\mu) , \tag{139}$$

with $p^\mu = \gamma_\mu$. It is a diffeomorphism since the Lagrangian function is hyperregular. The Hamiltonian function is

$$H = \frac{1}{2}p^\mu p_\mu + \frac{1}{2}m^2\gamma^2 - \gamma_\mu s^\mu , \tag{140}$$

the contact Hamiltonian m-form (48) is

$$\Theta_{\mathcal{H}} = p^\mu \mathrm{d}\gamma \wedge \mathrm{d}x_\mu + H\,\mathrm{d}^4x + \mathrm{d}s^\mu \wedge \mathrm{d}x_\mu , \tag{141}$$

and now $\sigma_{\Theta_H} = -\gamma_\mu \mathrm{d}x^\mu$. The Herglotz–Hamilton–De Donder–Weyl Equations (56) for sections $\boldsymbol{\psi}(x^\nu) = (x^\mu, y(x^\nu), p^\mu(x^\nu), s^\mu(x^\nu))$ are

$$\frac{\partial s^\mu}{\partial x^\mu} = \frac{1}{2}p^\mu p_\mu - \frac{1}{2}m^2 y^2 + \gamma_\mu s^\mu \,, \tag{142}$$

$$\frac{\partial y}{\partial x^\mu} = p_\mu \,, \quad \frac{\partial p^\mu}{\partial x^\mu} = -m^2 y + \gamma_\mu p^\mu \,. \tag{143}$$

Combining the last two equations above, we obtain Equation (134). Thus, the Lagrangian and Hamiltonian formalisms are equivalent.

As pointed out at the end of the previous section, taking $\gamma_\mu = (-\gamma, 0, 0, 0)$, we recover the telegrapher's Equation (138) as a particular case of the Klein–Gordon equation with damping.

5.3. Maxwell's Electromagnetism

Action-dependent Maxwell's equations have been studied in the context of electromagnetism in matter. A variational derivation was first done in ref. [57] and later formalized using k-contact geometry in refs. [75, 94]. The damped/forced electromagnetic waves have been studied in ref. [57] and some applications to materials are explored in ref. [75]. We present the multicontact description based on ref. [52].

5.3.1. Lagrangian Formalism

We use the description of Electromagnetism in terms of principle bundles, that is, as a Yang–Mills theory. Consider the principle bundle $P \to M$ with structure group $U(1)$ over a four-dimensional space-time M. The associated bundle of connections is denoted as $\pi : C \to M$ (see ref. [95] for more details). The multicontact Lagrangian formalism takes place in the space $\mathcal{P} = J^1\pi \times_M \Lambda^3(\mathrm{T}^*M)$, with local coordinates $(x^\mu, A_\mu, A_{\mu,\nu}, s^\mu)$ such that $\omega = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. For this example, we consider the electromagnetic Lagrangian with a linear dissipation term:

$$L = -\frac{1}{4\mu_0} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\alpha\beta} - A_\alpha J^\alpha - \gamma_\alpha s^\alpha \,, \tag{144}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic tensor field, $g^{\mu\nu}$ is a constant metric on M with signature $(-,+,+,+)$, J^α and $\gamma_\alpha \in \mathscr{C}^\infty(M)$ are smooth functions (for $0 \le \alpha \le 3$) and μ_0 is a constant.[75] Notice that it is a singular Lagrangian.

The Lagrangian energy is

$$E_{\mathcal{L}} = -\frac{1}{4\mu_0} g^{\mu\nu} g^{\alpha\beta} F_{\beta\nu} F_{\alpha\mu} + A_\alpha J^\alpha + \gamma_\alpha s^\alpha \,, \tag{145}$$

and the Lagrangian 4-form is

$$\Theta_{\mathcal{L}} = \frac{1}{\mu_0} g^{\alpha\beta} g^{\mu\nu} F_{\beta\nu} \mathrm{d}A_\alpha \wedge \mathrm{d}^3 x_\mu + E_{\mathcal{L}} \mathrm{d}^4 x + \mathrm{d}s^\mu \wedge \mathrm{d}^3 x_\mu \,. \tag{146}$$

As the Lagrangian is singular, we should not expect the system to be multicontact, but it could be premulticontact if some extra compatibility conditions are met (see B.4). To check it, we need to compute the characteristic and Reeb distributions:

$$C = \left\langle \frac{\partial}{\partial A_{\mu,\nu}} + \frac{\partial}{\partial A_{\nu,\mu}} \right\rangle_{\mu,\nu=0,1,2,3} ,$$

$$\mathcal{D}^{\mathfrak{R}} = \left\langle \frac{\partial}{\partial A_{\mu,\nu}} + \frac{\partial}{\partial A_{\nu,\mu}}, \frac{\partial}{\partial s^\mu} \right\rangle_{\mu,\nu=0,1,2,3} . \tag{147}$$

We see that rank $(C) = 10$ and rank $(\mathcal{D}^{\mathfrak{R}}) = 14$, which are compatible with a space-time of dimension $m = 4$ and $k = 4$. The last condition holds because $\imath\left(\frac{\partial}{\partial s^\mu}\right)\Theta_{\mathcal{L}} = \mathrm{d}^3 x_\mu$. Therefore, it is a premulticontact Lagrangian system. The corresponding dissipation form is $\sigma_{\Theta_{\mathcal{L}}} = \gamma_\alpha \mathrm{d}x^\alpha$.

For the integral holonomic sections $\boldsymbol{\psi}(x^\mu) = \left(x^\mu, A_\alpha(x^\mu), A_{\alpha,\beta}(x^\mu) = \frac{\partial A_\alpha}{\partial x^\beta}, s^\nu(x^\mu)\right)$, Equations (80) read,

$$\begin{cases} \dfrac{\partial s^\mu}{\partial x^\mu} = L \,, \\ \mu_0 J^\mu = g^{\nu\sigma} g^{\mu\alpha} \left(\dfrac{\partial F_{\sigma\alpha}}{\partial x^\nu} + \gamma_\nu F_{\sigma\alpha} \right) . \end{cases} \tag{148}$$

To recover an expression which we can interpret as an electromagnetic field in matter, we set $g^{\mu\nu} = \frac{1}{\sqrt{1+\chi_m}} \mathrm{diag}\left(-(1+\chi_e)(1+\chi_m), 1, 1, 1\right)$, where χ_e and χ_m are the electric and magnetic susceptibilities respectively. Moreover, considering $x = 0 = ct$ and denoting $\gamma_\mu = \left(\frac{\gamma}{c}, \boldsymbol{\gamma}\right)$ and $J^\mu = (c\rho, \mathbf{j})$, the last equation of (148) reads in vector notation as [75]

$$\rho = (1+\chi_e)\epsilon_0 (\nabla \cdot \mathbf{E} + \boldsymbol{\gamma} \cdot \mathbf{E}) \,, \tag{149}$$

$$\mathbf{j} = -(1+\chi_e)\epsilon_0 \left(\frac{\partial \mathbf{E}}{\partial t} + \gamma \mathbf{E} \right) + \frac{1}{(1+\chi_m)\mu_0} (\nabla \times \mathbf{B} + \boldsymbol{\gamma} \times \mathbf{B}) \,. \tag{150}$$

When $\gamma_\nu = 0$, we recover the *Gauss Law (for electric fields)* and the *Ampère–Maxwell Law* for linear materials. The other Maxwell's equations, *Gauss Law (for magnetic fields)* and *Faraday–Henry–Lenz Law*, are the same as in the non-action-dependent case since they just state that the curvature of the connection is closed.

If $J^\mu = 0$, the Lagrangian and the equations are invariant under the transformation $A_\mu \to A_\mu + \frac{\partial f}{\partial x^\mu}$, for any smooth function f in M. To compute the dissipated quantities associated with this gauge group, we first need the infinitesimal symmetry:

$$Y = \frac{\partial f}{\partial x^\mu} \frac{\partial}{\partial A_\mu} + \frac{\partial^2 f}{\partial x^\mu x^\nu} \left(\frac{\partial}{\partial A_{\mu\nu}} + \frac{\partial}{\partial A_{\nu\mu}} \right) . \tag{151}$$

Then, the dissipated quantity is

$$\imath(Y)\Theta_{\mathcal{L}} = \frac{1}{\mu_0} g^{\alpha\beta} g^{\mu\nu} F_{\beta\nu} \frac{\partial f}{\partial x^\alpha} \mathrm{d}^3 x_\mu = \xi^\mu \mathrm{d}^3 x_\mu \,. \tag{152}$$

These are the same functions ξ^μ described in (15). The conservation laws are replaced by the so-called dissipation laws (15). The geometric formulation using multicontact geometry is, for solutions ψ,

$$\psi^* \mathrm{d}\left(\xi^\mu \wedge \mathrm{d}^3 x_\mu\right) = \psi^* \left(\sigma_{\Theta_{\mathcal{L}}} \wedge \left(\xi^\mu \mathrm{d}^3 x_\mu\right)\right), \tag{153}$$

which are equivalent to the second set of field Equations (148).

5.4. Metric-Affine Gravity

Gravity is one of the first field theories where action-dependent sources were considered. There are two promising applications. First, to describe dark energy as a nonconservative effect produced by an action-dependent term in the Lagrangian. Second, to describe open gravitational systems, like those emitting gravitational waves, as an "effective" field theory that is action-dependent.

The earliest proposal was ref. [96], adding a linear action term to the Einstein–Hilbert action. Due to difficulties in Herglotz's variational principle for field theories, the model was later upgraded in ref. [76]. This variational principle was clarified in refs. [55, 56], where the field equations are derived geometrically to ensure covariance. Another approach is to consider cosmology as a dynamical system with a scaling symmetry (or dynamical similarity).[97–99] The presence of a symmetry of this kind provides the system with a natural contact structure. For a general approach, see ref. [81].

The metric-affine model (or Einstein–Palatini) is a first-order singular field theory for General Relativity. A multisymplectic formulation of the model has been developed in several works (see, for instance, refs. [87, 89, 100, 101]). The metric-affine action is equivalent to the Einstein–Hilbert action in the final constraint surface, after performing a gauge reduction (see ref. [89] for a geometrical description). This property makes it an interesting nontrivial example for action-dependent field theories.

Given the relevance of action-dependent gravity, here we present the multicontact formalism for the metric-affine action with a linear term in the action.

5.4.1. Lagrangian Formalism

The configuration bundle is $\pi : \mathrm{E} \to M$, where M is a connected orientable 4-dimensional manifold representing space-time, with volume form denoted as $\omega \in \Omega^4(M)$, and $\mathrm{E} = \Sigma \times_M C(LM)$, where Σ is the manifold of Lorentzian metrics on M, with signature $(-+++)$, and $C(LM)$ is the bundle of connections on M; that is, linear connections in $\mathrm{T}M$.

Natural coordinates (x^μ) in M are taken such that $\omega = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \equiv \mathrm{d}^4x$. The adapted fiber coordinates in $J^1\pi$ are $(x^\mu, g_{\alpha\beta}, \Gamma^\nu_{\lambda\gamma}, g_{\alpha\beta,\mu}, \Gamma^\nu_{\lambda\gamma,\mu})$, with $\mu, \nu, \gamma, \lambda = 0, 1, 2, 3$ and $0 \le \alpha \le \beta \le 3$. The functions $g_{\alpha\beta}$ are the components of the metric and $\Gamma^\nu_{\lambda\gamma}$ are the Christoffel symbols of the connection. We do not assume torsionless connections and thus $\Gamma^\nu_{\lambda\gamma} \neq \Gamma^\nu_{\gamma\lambda}$, in general. Let $\lambda = \lambda_\mu \mathrm{d}x^\mu$ be a 1-form of M. The metric-affine Lagrangian is

$$L_{\mathrm{EP}} = \mathrm{g}R - \lambda_\mu s^\mu \,, \tag{154}$$

where $\mathrm{g} = \sqrt{|\det(g_{\alpha\beta})|}$, $R = g^{\alpha\beta}R_{\alpha\beta}$ is the *scalar curvature*, the components of the *Ricci tensor* are

$$R_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha,\gamma} - \Gamma^\gamma_{\gamma\alpha,\beta} + \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma} - \Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha} \,, \tag{155}$$

which depend only on the connection, and $g^{\alpha\beta}$ denotes the inverse matrix of g.

It is an affine Lagrangian, so we follow the general results presented in Section 4.2. We have that:

$$f^{\beta\gamma,\mu}_\alpha = \frac{\partial L}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} = \mathrm{g}\,(\delta^\mu_\alpha g^{\beta\gamma} - \delta^\beta_\alpha g^{\mu\gamma}) \,, \tag{156}$$

$$f^{\alpha\beta,\mu} = \frac{\partial L}{\partial g_{\alpha\beta,\mu}} = 0 \,, \tag{157}$$

$$V = \mathrm{g}\, g^{\alpha\beta}\left(\Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma}\right) + \lambda_\mu s^\mu \,. \tag{158}$$

This Lagrangian leads to a premulticontact Lagrangian because $\frac{\partial f^{\beta\gamma,\mu}_\alpha}{\partial s^\nu} = 0$, and it fulfills condition (92) as follows. We need two set of functions: $J_{\alpha\beta,\mu}$ with respect to the components of the metric $g_{\alpha\beta}$, and $J^\alpha_{\beta\gamma,\mu}$ with respect to the components of the connection $\Gamma^\alpha_{\beta\gamma}$. Their expressions can be computed explicitly (although they are not unique):

$$J_{\alpha\beta,\mu} = \frac{1}{\mathrm{g}\, n(\rho\sigma)}\left(-\frac{1}{2}g_{\rho\sigma}g_{\alpha\beta}\delta^\tau_\gamma + \frac{1}{6}g_{\alpha\beta}g_{\gamma\sigma}\delta^\tau_\rho - \frac{1}{3}g_{\alpha\gamma}g_{\beta\sigma}\delta^\tau_\rho + g_{\beta\sigma}g_{\alpha\rho}\delta^\tau_\gamma\right) \times \left(\lambda_\nu f^{\rho\sigma,\nu}_\tau + \frac{\partial V}{\partial \Gamma^\tau_{\rho\sigma}}\right), \tag{159}$$

$$J^\alpha_{\beta\gamma,\mu} = -\frac{4}{3\mathrm{g}\, n(\rho\sigma)}\delta^\alpha_\mu\left(g_{\beta\rho}g_{\gamma\sigma}\frac{\partial V}{\partial g_{\rho\sigma}} - \frac{1}{2}g_{\beta\gamma}g_{\rho\sigma}\frac{\partial V}{\partial g_{\rho\sigma}}\right). \tag{160}$$

The number $n(\rho\sigma)$ is 1, if $\rho = \sigma$, and 2, if $\rho \neq \sigma$. The Lagrangian premulticontact 4-form (25) is

$$\Theta_{\mathcal{L}} = -f^{\beta\gamma,\mu}_\alpha \,\mathrm{d}\Gamma^\alpha_{\beta\gamma} \wedge \mathrm{d}^3x_\mu + V\,\mathrm{d}^4x + \mathrm{d}s^\mu \wedge \mathrm{d}^3x_\mu \,, \tag{161}$$

and $\sigma_{\Theta_{\mathcal{L}}} = \lambda_\mu \mathrm{d}x^\mu$. For a holonomic section $\boldsymbol{\psi} : M \to J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M)$

$$\boldsymbol{\psi}(x^\nu) = \left(x^\mu, g_{\alpha\beta}(x^\nu), \frac{\partial g_{\alpha\beta}}{\partial x^\mu}(x^\nu), \Gamma^\alpha_{\beta\gamma}(x^\nu), \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\mu}(x^\mu), s^\mu(x^\nu)\right) \tag{162}$$

the Lagrangian Equations (44) and (45) are:

$$\frac{\partial s^\mu}{\partial x^\mu} = L \,, \tag{163}$$

$$\frac{\partial V}{\partial g_{\sigma\rho}} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\mu}\frac{\partial f^{\beta\gamma,\mu}_\alpha}{\partial g_{\sigma\rho}} = 0 \,, \tag{164}$$

$$\frac{\partial V}{\partial \Gamma^\alpha_{\beta\gamma}} + \sum_{\rho\le\sigma}\frac{\partial g_{\rho\sigma}}{\partial x^\mu}\frac{\partial f^{\beta\gamma,\mu}_\alpha}{\partial g_{\rho\sigma}} + \lambda_\mu f^{\beta\gamma,\mu}_\alpha = 0 \,. \tag{165}$$

Equation (164) has two distinct components. First, if we multiply by

$$-\frac{1}{3}g_{\epsilon\nu}g_{\zeta\gamma}\delta^\alpha_\beta + g_{\zeta\gamma}g_{\epsilon\beta}\delta^\alpha_\nu + \frac{1}{3}g_{\zeta\nu}g_{\epsilon\gamma}\delta^\alpha_\beta - g_{\epsilon\gamma}g_{\zeta\beta}\delta^\alpha_\nu \,, \tag{166}$$

and rearrange the terms, we obtain:

$$t^\alpha_{\beta\gamma} \equiv T^\alpha_{\beta\gamma} - \frac{1}{3}\delta^\alpha_\beta T^\nu_{\nu\gamma} + \frac{1}{3}\delta^\alpha_\gamma T^\nu_{\nu\beta} \,, \tag{167}$$

where $T^{\alpha}_{\beta\gamma} = \frac{1}{2}(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta})$. These are the torsion constraints and have the same expression as the action-independent case.[89] Assuming these constraints, the Equations (165) can be rewritten as:

$$m_{\rho\sigma,\mu} \equiv g_{\rho\sigma,\mu} - g_{\sigma\nu}\Gamma^{\nu}_{\mu\rho} - g_{\rho\nu}\Gamma^{\nu}_{\mu\sigma} - \frac{4}{3}g_{\rho\sigma}T^{\nu}_{\nu\mu} + g_{\rho\sigma}\lambda_{\mu} \Rightarrow \left(\nabla^{\Gamma}g\right)_{\rho\sigma,\mu} = \frac{4}{3}g_{\rho\sigma}T^{\nu}_{\nu\mu} - g_{\rho\sigma}\lambda_{\mu}, \tag{168}$$

which can be considered as a nonmetricity condition.

The covariant constraint algorithm requires us to compute the tangency conditions in all space-time directions. These are given by semiholonomic locally decomposable multivector fields $\mathbf{X}_{\mathcal{L}} \in \mathsf{X}^{m}(J^{1}\pi)$ whose integral sections are the solutions of the field equations. In this case,

$$\mathbf{X}_{\mathcal{L}} = \bigwedge_{\mu=0}^{3} X_{\mu} = \bigwedge_{\mu=0}^{3}\left(\frac{\partial}{\partial x^{\mu}} + g_{\alpha\beta,\mu}\frac{\partial}{\partial g_{\alpha\beta}} + X_{\alpha\beta\nu,\mu}\frac{\partial}{\partial g_{\alpha\beta,\nu}} + \Gamma^{\alpha}_{\beta\gamma,\mu}\frac{\partial}{\partial \Gamma^{\alpha}_{\beta\gamma}} + X^{\alpha}_{\beta\gamma\nu,\mu}\frac{\partial}{\partial \Gamma^{\alpha}_{\beta\gamma,\nu}} + X^{\nu}_{\mu}\frac{\partial}{\partial s^{\nu}}\right), \tag{169}$$

where $X_{\alpha\beta\nu,\mu}, X^{\alpha}_{\beta\gamma\nu,\mu}$ and X^{ν}_{μ} are smooth functions of $J^{1}\pi \times_{M} \Lambda^{m-1}(\mathrm{T}^{*}M)$. The tangency condition on the constraints leads to

$$0 = X_{\mu}(t^{\alpha}_{\beta\gamma}) = T^{\alpha}_{\beta\gamma,\mu} - \frac{1}{3}\delta^{\alpha}_{\beta}T^{\nu}_{\nu\gamma,\mu} + \frac{1}{3}\delta^{\alpha}_{\gamma}T^{\nu}_{\nu\beta,\mu}, \tag{170}$$

$$0 = X_{\tau}(m_{\rho\sigma,\mu}) = X_{\rho\sigma\mu,\tau} - g_{\sigma\nu,\tau}\Gamma^{\nu}_{\mu\rho} - g_{\sigma\nu}\Gamma^{\nu}_{\mu\rho,\tau} - g_{\rho\nu,\tau}\Gamma^{\nu}_{\mu\sigma} - g_{\rho\nu}\Gamma^{\nu}_{\mu\sigma,\tau} - \frac{4}{3}g_{\rho\sigma,\tau}T^{\nu}_{\nu\mu} - \frac{4}{3}g_{\rho\sigma}T^{\nu}_{\nu\mu,\tau} + g_{\rho\sigma,\tau}\lambda_{\mu} + g_{\rho\sigma}\frac{\partial\lambda_{\mu}}{\partial x^{\tau}}. \tag{171}$$

The first set of equations are new constraints. The subsequent tangency condition is:

$$0 = X_{\tau}(X_{\mu}(t^{\alpha}_{\beta\gamma})) = \frac{1}{2}\left(X^{\alpha}_{\beta\gamma\mu,\tau} - X^{\alpha}_{\gamma\beta\mu,\tau}\right) + \frac{1}{6}\left(\delta^{\alpha}_{\beta}X^{\nu}_{\nu\gamma\mu,\tau} - X^{\nu}_{\gamma\nu\mu,\tau} + \delta^{\alpha}_{\gamma}X^{\nu}_{\nu\beta\mu,\tau} - \delta^{\alpha}_{\gamma}X^{\nu}_{\beta\nu\mu,\tau}\right), \tag{172}$$

which are not constraints.

It is well-known that the trace of the torsion of the connection is a gauge freedom of the Metric-Affine gravity (see, for instance, refs. [89, 102]). We can fix the gauge condition $T^{\nu}_{\nu\mu} = \frac{3}{4}\lambda_{\mu}$, so that the metricity condition (170) holds, although the connection is not traceless. Alternatively, we can impose a torsionless connection, and the constraint (170) becomes:

$$\left(\nabla^{\Gamma}g\right)_{\rho\sigma,\mu} = -\lambda_{\mu}g_{\rho\sigma}, \tag{173}$$

which can be thought of as a dissipation law for the components of the metric. Here ∇^{Γ} denotes the covariant derivative generated by the connection Γ. In particular, if $\lambda \neq 0$, the connection Γ is not the Levi-Civita connection of g. This fact signals that, in the action-dependent case, one cannot generically recover the Einstein–Hilbert Lagrangian from the metric-affine one, as it is the case when λ vanishes.[89]

To investigate this point, we compare the other Equation (164) and the equations derived from the Einstein–Hilbert Lagrangian. Denote by $R(\Gamma)$, $R(g)$ the Ricci tensor of the connection Γ and of the metric g respectively. After minor algebraic effort, one can see that Equation (164) is equivalent to $R(\Gamma)_{\alpha\beta} = 0$. Imposing, for instance, that $T^{\nu}_{\nu\mu} = 0$, and considering the case where λ is closed, it can also be written as:

$$0 = R(\Gamma)_{\alpha\beta} = R(g)_{\alpha\beta} - \frac{1}{2}\left(\nabla^{g}_{\beta}\lambda_{\alpha} + \nabla^{g}_{\alpha}\lambda_{\beta} + g_{\alpha\beta}g^{\rho\sigma}\nabla^{g}_{\sigma}\lambda_{\rho} - \lambda_{\alpha}\lambda_{\beta} + g_{\alpha\beta}g^{\rho\sigma}\lambda_{\rho}\lambda_{\sigma}\right), \tag{174}$$

where ∇^{g} is the covariant derivative generated by the Levi-Civita connection of g. This equation is different from the one derived from the Einstein–Hilbert Lagrangian with the same action-dependent term:[55,76]

$$R(g)_{\alpha\beta} - \frac{1}{2}\left(\nabla^{g}_{\beta}\lambda_{\alpha} + \nabla^{g}_{\alpha}\lambda_{\beta} + g_{\alpha\beta}g^{\rho\sigma}\nabla^{g}_{\sigma}\lambda_{\rho} + 2\lambda_{\alpha}\lambda_{\beta} + g_{\alpha\beta}g^{\rho\sigma}\lambda_{\rho}\lambda_{\sigma}\right) = 0, \tag{175}$$

at least in the points where $T^{\nu}_{\nu\mu} = 0$. Thus, in general, the action-dependent metric–affine gravity is not equivalent to the action-dependent Einstein–Hilbert gravity, in the sense that the set of metrics that are solutions to the field equations is different.

An interpretation of this nonequivalence is the following. The metric-affine formulation has more solutions than the Einstein–Hilbert one, but it also contains a gauge symmetry. Each equivalent class of solutions contains exactly one solution of the Einstein–Hilbert field equations, and every solution to the latter is inside one equivalence class of the former. The gauge fixing that recovers this solution is the vanishing of the trace of the torsion, as already noticed by Einstein.[88] When we add an action-dependent term in the metric-affine Lagrangian that is no longer the case: in a class of solutions of the metric-affine Lagrangian, in general there is no solution to the Einstein–Hilbert Lagrangian with the same action-dependent term.

5.4.2. Hamiltonian Formalism

As it is explained in Section 4.2.2, the Legendre map induces the following relations on the multimomenta:

$$p^{\beta\gamma,\mu}_{\alpha} = \frac{\partial L}{\partial \Gamma^{\alpha}_{\beta\gamma,\mu}} = \mathrm{g}(\delta^{\mu}_{\alpha}g^{\beta\gamma} - \delta^{\beta}_{\alpha}g^{\mu\gamma}), \quad p^{\alpha\beta,\mu} = \frac{\partial L}{\partial g_{\alpha\beta,\mu}} = 0. \tag{176}$$

These are restrictions that determine all the multimomenta as functions of the positions and define locally the submanifold $\mathcal{P}^{*} \subset \mathcal{P}$. Therefore, we can use $(x^{\mu}, g_{\alpha\beta}, \Gamma^{\alpha}_{\beta\gamma}, s^{\mu})$ as coordinates on $\mathcal{P}^{*}$. The Hamiltonian 4-form $\Theta^{0}_{\mathcal{H}}$ has the same local expression as the Lagrangian one. The equations in coordinates are the same

as in the Lagrangian case, but now the velocities are not coordinates of the system. The torsion constraint is also present in the Hamiltonian formalism, as it is projectable by the Legendre map. Furthermore, the nonmetricity condition becomes an equation of motion.

5.5. Burgers' Equation from the Heat Equation

Burgers' equation[103] is a remarkable nonlinear partial differential equation.[93] It appears in applied mathematics, physics, fluid dynamics, etc. It reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2}\,, \tag{177}$$

where t, x are the independent variables, $u = u(t, x)$ is the dependent variable, and $k \geq 0$ is a diffusion coefficient. Burgers' equation is closely related to the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\,, \tag{178}$$

In this example we see that Burgers' equation can be obtained as a contactification of the heat equation (see also ref. [58]).

5.5.1. Lagrangian Formalism

Before contactifying the heat equation, we need a Lagrangian or Hamiltonian formulation of the heat equation. It is well known that the heat equation is not variational. However, it can be made variational by introducing an additional dependent variable v. The configuration bundle is $\pi : E \to \mathbb{R}^2$, where the coordinates in $\mathbb{R}^2$ and E are (t, x) and $(t, x; u, v)$, respectively. The phase bundle $J^1\pi$ has coordinates $(t, x; u, v, u_t, u_x, v_t, v_x)$. Consider the Lagrangian function $L_\circ \in \mathscr{C}^\infty(J^1\pi)$ given by

$$L_\circ = -ku_x v_x - \frac{1}{2}(vu_t - uv_t)\,, \tag{179}$$

whose Euler–Lagrange equations read

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0\,, \quad \frac{\partial v}{\partial t} + k\frac{\partial^2 v}{\partial x^2} = 0\,,. \tag{180}$$

Since the second equation is linear homogeneous, it always has solutions, i.e. $v = 0$. Thus, there is a one-to-one correspondence between the solutions to the heat Equation (178) and the solutions to the Euler–Lagrange Equations (180) with $v = 0$.

To perform the contactification, consider the fiber bundle $\tau : \mathcal{P} \to \mathbb{R}^2$, with adapted coordinates $(t, x; u, v, u_t, u_x, v_t, v_x, s^t, s^x)$, $\omega = \mathrm{d}t \wedge \mathrm{d}x$, and the singular Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$ given by

$$L = L_\circ - \gamma u s^x\,, \tag{181}$$

where $L_\circ$ is the Lagrangian function given in (179) and $\gamma \in \mathbb{R}$ is a constant. Note that we have added a term that, unlike in previous examples, is not of the simple form $\gamma_\mu(x^\nu)s^\mu$. This additional term will give rise to the nonlinear term in the Burgers' equation. The Lagrangian energy is

$$E_{\mathcal{L}} = -ku_x v_x + \gamma u s^x\,, \tag{182}$$

and the Lagrangian 2-form (25) is

$$\begin{aligned}\Theta_{\mathcal{L}} = &- kv_x \mathrm{d}u \wedge \mathrm{d}t - ku_x \mathrm{d}v \wedge \mathrm{d}t + \frac{1}{2}v\mathrm{d}u \wedge \mathrm{d}x - \frac{1}{2}u\mathrm{d}v \wedge \mathrm{d}x \\ &+ (-ku_x v_x + \gamma u s^x)\mathrm{d}t \wedge \mathrm{d}x + \mathrm{d}s^t \wedge \mathrm{d}x - \mathrm{d}s^x \wedge \mathrm{d}t\,,\end{aligned} \tag{183}$$

and $\sigma_{\Theta_{\mathcal{L}}} = \gamma u \mathrm{d}x$. Note that $(\Theta_{\mathcal{L}}, \omega)$ define a 2-premulticontact structure on $\mathcal{P}$ with

$$\mathcal{C} = \left\langle \frac{\partial}{\partial u_t}, \frac{\partial}{\partial v_t} \right\rangle\,, \quad D^{\mathfrak{R}} = \left\langle \frac{\partial}{\partial u_t}, \frac{\partial}{\partial v_t}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial s^x} \right\rangle.$$

For holonomic sections $\boldsymbol{\psi}(t, x) = (t, x, u(t, x), v(t, x), \frac{\partial u}{\partial t}(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial v}{\partial t}(t, x), \frac{\partial v}{\partial x}(t, x), s^t(t, x), s^x(t, x))$, the Herglotz–Euler–Lagrange Equations (44) and (45) are:

$$\begin{aligned}&\frac{\partial u}{\partial t} - k\gamma u\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2}\,,\\ &\frac{\partial v}{\partial t} + k\gamma u\frac{\partial v}{\partial x} = -k\frac{\partial^2 v}{\partial x^2} + \gamma s^x\,, \quad \frac{\partial s^t}{\partial t} + \frac{\partial s^x}{\partial x} = L\,.\end{aligned} \tag{184}$$

Note that there exists a correspondence between the solutions to Burgers' Equation (177) and the solutions $\left(u, v, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}, s^t, s^x\right)$ to the Herglotz–Euler–Lagrange Equations (184) for the Lagrangian (181), with $\gamma = -\frac{1}{k}$, $v = s^x = 0$ and $\frac{\partial s^t}{\partial t} = L$.

It is important to highlight that, although the Lagrangian (181) is singular, no compatibility constraints arise from the Herglotz–Euler–Lagrange equations.

5.5.2. Hamiltonian Formalism

We have provided a Lagrangian formulation for both the heat equation and Burgers' equation. Now, we develop the Hamilton–de Donder–Weyl formalism to the Lagrangian L defined in (181). The Legendre map associated with the Lagrangian L is the map

$$\mathcal{F}L(t, x; u, v, u_t, u_x, v_t, v_x, s^t, s^x) = (t, x; u, v, p^t, p^x, q^t, q^x, s^t, s^x), \tag{185}$$

where,

$$\begin{aligned}&p^t = \frac{\partial L}{\partial u_t} = -\frac{1}{2}v\,, \quad p^x = \frac{\partial L}{\partial u_x} = -kv_x\,, \quad q^t = \frac{\partial L}{\partial v_t} = \frac{1}{2}u\,,\\ &q^x = \frac{\partial L}{\partial v_x} = -ku_x\,.\end{aligned} \tag{186}$$

Hence, the image of the Legendre map $\mathcal{P}_0^* = \mathcal{F}L(\mathcal{P})$ is given by the two constraints

$$p^t + \frac{1}{2}v = 0\,, \quad q^t - \frac{1}{2}u = 0\,, \tag{187}$$

and the Lagrangian function L is almost-regular. We use coordinates $(t, x; u, v, p^x, q^x, s^t, s^x)$ on $\mathcal{P}_0^*$. Hence, the Hamiltonian on $\mathcal{P}_0^*$ is

$$H_0 = -\frac{1}{k}p^x q^x + \gamma u s^x . \tag{188}$$

The Hamiltonian 2-form (58) is

$$\Theta^0_{\mathcal{H}} = p^x \mathrm{d}u \wedge \mathrm{d}t + q^x \mathrm{d}v \wedge \mathrm{d}t + \frac{1}{2}v\mathrm{d}u \wedge \mathrm{d}x - \frac{1}{2}u\mathrm{d}v \wedge \mathrm{d}x + \left(-\frac{1}{k}p^x q^x + \gamma u s^x\right)\mathrm{d}t \wedge \mathrm{d}x + \mathrm{d}s^t \wedge \mathrm{d}x - \mathrm{d}s^x \wedge \mathrm{d}t . \tag{189}$$

Note that $(\Theta^0_{\mathcal{H}}, \omega)$ define a 2-multicontact structure on $\mathcal{P}_0^*$. We have $\sigma_{\Theta^0_{\mathcal{H}}} = \gamma\, u\mathrm{d}x$. For integral sections $\boldsymbol{\psi}(t,x) = (t, x, u(t,x), v(t,x), p^x(t,x), q^x(t,x), s^t(t,x), s^x(t,x))$, the Herglotz–Hamilton–de Donder–Weyl Equations (55) for this Hamiltonian:

$$\frac{\partial v}{\partial t} = \frac{\partial p^x}{\partial x} + \gamma(s^x + u p^x), \quad \frac{\partial u}{\partial t} = -\frac{\partial q^x}{\partial x} - \gamma u q^x ,$$
$$\frac{\partial u}{\partial x} = -\frac{1}{k}q^x , \quad \frac{\partial v}{\partial x} = -\frac{1}{k}p^x , \tag{190}$$

$$\frac{\partial s^t}{\partial t} + \frac{\partial s_x}{\partial x} = p^x\frac{\partial u}{\partial x} + q^x\frac{\partial v}{\partial x} - \frac{1}{2}v\frac{\partial u}{\partial t} + \frac{1}{2}u\frac{\partial v}{\partial t} + \frac{1}{k}p^x q^x - \gamma u s^x , \tag{191}$$

and there are no compatibility constraints. Combining the first four equations, we obtain

$$\frac{\partial u}{\partial t} - \gamma k u\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2} , \tag{192}$$

$$\frac{\partial v}{\partial t} + \gamma k u\frac{\partial v}{\partial x} = -k\frac{\partial^2 v}{\partial x^2} + \gamma s^x . \tag{193}$$

If we set the value of the constant γ to be $\gamma = -\frac{1}{k}$, Equation (192) becomes Burgers' equation (177).

Note that Equation (191) admits solutions $(u, v, p^x, q^x, s^t, s^x)$ with u being a solution to Burgers' Equation (177), $v = p^x = s^t = s^x = 0$, and $q^x = -k\frac{\partial u}{\partial x}$. As in the Lagrangian counterpart, no compatibility constraints arise.

Remark 5.2. The heat equation can also be described via a Hamiltonian formalism by taking the Legendre map associated with the Lagrangian (179).

6. Summary and Outlook

We have presented a multilayered introduction to action-dependent field theories. In a friendly presentation, we have discussed the interpretation of the "action-variables" and the field equations for regular Lagrangians (6) and Hamiltonians (12).

Necessitated by the intricacies of physically-motivated field theories, we have provided a quick presentation of the geometric framework underlying action-dependent field theory: multicontact geometry.[52] Geometric formulations are powerful techniques to study field theories, especially when they are not regular and/or symmetries are involved. Using the multicontact formalism, we have reviewed how to compute the dynamics of regular and singular Lagrangians. We have also shown how to construct the associated Hamiltonian formalism and how to apply the constraint algorithm.

We have developed a large collection of examples, illustrating how to derive the field equations and how to use the geometric structures. Both the Lagrangian and Hamiltonian approaches have been computed. The examples have been selected to showcase the influence of action-dependent terms on prototypical Lagrangians in physics.

First, we have studied the generic cases of field theories described by quadratic (regular and singular) Lagrangians, and by affine Lagrangians. Second, we have analyzed some well-known theories for which the standard Lagrangians have been modified by adding terms depending on the "action variables". In particular, we have considered two regular systems: the one-dimensional wave equation and the Klein–Gordon equation (and the telegrapher equation as a particular case), and three singular ones: Metric-affine gravity, Maxwell's electromagnetism, and the Burgers equation obtained from the heat equation. (Two additional examples are discussed in the following appendix: the Bosonic string and (2 + 1)-dimensional gravity and Chern–Simons equation.)

In the context of action-dependent field theories, the examples of metric-affine gravity, bosonic string, Chern–Simons, as well as the generic cases of quadratic and affine Lagrangians are new in the literature. The Klein–Gordon equation and Burger's equation were studied previously in the action-dependent context (see, for instance, refs. [94] and [58] respectively), but here we have described them using the multicontact formalism for the first time. All in all, this work represents the largest increase of new instances of action-dependent field theories so far.

The examples have been chosen to explore the behavior of action-dependent field theories. For instance, we observe that action-dependent terms have an intricate influence in General Relativity and produce new terms in the field equations that cannot be obtained by other kinds of particles. We have shown how equivalent Lagrangians lead to different field equations when an action-dependent term is added. Moreover, the nonconservative behavior of these theories makes them suitable for modeling open gravitational systems. These considerations make action-dependent gravity a promising field of research.

The study of the symmetries, with their associated dissipated quantities, remains a topic of great interest. The reduction of symmetries is currently being developed for dissipative mechanical systems,[104] but no similar procedure exists in the context of multicontact geometry yet. Another interesting object is the scaling symmetry or dynamical similarity. The recent work[81] points out that usual systems with this special symmetry benefit from a description using contact geometry.

On a more topological note, the "action variables" provide a connection between the boundary and the bulk. One can foreshadow that the boundary terms play an important role in action-dependent field theories. Moreover, the dissipation form, that encodes the nonconservation, has a strong link to the cohomology of the manifold. How these elements relate to each other and their effects on the dynamics remain to be scrutinized.

Another open problem of the multicontact framework of action-dependent classical field theory is developing an space-time split formalism. This will provide a Hamiltonian formula-

tion more akin to the commonly used one in theoretical physics. This approach will be presented in a future work.

Finally, multiple properties of multicontact geometry remain to be studied. To highlight two, Legendre submanifolds and the graded Jacobi algebra, which will pave the way to stronger geometric theorems to characterize action-dependent field theories.

Appendix A: Other Applications

In this appendix, we analyze two other interesting physical theories modified with action-dependent terms, whose multicontact treatment is similar to some of the cases previously studied. These are the Bosonic string, which is a regular theory, and the (2 + 1)-dimensional gravity and the Chern–Simons equation, which is another affine (and then singular) case.

A.1. Bosonic String Theory

The multisymplectic formulation of the bosonic string has been recently developed in refs. [73, 74, 91]. See ref. [105] for more details on string theory. In this theory, space-time is a $(d+1)$-dimensional manifold M endowed with a space-time metric $G_{\mu\nu}$ with signature $(-+\cdots+)$. Local coordinates on M are x^μ $(\mu = 0, 1, \dots, d)$. The string worldsheet Σ is a 2-dimensional manifold, with local coordinates σ^a, with $a = 0, 1$, which is endowed with the volume form $\omega = \mathrm{d}^2\sigma$. The fields $x^\mu(\sigma)$ are scalar fields on Σ given by the embedding maps $\Sigma \to M : \sigma^a \mapsto x^\mu(\sigma)$. Hence, the configuration bundle is $\pi : E = \Sigma \times M \to \Sigma$ and its sections are $\phi : \Sigma \to \Sigma \times M : \sigma^a \mapsto (\sigma^a, x^\mu(\sigma))$. The Lagrangian and Hamiltonian phase bundles $J^1\pi \to \Sigma$ and $J^{1*}\pi \to \Sigma$ have local coordinates $(\sigma^a, x^\mu, x^\mu_a)$ and $(\sigma^a, x^\mu, p^a_\mu)$, respectively. On $J^1\pi$ we also have a 2-form $g = \frac{1}{2} g_{ab} \mathrm{d}\sigma^a \wedge \mathrm{d}\sigma^b$, whose pullback by jet prolongations of sections $j^1\phi(\sigma^a) = \left(\sigma^a, x^\mu(\sigma^a), \frac{\partial x^\mu}{\partial \sigma^a}(\sigma^a)\right)$ gives the induced metric on Σ,

$$(j^1\phi)^* g = h \equiv \frac{1}{2} h_{ab} \mathrm{d}\sigma^a \wedge \mathrm{d}\sigma^b\,, \quad \text{where} \quad h_{ab} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}\,. \tag{194}$$

The bosonic string theory is described by the *Nambu–Goto Lagrangian*,

$$L_0 = -T\sqrt{-\det g} = -T\sqrt{-\det(G_{\mu\nu} x^\mu_a x^\nu_b)}\,, \tag{195}$$

where T is a constant called the *string tension*. This Lagrangian is regular since

$$\frac{\partial^2 L_0}{\partial x^\mu_a \partial x^\nu_b} = -T\sqrt{-\det g}\left[G_{\mu\nu} g^{ba} - G_{\mu\alpha} G_{\rho\nu} x^\alpha_c x^\rho_d \left(g^{ba} g^{cd} + g^{bc} g^{ad} - g^{ca} g^{bd}\right)\right], \tag{196}$$

where $g^{ba} \equiv (g^{-1})^{ba} = \frac{1}{\det g} \epsilon^{bc} \epsilon^{ad} g_{dc}$, is a regular matrix everywhere. Then, the Legendre map, which is a diffeomorphism, allows us to translate the 2-form g to $J^{1*}\pi$ as $\mathcal{FL}_{\circ *} g = g \in \Omega^2(J^{1*}\pi)$.

A.1.1. Lagrangian Formalism: For this theory, the bundle $\tau : \mathcal{P} \simeq J^1\pi \times \mathbb{R}^2 \to \Sigma$ has adapted coordinates $(\sigma^a, x^\mu, x^\mu_a, s^a)$. The contactified Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$ we propose is

$$\begin{aligned} &L(\sigma^a, x^\mu, x^\mu_a, s^a) \\ &\quad = L_0(\sigma^a, x^\mu, x^\mu_a) + \gamma_a s^a = -T\sqrt{-\det(G_{\mu\nu} x^\mu_a x^\nu_b)} + \gamma_a s^a \\ &\quad = -T\sqrt{-\det g} + \gamma_a s^a\,, \end{aligned} \tag{197}$$

where $\gamma \equiv (\gamma_a) \in \mathbb{R}^2$ is a constant vector, and it is a regular Lagrangian function. The Lagrangian energy is

$$\begin{aligned} E_{\mathcal{L}} &\equiv \frac{\partial L}{\partial x^\mu_a} x^\mu_a - L = -T\sqrt{-\det g}\,(g^{ba} g_{ab} - 1) - \gamma_a s^a \\ &= -T\sqrt{-\det g} - \gamma_a s^a \in \mathscr{C}^\infty(\mathcal{P})\,, \end{aligned} \tag{198}$$

where the last equality follows because g_{ab} is a 2×2 matrix in the case of the string. The Lagrangian multicontact 2-form (25) is

$$\begin{aligned} \Theta_{\mathcal{L}} &= \frac{\partial L}{\partial x^\mu_a} \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a - E_{\mathcal{L}}\, \mathrm{d}^2\sigma + \mathrm{d}s^a \wedge \mathrm{d}^1\sigma_a \\ &= -T\sqrt{-\det g}\, G_{\mu\nu} g^{ba} x^\nu_b \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a \\ &\quad - \left(T\sqrt{-\det g} + \gamma_a s^a\right) \mathrm{d}^2\sigma + \mathrm{d}s^a \wedge \mathrm{d}^1\sigma_a\,, \end{aligned} \tag{199}$$

where $\mathrm{d}^1\sigma_a = \imath\left(\frac{\partial}{\partial \sigma^a}\right) \mathrm{d}^2\sigma$. In this case, we have $\sigma_{\Theta_{\mathcal{L}}} = -\gamma_a \mathrm{d}\sigma^a$.

For holonomic sections $\boldsymbol{\psi}(\sigma) = \left(\sigma^a, x^\mu(\sigma), \frac{\partial x^\mu}{\partial \sigma^a}(\sigma), s^a(\sigma)\right)$ the Lagrangian Equations (41) and (42) lead to

$$\frac{\partial s^a}{\partial \sigma^a} = L\,, \tag{200}$$

$$\begin{aligned} &T\sqrt{-\det g}\, G_{\mu\nu} g^{ba} \gamma_a \frac{\partial x^\nu}{\partial \sigma^b} \\ &= \frac{\partial x^\rho}{\partial \sigma^a}\left[\frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\, G_{\mu\nu} g^{ba} \frac{\partial x^\nu}{\partial \sigma^b}\right) - \frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\, G_{\rho\nu} g^{ba} \frac{\partial x^\nu}{\partial \sigma^b}\right)\right] \\ &\quad + \sqrt{-\det g}\left[G_{\mu\nu} g^{ba} - G_{\mu\alpha} G_{\beta\nu}\left(g^{ba} g^{cd} + g^{cb} g^{ad} - g^{ca} g^{bd}\right) \frac{\partial x^\alpha}{\partial \sigma^c} \frac{\partial x^\beta}{\partial \sigma^d}\right] \\ &\quad \times \frac{\partial^2 x^\nu}{\partial \sigma^a \partial \sigma^b} + \left[\frac{1}{2}\sqrt{-\det g}\, g^{ba} \frac{\partial G_{\alpha\beta}}{\partial x^\mu} \frac{\partial x^\alpha}{\partial \sigma^a} \frac{\partial x^\beta}{\partial \sigma^b}\right. \\ &\quad \left. + \frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det g}\, G_{\mu\nu} g^{ba} \frac{\partial x^\nu}{\partial \sigma^b}\right)\right]. \end{aligned} \tag{201}$$

where (201) are the Herglotz–Euler–Lagrange Equations (44) and (45) which, using (194) and after some nontrivial calculations, can be written as

$$\begin{aligned} T\sqrt{-\det g}\, G_{\mu\nu} g^{ba} \gamma_a \frac{\partial x^\nu}{\partial \sigma^b} &= -\sqrt{-\det h}\, h^{ba} \frac{\partial G_{\alpha\beta}}{\partial x^\mu} \frac{\partial x^\alpha}{\partial \sigma^a} \frac{\partial x^\beta}{\partial \sigma^b} \\ &\quad + \frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det h}\, G_{\mu\nu} h^{ba} \frac{\partial x^\nu}{\partial \sigma^b}\right) \\ &\quad + \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det h}\, G_{\mu\nu} h^{ba} \frac{\partial x^\nu}{\partial \sigma^b}\right) \frac{\partial x^\rho}{\partial \sigma^a} \end{aligned}$$

$$+\sqrt{-\det h}\left[G_{\mu\nu}h^{ba}-G_{\mu\alpha}G_{\beta\nu}\left(h^{ba}h^{cd}+h^{cb}h^{ad}-h^{ca}h^{bd}\right)\right.$$
$$\left.\times\frac{\partial x^{\alpha}}{\partial\sigma^{c}}\frac{\partial x^{\beta}}{\partial\sigma^{d}}\right]\frac{\partial^{2}x^{\nu}}{\partial\sigma^{a}\partial\sigma^{b}}\,.$$

A.1.2. Hamiltonian Formalism: The Hamiltonian formalism is developed starting from the Legendre map, $\mathcal{FL}:\mathcal{P}\to\mathcal{P}^{*}$, which is

$$\mathcal{FL}^{*}\sigma^{a}=\sigma^{a}\,,\quad \mathcal{FL}^{*}x^{\mu}=x^{\mu}\,,\quad \mathcal{FL}^{*}p^{a}_{\mu}=-T\sqrt{-\det g}\;G_{\mu\nu}g^{ba}x^{\nu}_{b}\,. \tag{203}$$

This Legendre map is invertible as expected from the regularity of the Hessian matrix (196). The Hamiltonian function is

$$H(\sigma^{a},x^{\mu},p^{a}_{\mu},s^{a})\equiv p^{a}_{\mu}(\mathcal{FL}^{-1})^{*}x^{\mu}_{a}-(\mathcal{FL}^{-1})^{*}L\in\mathscr{C}^{\infty}(\mathcal{P}^{*})\,. \tag{204}$$

Introducing the matrix Π whose components are $\Pi^{ab}\equiv G^{\mu\nu}p^{a}_{\mu}p^{b}_{\nu}$, then

$$\begin{aligned}\mathcal{FL}^{*}\Pi^{ab}&=-T^{2}\det g\,g^{ba}\iff\mathcal{FL}^{*}\det\Pi\\&=(-T^{2}\det g)^{2}\det(g^{-1})=T^{4}\det g\,,\end{aligned} \tag{205}$$

from which it follows that $(\mathcal{FL}^{-1})^{*}x^{\nu}_{b}=-\frac{1}{T}\sqrt{-\det\Pi}\;G^{\mu\nu}\Pi_{ab}p^{a}_{\mu}$, and the Hamiltonian function can be written as

$$H(\sigma^{a},x^{\mu},p^{a}_{\mu},s^{a})=-\frac{1}{T}\sqrt{-\det\Pi}-\gamma_{a}s^{a}\,. \tag{206}$$

where $\Pi_{ab}=\frac{1}{\det\Pi}\epsilon_{cb}\epsilon_{da}\Pi^{cd}$.

The multicontact Hamiltonian 2–form is

$$\begin{aligned}\Theta_{\mathcal{H}}&=p^{a}_{\mu}\mathrm{d}x^{\mu}\wedge\mathrm{d}^{1}\sigma_{a}-H\,\mathrm{d}^{2}\sigma=p^{a}_{\mu}\mathrm{d}x^{\mu}\wedge\mathrm{d}^{1}\sigma_{a}\\&+\left(\frac{1}{T}\sqrt{-\det\Pi}+\gamma_{a}s^{a}\right)\mathrm{d}^{2}\sigma\,,\end{aligned} \tag{207}$$

and $\sigma_{\Theta_{\mathcal{H}}}=-\gamma_{a}\,\mathrm{d}\sigma^{a}$. For sections $\boldsymbol{\psi}(\sigma^{b})=(\sigma^{a},x^{\mu}(\sigma^{b}),p^{a}_{\mu}(\sigma^{b}),s^{a}(\sigma^{b}))$ the field equations become

$$\begin{aligned}\frac{\partial s^{a}}{\partial\sigma^{a}}&=\frac{\sqrt{-\det\Pi}}{T}\left(1-\Pi_{ba}G^{\mu\nu}p^{a}_{\mu}p^{b}_{\nu}\right)+\gamma_{a}s^{a}\\&=\frac{\sqrt{-\det\Pi}}{T}\left(1-\Pi_{ba}\Pi^{ab}\right)+\gamma_{a}s^{a}\,,\end{aligned} \tag{208}$$

$$\frac{\partial x^{\mu}}{\partial\sigma^{a}}=-\frac{\sqrt{-\det\Pi}}{T}\Pi_{ba}G^{\mu\nu}p^{b}_{\nu}\,, \tag{209}$$

$$\frac{\partial p^{a}_{\mu}}{\partial\sigma^{a}}=\frac{\sqrt{-\det\Pi}}{2T}\Pi_{ba}\frac{\partial G^{\rho\alpha}}{\partial x^{\mu}}p^{a}_{\rho}p^{b}_{\alpha}+\gamma_{a}p^{a}_{\mu}\,, \tag{210}$$

which are the corresponding Herglotz–Hamilton–De Donder–Weyl equations for the bosonic string. As usual, the Legendre map allows us to transform these equations into (200) and (201) along with the holonomy condition and thus the Lagrangian and Hamiltonian formalisms are equivalent.

A.2. (2 + 1)-dimensional Gravity Chern–Simons Equation

See refs. [73, 91] for the multisymplectic formulation of this theory. Gravity with cosmological constant λ in a 2 + 1-dimensional space-time M is developed in the tetrad formalism using the *vierbein* e^{a}_{μ} given by $g_{\mu\nu}=e^{a}_{\mu}e^{b}_{\nu}\eta_{ab}$ ($\mu=0,1,2$, $a=0,1,2$), and the Hodge dual spin connection $\omega^{a}_{\mu}=\frac{1}{2}\epsilon^{abc}\omega_{\mu bc}$, which are treated as the variational fields of the theory as in ref. [106].

A.2.1. Lagrangian Formalism: The configuration bundle $\pi:E\to M$ has coordinates $(x^{\mu},e^{a}_{\mu},\omega^{c}_{\rho})$. The multicontact Lagrangian formalism is established on the space $\mathcal{P}=J^{1}\pi\times_{M}\Lambda^{2}(\mathrm{T}^{*}M)$, with coordinates $(x^{\mu},e^{a}_{\mu},\omega^{c}_{\rho},e^{a}_{\sigma,\mu},\omega^{c}_{\sigma,\rho},s^{\alpha})$ such that the volume form is $\omega=\mathrm{d}x^{0}\wedge\mathrm{d}x^{1}\wedge\mathrm{d}x^{2}$. The Lagrangian of the Chern–Simons theory[106] with a linear dissipation term is given by

$$\begin{aligned}&L(x^{\mu},e^{a}_{\mu},\omega^{c}_{\rho},e^{a}_{\sigma,\mu},\omega^{c}_{\sigma,\rho},s^{\alpha})\\&\quad=2\epsilon^{\mu\nu\rho}\eta_{ac}e^{a}_{\mu}\omega^{c}_{\nu,\rho}+\epsilon^{\mu\nu\rho}\epsilon_{abc}\left(e^{a}_{\mu}\omega^{b}_{\nu}\omega^{c}_{\rho}+\frac{1}{3}\lambda e^{a}_{\mu}e^{b}_{\nu}e^{c}_{\rho}\right)-\gamma_{\alpha}s^{\alpha},\end{aligned} \tag{211}$$

which is an affine Lagrangian, where λ is the cosmological constant and $\gamma_{a}\in\mathbb{R}$. The Lagrangian energy is

$$E_{\mathcal{L}}=-\epsilon^{\mu\nu\rho}\epsilon_{abc}\left(e^{a}_{\mu}\omega^{b}_{\nu}\omega^{c}_{\rho}+\frac{1}{3}\lambda e^{a}_{\mu}e^{b}_{\nu}e^{c}_{\rho}\right)+\gamma_{\alpha}s^{\alpha}\,, \tag{212}$$

the Lagrangian 2-form (25) is

$$\begin{aligned}\Theta_{\mathcal{L}}&=2\epsilon^{\mu\nu\rho}\eta_{ac}e^{a}_{\mu}\mathrm{d}\omega^{c}_{\rho}\wedge\mathrm{d}^{2}x_{\nu}\\&+\left[-\epsilon^{\mu\nu\rho}\epsilon_{abc}\left(e^{a}_{\mu}\omega^{b}_{\nu}\omega^{c}_{\rho}+\frac{1}{3}\lambda e^{a}_{\mu}e^{b}_{\nu}e^{c}_{\rho}\right)+\gamma_{\alpha}s^{\alpha}\right]\mathrm{d}^{3}x\\&+\mathrm{d}s^{\mu}\wedge\mathrm{d}^{m-1}x_{\mu}\,,\end{aligned} \tag{213}$$

which is a premulticontact form, and now $\sigma_{\Theta_{\mathcal{L}}}=\gamma_{\mu}\mathrm{d}x^{\mu}$.

We can derive directly the equations of motion for multivector fields using the derived equations for affine Lagrangians (97) and (98). We have two sets of variables, the vierbein e^{a}_{μ} and the spin connection ω^{a}_{μ}. So, we have the semiholonomic multivector fields on $\mathcal{P}$ expressed as

$$\begin{aligned}\mathbf{X}_{\mathcal{L}}&=\bigwedge_{\sigma=0}^{2}X_{\sigma}\\&=\bigwedge_{\sigma=0}^{2}\left(\frac{\partial}{\partial x^{\sigma}}+e^{a}_{\mu,\sigma}\frac{\partial}{\partial e^{a}_{\mu}}+\omega^{c}_{\nu,\sigma}\frac{\partial}{\partial\omega^{c}_{\nu}}+F^{a}_{\mu\rho\sigma}\frac{\partial}{\partial e^{a}_{\mu,\rho}}\right.\\&\left.\qquad+G^{c}_{\nu\rho\sigma}\frac{\partial}{\partial\omega^{c}_{\nu,\rho}}+X^{\mu}_{\sigma}\frac{\partial}{\partial s^{\mu}}\right).\end{aligned} \tag{214}$$

Then, the Lagrangian field Equations (41) and (42) are

$$X^{\mu}_{\mu}=L\,, \tag{215}$$

$$\zeta_{1}{}^{\mu}_{a}\equiv\epsilon^{\mu\nu\rho}\left[\eta_{ac}\omega^{c}_{\nu,\rho}+\frac{1}{2}\epsilon_{abc}\left(\omega^{b}_{\nu}\omega^{c}_{\rho}+\lambda e^{b}_{\nu}e^{c}_{\rho}\right)\right]=0\,, \tag{216}$$

$$\zeta_{2}{}^{\nu}_{c}\equiv\epsilon^{\mu\nu\rho}\left[\eta_{ac}e^{a}_{\mu,\rho}-\epsilon_{abc}e^{a}_{\mu}\omega^{b}_{\rho}-\gamma_{\rho}\eta_{ac}e^{a}_{\mu}\right]=0\,. \tag{217}$$

The two last equations are constraints because they do not depend on the second derivatives of the fields, and they define the submanifold $\mathcal{P}_1 \subset \mathcal{P}$. The tangency conditions are:

$$X_\sigma({\zeta_1}_a^{\ \mu}) = \epsilon^{\mu\nu\rho}\left[\eta_{ac}G^c_{\nu,\rho\sigma} + \epsilon_{abc}\left(\omega^b_\nu\omega^c_{\rho,\sigma} + \lambda e^b_\nu e^c_{\rho,\sigma}\right)\right]\Big|_{\mathcal{P}_1} = 0\,, \tag{218}$$

$$X_\sigma({\zeta_2}_c^{\ \nu}) = \epsilon^{\mu\nu\rho}\Big[\eta_{ac}F^a_{\mu,\rho\sigma} - \epsilon_{abc}e^a_{\mu,\sigma}\omega^b_\rho - \epsilon_{abc}e^a_\mu\omega^b_{\rho,\sigma} - \frac{\partial\gamma_\rho}{\partial x^\sigma}\eta_{ac}e^a_\mu - \gamma_\rho\eta_{ac}e^a_{\mu,\sigma}\Big]\Big|_{\mathcal{P}_1} = 0. \tag{219}$$

These are compatible equations for $F^{\ a}_{e\,\mu,\rho\sigma}$ and $F^{\ c}_{\omega\,\nu,\rho\sigma}$, so the constraint algorithm finishes.

For the holonomic sections

$$\boldsymbol{\psi}(x^\mu) = \left(x^\mu, e^a_\mu(x^\mu), \omega^c_\rho(x^\mu), \frac{\partial e^a_\sigma}{\partial x^\mu}(x^\mu), \frac{\partial\omega^c_\sigma}{\partial x^\rho}(x^\mu), s^\alpha(x^\mu)\right) \tag{220}$$

of $\mathbf{X}_{\mathcal{L}}$, the Equations (215) become:

$$\frac{\partial s^\mu}{\partial x^\mu} = L\,, \quad {\zeta_1}_a^{\ \mu}(\psi) = 0\,, \quad {\zeta_2}_c^{\ \nu}(\psi) = 0\,. \tag{221}$$

As expected, the last two equations are constraints. The tangency conditions lead to the following equations for sections:

$$\epsilon^{\mu\nu\rho}\left[\eta_{ac}\frac{\partial^2\omega^c_\nu}{\partial x^\rho\partial x^\sigma} + \epsilon_{abc}\left(\omega^b_\nu\frac{\partial\omega^c_\rho}{\partial x^\sigma} + \lambda e^b_\nu\frac{\partial e^c_\rho}{\partial x^\sigma}\right)\right]\Big|_{\mathcal{P}_1} = 0\,, \tag{222}$$

$$\epsilon^{\mu\nu\rho}\Big[\eta_{ac}\frac{\partial^2 e^a_\mu}{\partial x^\rho\partial x^\sigma} - \epsilon_{abc}\frac{\partial e^a_\mu}{\partial x^\sigma}\omega^b_\rho - \epsilon_{abc}e^a_\mu\frac{\partial\omega^b_\rho}{\partial x^\sigma} - \frac{\partial\gamma_\rho}{\partial x^\sigma}\eta_{ac}e^a_\mu - \gamma_\rho\eta_{ac}\frac{\partial e^a_\mu}{\partial x^\sigma}\Big]\Big|_{\mathcal{P}_1} = 0\,. \tag{223}$$

A.2.2. Hamiltonian Formalism: For the multicontact Hamiltonian formalism we have $\mathcal{P}^* = J^{1*}\pi \times_M \Lambda^2(\mathrm{T}^*M)$, with coordinates $(x^\mu, e^a_\mu, \omega^c_\rho, p^{\mu\sigma}_a, \pi^{\ c}_{\nu\rho}, s^\alpha)$. The image of the Legendre transformation leads to the following constraints:

$$p^{\mu\sigma}_a = \frac{\partial L}{\partial e^a_{\mu,\sigma}} = 0\,, \quad \pi^{\ \nu\rho}_c = \frac{\partial L}{\partial\omega^c_{\nu,\rho}} = 2\epsilon^{\mu\nu\rho}\eta_{ac}e^a_\mu\,. \tag{224}$$

These constraints define a submanifold $\mathcal{P}^*_0 \subset \mathcal{P}^*$ and, as they are lineal relationships of the fields, the Lagrangian is almost-regular. Moreover, as they do not depend on s^μ, we can use the coordinates $(x^\mu, e^a_\mu, \omega^c_\nu, s^\mu)$ on $\mathcal{P}^*_0$.

The Hamiltonian function is

$$H_0 = -\epsilon^{\mu\nu\rho}\epsilon_{abc}\left(e^a_\mu\omega^b_\nu\omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\right) + \gamma_\alpha\mathrm{d}s^\alpha\,, \tag{225}$$

the Hamiltonian 3-form is

$$\Theta^0_{\mathcal{H}} = 2\epsilon^{\mu\nu\rho}\eta_{ac}e^a_\mu\mathrm{d}\omega^c_\rho\wedge\mathrm{d}^2x_\nu - \left[\epsilon^{\mu\nu\rho}\epsilon_{abc}\left(e^a_\mu\omega^b_\nu\omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\right) - \gamma_\alpha\mathrm{d}s^\alpha\right]\mathrm{d}^3x + \mathrm{d}s^\mu\wedge\mathrm{d}^{m-1}x_\mu\,, \tag{226}$$

which is a premulticontact form and $\sigma_{\Theta^0_{\mathcal{H}}} = \gamma_\mu\,\mathrm{d}x^\mu$.

For an integral section $\boldsymbol{\psi}(x^\mu) = \left(x^\mu, e^a_\mu(x^\mu), \omega^c_\rho(x^\mu), s^\alpha(x^\mu)\right)$, the above equations become:

$$\frac{\partial s^\mu}{\partial x^\mu} = L\,, \tag{227}$$

$$\epsilon^{\mu\nu\rho}\left[\eta_{ac}\frac{\partial\omega^c_\nu}{\partial x^\rho} + \frac{1}{2}\epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho + \lambda e^b_\nu e^c_\rho\right)\right] = 0\,, \tag{228}$$

$$\epsilon^{\mu\nu\rho}\left[\eta_{ac}\frac{\partial e^a_\mu}{\partial x^\rho} - \epsilon_{abc}e^a_\mu\omega^b_\rho - \gamma_\rho\eta_{ac}e^a_\mu\right] = 0\,. \tag{229}$$

Appendix B: Multicontact and Premulticontact Structures

See ref. [52] for an exhaustive exposition on multicontact and premulticontact structures and their application to describe non-conservative (or action-dependent) first-order field theories.

Let P be a manifold with $\dim P = m + N$ and $N \geq m \geq 1$, and two forms $\Theta, \omega \in \Omega^m(P)$ with constant rank. If $\mathcal{D} \subset \mathrm{T}P$ is a regular distribution and $\Gamma(\mathcal{D})$ denotes the set of sections of $\mathcal{D}$ (vector fields on $\mathcal{D}$)); for every $k \in \mathbb{N}$, let

$$\begin{aligned}\mathcal{A}^k(\mathcal{D}) &= \{\alpha \in \Omega^k(P) \mid \imath(Z)\alpha = 0\,;\, \forall Z \in \Gamma(\mathcal{D})\}\\ &= \{\alpha \in \Omega^k(P) \mid \Gamma(\mathcal{D}) \subset \ker\,\alpha\}\,,\end{aligned} \tag{230}$$

where $\ker\alpha = \{Z \in \mathsf{X}(P) \mid \imath(Z)\,\alpha = 0\}$.

The **Reeb distribution** $\mathcal{D}^{\mathfrak{R}} \subset \mathrm{T}P$ associated with the couple (Θ, ω) is defined, defined, at every point $\mathrm{p} \in P$, as

$$\mathcal{D}^{\mathfrak{R}}_{\mathrm{p}} = \left\{v \in (\ker\omega)|_{\mathrm{p}} \;\mid\; \imath(v)\mathrm{d}\Theta_{\mathrm{p}} \in \mathcal{A}^m_{\mathrm{p}}(\ker\omega)\right\}. \tag{231}$$

The set of sections of the Reeb distributions is denoted $\mathfrak{R} = \Gamma(\mathcal{D}^{\mathfrak{R}})$ and its elements $R \in \mathfrak{R}$ are called **Reeb vector fields**:

$$\mathfrak{R} = \{R \in \Gamma(\ker\omega) \mid \imath(R)\mathrm{d}\Theta \in \mathcal{A}^m(\ker\omega)\}\,. \tag{232}$$

Observe that $\ker\omega \cap \ker\mathrm{d}\Theta \subset \mathcal{D}^{\mathfrak{R}}$. If ω is a closed form, then $\mathfrak{R}$ is involutive.

Definition B.1. *The couple (Θ, ω) is a **premulticontact structure** if ω is a closed form and, for $0 \leq k \leq N - m$, we have that:*

(1) $\operatorname{rank}\ker\omega = N$.
(2) $\operatorname{rank}\mathcal{D}^{\mathfrak{R}} = m + k$.
(3) $\operatorname{rank}(\ker\omega \cap \ker\Theta \cap \ker\mathrm{d}\Theta) = k$.
(4) $\mathcal{A}^{m-1}(\ker\omega) = \{\imath(R)\,\Theta \mid R \in \mathfrak{R}\}$,

*The triple (P, Θ, ω) is a **premulticontact manifold** and Θ is a **premulticontact form** on P. The distribution $C \equiv \ker\omega \cap \ker\Theta \cap \ker\mathrm{d}\Theta$ is the **characteristic distribution** of (P, Θ, ω).*

*If $k = 0$, the couple (Θ, ω) is a **multicontact structure**, (P, Θ, ω) is a **multicontact manifold** and Θ is a **multicontact form** on P.*

Given a (pre)multicontact manifold (P, Θ, ω), there exists a unique 1-form $\sigma_\Theta \in \Omega^1(P)$, which is called the **dissipation form**, verifying that

$$\sigma_\Theta \wedge \imath(R)\Theta = \imath(R)\mathrm{d}\Theta\,, \quad \text{for every } R \in \mathfrak{R}\,. \tag{233}$$

Then, we define the operator

$$\overline{\mathrm{d}} : \Omega^k(P) \to \Omega^{k+1}(P)$$

$$\beta \mapsto \overline{\mathrm{d}}\beta = \mathrm{d}\beta + \sigma_\Theta \wedge \beta\,. \tag{234}$$

Theorem B.2. *Around every point* $\mathrm{p} \in P$ *of a premulticontact manifold* (P, Θ, ω)*, there exists a local chart of adapted coordinates* $(U; x^1, \dots, x^m, u^1 \dots, u^{N-m-k}, s^1 \dots, s^m, w^1, \dots, w^k)$ *such that*

$$\ker \omega|_U = \left\langle \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^{N-m-k}}, \frac{\partial}{\partial s^1}, \dots, \frac{\partial}{\partial s^m}, \frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^k} \right\rangle , \tag{235}$$

$$D^{\mathfrak{R}}|_U = \left\langle \frac{\partial}{\partial s^1}, \dots, \frac{\partial}{\partial s^m}, \frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^k} \right\rangle , \tag{236}$$

$$C|_U = \left\langle \frac{\partial}{\partial w^1}, \dots , \frac{\partial}{\partial w^k} \right\rangle . \tag{237}$$

On these charts, the coordinates (x^μ) can be chosen in such a way that the form ω can be expressed as $\omega|_U = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$.

If (P, Θ, ω) is a multicontact manifold, in the above chart of coordinates, there exists a unique local basis $\{R_\mu\}$ of $\mathfrak{R}$ such that

$$\imath\big(R_\mu\big)\Theta = \mathrm{d}^{m-1}x_\mu\,. \tag{238}$$

Moreover, $[R_\mu, R_\nu] = 0$. These vector fields $R_\mu \in \mathfrak{R}$ are the **local Reeb vector fields** of the multicontact manifold (P, Θ, ω) in the chart $U \subset P$. Furthermore, from (232), we have that there are local functions $\Gamma_\mu \in \mathscr{C}^\infty(U)$ associated with the basis $\{R_\mu\}$ which are given by

$$\imath\big(R_\mu\big)\mathrm{d}\Theta = \Gamma_\mu\,\omega\,, \quad \forall \mu\,, \tag{239}$$

because $\imath\left(R_\mu\right)\mathrm{d}\Theta \in \mathcal{A}^m_{\mathrm{p}}(\ker\omega) = \langle\omega\rangle$. As a consequence, the dissipation form can be locally expressed on these charts as

$$\sigma_\Theta = \Gamma_\mu \mathrm{d}x^\mu\,, \tag{240}$$

because $\sigma_\Theta \wedge \mathrm{d}^{m-1}x_\mu = \Gamma_\mu\,\omega = \Gamma_\mu\,\mathrm{d}^m x$, for every μ.

If (P, Θ, ω) is a premulticontact manifold, then there exist local vector fields $\{R_\mu\}$ of $\mathfrak{R}$ such that $\mathfrak{R} = \left\langle R_\mu \right\rangle + C$ and $\imath\left(R_\mu\right)\Theta = \mathrm{d}^{m-1}x_\mu$. They are unique up to a term in the characteristic distribution. Moreover $[R_\mu, R_\nu] \in \Gamma(C)$.

Observe that, using the adapted coordinates, the local Reeb vector fields are $R_\mu = \frac{\partial}{\partial s^\mu}$.

The local structure of (pre)multicontact manifolds is stated in the following:

Proposition B.3. *Every (pre)multicontact manifold* (P, Θ, ω) *is locally diffeomorphic to a fiber bundle* $\tau : P \to M$*, where* M *is an orientable manifold with volume form* ω_M*, and* $\omega = \tau^*\omega_M$.

This is the canonical model for (pre)multicontact manifolds and it is the situation which is interesting in field theories.

Definition B.4.

1. *The couple* (Θ, ω) *is a* ***multicontact bundle structure*** *and* (P, Θ, ω) *is said to be a* ***multicontact bundle*** *if:*
   (1) $\operatorname{rank} D^{\mathfrak{R}} = m$.
   (2) $\ker\omega \cap \ker\Theta \cap \ker\mathrm{d}\Theta = \{0\}$.
   (3) $\mathcal{A}^{m-1}(\ker\omega) = \{\imath(R)\,\Theta\,,\ R \in \mathfrak{R}\}$.
2. *The couple* (Θ, ω) *is a* ***premulticontact bundle structure*** *and* (P, Θ, ω) *is said to be a* ***premulticontact bundle*** *if, for* $0 < k \leq N - m$*, we have that:*
   (1) $\operatorname{rank} D^{\mathfrak{R}} = m + k$.
   (2) $\operatorname{rank}(\ker\omega \cap \ker\Theta \cap \ker\mathrm{d}\Theta) = k$.
   (3) $\mathcal{A}^{m-1}(\ker\omega) = \{\imath(R)\,\Theta\,,\ R \in \mathfrak{R}\}$,

Furthermore, for field theories, the (pre)multisymplectic structures satisfy an additional requirement:

Definition B.5. *If* (P, Θ, ω) *is a (pre)multicontact manifold such that*

$$\imath(X)\imath(Y)\Theta = 0\,, \quad \textit{for every } X, Y \in \Gamma(\ker\omega)\,, \tag{241}$$

then (Θ, ω) *is said to be a* ***variational (pre)multicontact structure*** *and* (P, Θ, ω) *is a* ***variational (pre)multicontact manifold****.*

The terminology comes from the fact that this last condition is what is imposed to the multicontact form to ensure that the theory is variational and hence the field equations derive from a Lagrangian (see ref. [56]).

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Conflict of Interest

The authors declare no conflicts of interest.

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