Journal of Nonlinear Science (2025) 35:42
https://doi.org/10.1007/s00332-025-10135-w

An Energy–Momentum Method for Ordinary Differential Equations with an Underlying k-Polysymplectic Manifold

Leonardo Colombo[1] · Javier de Lucas[2,3] · Xavier Rivas[4] · Bartosz M. Zawora[3]

Received: 28 November 2023 / Accepted: 25 January 2025

Abstract

This work presents a comprehensive review of the k-polysymplectic Marsden–Weinstein reduction theory, rectifying prior errors and inaccuracies in the literature while introducing novel findings. It also emphasises the genuine practical significance of seemingly minor technical details. On this basis, we introduce a novel k-polysymplectic energy–momentum method, new related stability analysis techniques, and apply them to Hamiltonian systems of ordinary differential equations relative to a k-polysymplectic manifold. We provide detailed examples of both physical and mathematical significance, including the study of complex Schwarz equations related to the Schwarz derivative, a series of isotropic oscillators, integrable Hamiltonian systems, quantum oscillators with dissipation, affine systems of differential equations, and polynomial dynamical systems.

Communicated by François Gay-Balmaz.

✉ Javier de Lucas
javier.de.lucas@fuw.edu.pl

Leonardo Colombo
leonardo.colombo@car.upm-csic.es

Xavier Rivas
xavier.rivas@urv.cat

Bartosz M. Zawora
b.zawora@uw.edu.pl

1 Centro de Automática y Robótica (CSIC-UPM), Carretera de Campo Real, km 0, 200, 28500 Arganda del Rey, Spain

2 Centre de Recherches Mathématiques, Université de Montréal, Succ. Centre-Ville, CP 6128, Montréal, QC H3C 3J7, Canada

3 Department of Mathematical Methods in Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland

4 Department of Computer Engineering and Mathematics, Universitat Rovira i Virgili, Avinguda Països Catalans 26, 43007 Tarragona, Spain

Published online: 12 February 2025

Keywords Energy–momentum method · k-polysymplectic manifold · Lie system · Marsden–Weinstein reduction · Relative equilibrium point · Stability

Mathematics Subject Classification Primary 34A26 · 34D20 · 37J39; Secondary 53B50 · 53C15

Contents

1 Introduction

The classical energy–momentum method is a technique for analysing a Hamiltonian system on a symplectic manifold, particularly in the region near solutions whose evolution is induced by the Lie symmetries of the Hamiltonian system (see Bloch 2003 for a historical introduction and Marsden and Simo 1988 for one of its foundational works). More specifically, it explores whether, over time, solutions converge towards or diverge from the solutions associated with the Lie symmetries of the Hamiltonian system. The classical energy–momentum method is grounded in the symplectic Marsden–Weinstein reduction theory and utilises stability analysis techniques.

The main ideas behind the energy–momentum method can be traced back to Routh, Poincaré, Lyapunov, Arnold, Lewis, and Smale, among others (see Bloch 2003, Section 3.14). Then, the classical energy–momentum method, devised and developed mainly by Marsden and Simo (1988), was successfully applied to many problems by numerous researchers (Abarbanel and Holm 1987; Marsden et al. 1990; Marsden and Ratiu 1999; Marsden et al. 1989; Ortega et al. 2005; Simo et al. 1991; Zenkov et al. 1998). Over the years, the energy–momentum method was extended to deal with more general differential equations, e.g. stochastic Hamiltonian systems (Bai and Zhang 2014), discrete systems (Marsden and Ratiu 1999; Simo and Tarnow 1992), etc (Wang and Krishnaprasad 1992). In this work, we develop a new energy–momentum method for Hamiltonian systems related to k-polysymplectic manifolds (Awane 1992; de León et al. 1988).

k-Symplectic geometry is a generalisation of symplectic geometry introduced by Awane (1992), Awane and Goze (2000). Posteriorly, de León et al. (1997, 1988), de León et al. (1988) and McLean and Norris (2000), Norris (1993) utilised k-symplectic geometry to describe first-order field theories (Búa et al. 2015; Echeverría-Enríquez et al. 1996; Román-Roy et al. 2007). k-Symplectic geometry is the same as the *polysymplectic geometry* described by Günther (1987), but differs from the polysymplectic geometry introduced by Giachetta et al. (1997), Sardanashvily (1995) and Kanatchikov (1998). k-Symplectic manifolds have been widely used to study physical systems governed by systems of partial differential equations. In particular, it gives a geometric description of the Euler–Lagrange and the Hamilton–de Donder–Weyl field equations and the systems described by them. For instance, k-symplectic geometry enables us to describe their symmetries, conservation laws, reductions, etc (Awane 1992; Günther 1987; Marrero et al. 2015; Román-Roy et al. 2007). As there are many k-symplectic-like definitions with related but mainly different and even contradictory meanings, it is relevant to fix properly the terminology. Hereafter, we will deal with *k-polysymplectic manifolds*, i.e. manifolds endowed with a closed nondegenerate differential two-form taking values in a k-dimensional vector space.

Remarkably, k-polysymplectic geometry has proved to be useful in the analysis of systems of ordinary differential equations and their so-called superposition rules (de Lucas and Vilariño 2015). Note that (de Lucas and Vilariño 2015) uses mainly the *k-symplectic structure notion*, namely a family of k differential closed two-forms $\omega_1, \ldots, \omega_k$ on a manifold M so that $\bigcap_{\alpha=1}^{k} \ker \omega_\alpha = 0$ and $\dim M = n(k+1)$ for some natural number n. It is also worth stressing that the study of systems of ordinary differential equations via k-polysymplectic geometry differs substantially from the standard framework, which is focused on systems of partial differential equations, and leads to new lines of research.

More specifically, this work focuses on studying systems of first-order differential equations describing the integral curves of a vector field. Moreover, we assume that the vector field is Hamiltonian relative to a k-polysymplectic manifold, which here amounts to the fact that it is Hamiltonian relative to a series of presymplectic forms whose kernels have zero intersection. We aim to develop an energy–momentum method for such systems of ordinary differential equations with an underlying k-polysymplectic geometry. To achieve this goal, we will begin by reviewing and improving previous works on k-polysymplectic Marsden–Weinstein reductions (Blacker 2019; de Lucas et al. 2023; García-Toraño Andrés and Mestdag 2023; Marrero et al. 2015; Munteanu et al. 2004), which is one of the basis of our k-polysymplectic energy–momentum method. Hopefully, our review will solve previous problems and inaccuracies in the k-polysymplectic reduction literature, and will allow us to understand the meaning of some of the findings of this work.

The first k-polysymplectic reduction was developed by Günther Awane (1992), de León et al. (2015), Günther (1987). Unfortunately, his work was flawed mainly due to the improper analysis of the double orthogonal relative to a k-polysymplectic form. More specifically, Günther (1987, Lemma 7.5 and Theorem 7.7) contain main Günther's mistakes, while Marrero et al. (2015, Section 2.2) provides an interesting counterexample showing Günther's error[1]. Another similarly flawed attempt to develop a k-polysymplectic reduction was accomplished in Munteanu et al. (2004). These mistakes were fixed in Marrero et al. (2015), where sufficient conditions to accomplish a k-polysymplectic reduction were established. Despite that, Marrero et al. (2015, Lemma 3.4) implicitly suggests that Sard's Theorem justifies that it is enough to assume that the k-polysymplectic momentum map is a submersion. Although this assumption works very well in the classical symplectic Marsden–Weinstein reduction theory and Sard's Theorem can be used to justify it Blankenstein and Ratiu (2004), our work proves that this condition is far from ideal in the k-polysymplectic geometry realm and why Sard's Theorem cannot be used in this new context. Moreover, practical examples showing that it is convenient to assume that the momentum map in k-polysymplectic geometry is not a submersion are provided. Then, we stress that it is appropriate to use a formalism with k-polysymplectic momentum maps that admit only weak regular points, as accomplished in de Lucas et al. (2023). We highlight that this provides a practical generalisation of the k-polysymplectic Marsden–Weinstein reduction and it completes the analysis performed in Blacker (2019), García-Toraño Andrés and Mestdag (2023), Günther (1987), Marrero et al. (2015).

Necessary and sufficient conditions for a k-polysymplectic Marsden–Weinstein reduction were described implicitly in Marrero et al. (2015, pg. 12) and spelled out in detail by Blacker in Blacker (2019). Unfortunately, one of Blacker's main theorems, namely (Blacker 2019, Theorem 3.22), has a small misleading typo in the statement of the conditions (as pointed out in García-Toraño Andrés and Mestdag (2023)), in its proof, and it presents other minor technical issues concerning the existence of certain submanifold structures. These latter facts are shown and explained in this work for the first time. It is also worth noting that Blacker analyses the occurrence of orbifolds in k-polysymplectic Marsden–Weinstein reductions for regular values of momentum maps related to pathological Lie group actions.

The need for the use of Ad^{*k}-equivariant momentum maps in the k-polysymplectic Marsden–Weinstein reductions was removed in de Lucas et al. (2023) by extending to the k-polysymplectic realm the classical theory of affine Lie group actions on symplectic manifolds (Ortega and Ratiu 2004). Next, García-Toraño and Mestdag reviewed in García-Toraño Andrés and Mestdag (2023) the sufficient conditions for the k-polysymplectic Marsden–Weinstein reduction devised in Marrero et al. (2015). They claimed that just one of the sufficient conditions for the k-polysymplectic reduction given in Marrero et al. (2015, Theorem 3.17, condition (3.6)) is enough to ensure the existence of a k-polysymplectic Marsden–Weinstein reduction. In this work, we show a mistake in the proof of one of the main results in García-Toraño Andrés and Mestdag (2023), used to justify the previous claim. Indeed, we here point out that García-Toraño Andrés and Mestdag (2023, Lemma 3.1) is false via a counterexample, and prove the general independence of the conditions in Marrero et al. (2015, Theorem

[1] There is a typo in Marrero et al. (2015, pg. 4) as its authors refer to Theorem 7.6 in Günther (1987), which does not exist: there is only Definition 7.6. Flawed Günther's reduction theorem is described in Günther (1987, Theorem 7.7).

3.17). Moreover, our work also explains other properties relative to such sufficient conditions.

In order to illustrate a relevant example of k-polysymplectic Marsden–Weinstein reduction, we review the construction of a k-polysymplectic manifold induced by k symplectic manifolds and a related k-polysymplectic Marsden–Weinstein reduction. It is worth noting that, in this case, and in our applications in Sect. 5, the sufficient conditions for the k-polysymplectic Marsden–Weinstein reduction given in Marrero et al. (2015) are generally simpler to apply than Blacker's necessary and sufficient conditions, as the conditions in Marrero et al. (2015) do not depend on double k-polysymplectic orthogonal spaces and can be verified using structures easily available in our examples.

Next, an energy–momentum method for Hamiltonian k-polysymplectic systems is developed. This entails the definition and characterisation of a relative equilibrium notion for k-polysymplectic Hamiltonian systems. In short, a relative equilibrium point for a k-polysymplectic Hamiltonian system is a point at which the dynamics is determined by a Hamiltonian Lie symmetry of the k-polysymplectic Hamiltonian system. After a k-polysymplectic Marsden–Weinstein reduction, relative equilibrium points project onto equilibrium points of a k-polysymplectic Hamiltonian system on a reduced k-polysymplectic manifold. Our k-polysymplectic energy–momentum method also requires the development of an appropriate modification of known symplectic stability techniques to a k-polysymplectic realm. In particular, the stability of relative equilibrium points for k-polysymplectic Hamiltonian systems is characterised by analysing the character of k different functions having, mainly, degenerate critical points, namely their Hessians are degenerate at critical points. As in the symplectic case, a formal stability giving sufficient but not necessary conditions for the stability in the reduced space are given. The interest in our formal stability condition is justified by our applications. Although the formal stability condition is easy to verify and can be used in many cases, it is worth noting that proving its properties is quite more difficult than in the symplectic case. Moreover, we here just sketch that it is possible to develop many other alternative sufficient conditions to ensure stability.

Then, some applications of our k-polysymplectic energy–momentum method are developed. In particular, the theory of Lie systems is used to transform certain automorphic Lie systems (Cariñena and de Lucas 2011; Cariñena et al. 2000, 2007; Winternitz 1983) into k-polysymplectic Hamiltonian systems. A Lie system is a non-autonomous system of first-order differential equations whose general solution can be written as an autonomous function, a *superposition rule*, of a generic family of particular solutions and some constants. Lie systems are very important due to their applications and mathematical properties (Cariñena and de Lucas 2011; de Lucas and Sardón 2020). Automorphic Lie systems are Lie systems in Lie groups of special relevance, in particular, in control theory (Cariñena and Ramos 2003). A k-polysymplectic manifold is used to study complex Schwarz equations, which are here studied through the theory of Lie systems and k-polysymplectic geometry for the first time (see de Lucas and Sardón 2020 for the analysis of the real, simpler, case). It is worth noting that the complex Schwarz equation provides the description, when written as a first-order system of differential equations, of certain properties of the Schwarz derivative, which has applications in string theory, modular forms, hypergeometric functions (Guieu and

Roger 2007; Hille 1997; Lehto 1979), and other related equations (Bozhkov and da Conceição 2020). Automorphic Lie systems related to quantum oscillators with dissipative terms are also studied via k-polysymplectic techniques. We develop methods to study certain dynamical systems via Hamiltonian k-polysymplectic systems. This is applied to a family of k particles in a three-dimensional space, that are under the effect of different isotropic potentials and have no interaction between them. In this case, the techniques of our k-polysymplectic energy–momentum method are illustrated. Furthermore, a particular type of affine Lie system is used to show certain aspects of our k-polysymplectic energy–momentum method. Potentially, the ideas used in this latter example could be used to study affine control systems of a similar type (Cariñena and Ramos 2003). Other examples related to differential equations with polynomial coefficients are presented and analysed.

The structure of the paper goes as follows. Section 2 presents the basic notions and terminology to be used in our work. More particularly, Sect. 2.1 provides a review of the fundamentals of Lyapunov stability. In Sect. 2.2, we delve into the theory of k-polysymplectic manifolds, introducing the concept of an $\boldsymbol{\omega}$-Hamiltonian vector field and function on such a manifold in Sect. 2.3, and studying k-polysymplectic momentum maps in Sect. 2.4. Section 3 is dedicated to enhancing the existing Marsden–Weinstein reduction procedures for k-polysymplectic manifolds and presenting a k-polysymplectic Marsden–Weinstein reduction of the dynamics governed by an $\boldsymbol{\omega}$-Hamiltonian vector field. Note that this implies that some previous results, like (Marrero et al. 2015, Theorem 4.4), are here slightly modified to analyse more efficiently systems of ordinary differential equations. Relevantly, this section surveys and corrects many inaccuracies and mistakes in the previous literature. Section 4 introduces an energy–momentum method for systems of ordinary differential equations (ODEs) with an underlying k-polysymplectic structure. We define and characterise the concept of a relative equilibrium point for such systems. A theory of stability for the analysis of relative equilibrium points for k-polysymplectic Hamiltonian systems is presented. In Sect. 5, we thoroughly examine several relevant examples, including the complex Schwarz equation, the product of multiple symplectic manifolds along with a related family of isotropic oscillators, an affine first-order system of differential equations related to Lie systems and, potentially, to control systems, and quantum harmonic oscillators with dissipative terms. Finally, Sect. 6 summarises the conclusions of our work and offers insights into potential avenues for further development.

2 Fundamentals

Let us set some general assumptions and notation to be used throughout this work. It is hereafter assumed that all structures are smooth. Manifolds are real, Hausdorff, connected, paracompact, and finite-dimensional. Differential forms are assumed to have constant rank unless otherwise stated. Summation over crossed repeated indices is understood, although it can be explicitly detailed at times to improve the clarity of our presentation. All our considerations are local to stress our main ideas and to avoid technical problems concerning the global manifold structure of quotient spaces and similar issues. Hereafter, $\mathfrak{X}(P)$ and $\Omega^k(P)$ stand for the $\mathscr{C}^\infty(P)$-modules of vector fields and differential k-forms on a manifold P.

2.1 Lyapunov Stability

Let us establish some fundamental notions and theorems on the stability of dynamical systems used in our k-polysymplectic formulation of the energy–momentum method (de Lucas and Zawora 2021; Zawora 2021).

Since all manifolds considered in this work are paracompact and Hausdorff, they admit a Riemannian metric $\mathbf{g}$ (Lee 2009). The topology induced by $\mathbf{g}$ is the one of the manifold (Lee 2009, 2012; Zawora 2021). The metric $\mathbf{g}$ induces a distance in P so that the distance between two points $x_1, x_2 \in P$ is given by

$$d_{\mathbf{g}}(x_1, x_2) := \inf \left\{ \ell_{\mathbf{g}}(\gamma) \;\mid\; \gamma : [0,1] \to P, \;\; \gamma(0) = x_1, \;\; \gamma(1) = x_2 \right\},$$

where $\ell_{\mathbf{g}}(\gamma)$ is the length of the smooth curve $\gamma : [0,1] \to P$ relative to the metric $\mathbf{g}$.

Moreover, consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x), \qquad \forall x \in P, \tag{2.1}$$

where X is a vector field on P. A point $x_e \in P$ such that $X(x_e) = 0$. Such a point is called an *equilibrium point* of (2.1), or indistinctly, of X. Furthermore, x_e is *stable* if, for every ball $B_{x_e,\varepsilon} := \{x \in P \mid d_{\mathbf{g}}(x, x_e) < \epsilon\}$, there exists a radius $\delta(\varepsilon, x_e)$ such that every solution $x(t)$ of (2.1) with initial condition $x(t_0) = x_0 \in B_{x_e,\delta(\varepsilon,x_e)}$ for some $t_0 \in \mathbb{R}$ is contained in $B_{x_e,\varepsilon}$ for $t > t_0$. An equilibrium point $x_e \in P$ is *unstable* if it is not stable.

The fact that the topology of a manifold is the same as the topology induced for any metric on it allows one to show that every $d_{\mathbf{g}}$, independently of the associated $\mathbf{g}$, induces the same stable and unstable points for (2.1).

Lyapunov theory studies the stability of equilibrium points of first-order differential equations. Let $\mathcal{M} : P \to \mathbb{R}$ be a function and let us define

$$\dot{\mathcal{M}}(x) := (X\mathcal{M})(x), \qquad \forall x \in P.$$

Let us recall the basic Lyapunov theorem for autonomous systems (2.1).

Theorem 2.1 *Let x_e be an equilibrium point of (2.1) and let $\mathcal{M} : P \to \mathbb{R}$ be a continuous function such that $\mathcal{M}(x_e) = 0$, $\mathcal{M}(x) > 0$, and $\dot{\mathcal{M}}(x) \leq 0$ for every $x \in B_{x_e,r} \backslash \{x_e\}$ and some $r \in \mathbb{R}^+$. Then, x_e is stable.*

In the literature, the function $\mathcal{M}$ is called a *Lyapunov function* (Vidyasagar 2002).

2.2 On *k*-Polysymplectic Manifolds

This section recalls the basic notions in k-polysymplectic geometry to be used later on. This is relevant as a single term may refer to different not equivalent geometric concepts in the literature.

Hereafter, we work with differential ℓ-forms on P that take values in $\mathbb{R}^k$. The space of such forms is denoted by $\Omega^\ell(P, \mathbb{R}^k)$, while its elements will be written in bold. Moreover, $\mathbb{R}^k$ has a fixed basis $\{e_1, \dots, e_k\}$ giving rise to a dual basis $\{e^1, \dots, e^k\}$ in $\mathbb{R}^{k*}$. Hence, an element $\boldsymbol{\omega} \in \Omega^\ell(P, \mathbb{R}^k)$ can always be written as $\boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha$ for some uniquely defined differential ℓ-forms $\omega^1, \dots, \omega^k$ on P. A differential ℓ-form on P taking values in $\mathbb{R}^k$, let us say $\boldsymbol{\omega}$, is nondegenerate if

$$\ker \boldsymbol{\omega} = \ker(\omega^\alpha \otimes e_\alpha) := \bigcap_{\alpha=1}^{k} \ker \omega^\alpha = 0.$$

Let us introduce the following definition that will be useful to simplify the notation of our further work. Let $\boldsymbol{\vartheta} = \vartheta^\alpha \otimes e_\alpha \in \Omega^\ell(P, \mathbb{R}^k)$ be an $\mathbb{R}^k$-valued differential ℓ-form on P. Then, the contraction of $\boldsymbol{\vartheta}$ with a vector field $X \in \mathfrak{X}(P)$ is defined as

$$\iota_X \boldsymbol{\vartheta} := (\iota_X \vartheta^\alpha) \otimes e_\alpha =: \langle \boldsymbol{\vartheta}, X \rangle \in \Omega^{\ell-1}(P, \mathbb{R}^k).$$

In short, the exterior differential, the Lie derivative with respect to vector fields, and many other operations on differential forms can naturally be extended to ℓ-differential forms taking values in vector spaces by considering the natural action of the above-mentioned operations on the components of $\mathbb{R}^k$-valued differential forms and extending them to $\Omega^\ell(P, \mathbb{R}^k)$ by linearity.

A k-vector field on a manifold P is, essentially, a family of k vector fields on P. We write $\mathfrak{X}(P, \mathbb{R}^k)$ for the space of k-vector fields on P and its elements will be written in bold. Moreover, a k-vector field, let us say $\boldsymbol{X}$, can always be written in a unique manner as $\boldsymbol{X} = X_\alpha \otimes e_\alpha$ for a family $X_1, \dots, X_k$ of vector fields on P. The contraction of a k-vector field $\boldsymbol{X} = X_\alpha \otimes e_\alpha$ with a k-differential form $\boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha$ is the function on P defined as follows

$$\iota_{\boldsymbol{X}} \boldsymbol{\omega} := \iota_{X_\alpha} \omega^\alpha =: \langle \boldsymbol{\omega}, \boldsymbol{X} \rangle.$$

Now, let us turn to one of the main fundamental notions to be studied in this paper.

Definition 2.2 A *k-polysymplectic form* on P is a closed nondegenerate $\mathbb{R}^k$-valued differential two-form $\boldsymbol{\omega}$ on P. The pair $(P, \boldsymbol{\omega})$ is called a *k-polysymplectic manifold*.

Consider a k-polysymplectic manifold $(P, \boldsymbol{\omega})$, and let $W_p \subset \mathrm{T}_p P$ at some $p \in P$. The *k-polysymplectic orthogonal complement* of W_p with respect to $(P, \boldsymbol{\omega})$ is

$$W_p^{\perp,k} := \{v_p \in \mathrm{T}_p P \mid \boldsymbol{\omega}(w_p, v_p) = 0, \ \forall w_p \in W_p\}.$$

k-Polysymplectic manifolds are called, for simplicity, polysymplectic manifolds in the literature (Marrero et al. 2015). Nevertheless, the latter term may be misleading as refers here to a different concept shown below. Hence, to avoid confusion, we will use the full term *k-polysymplectic manifold*. Let us define polysymplectic manifolds, k-polysymplectic manifolds, and related notions.

Definition 2.3 Let P be an $n(k+1)$-dimensional manifold. Then,

- A *polysymplectic form* on P is a nondegenerate differential two-form, $\boldsymbol{\omega}$, taking values in $\mathbb{R}^k$. We call $(P, \boldsymbol{\omega})$ a *polysymplectic manifold*.
- A *k-symplectic structure* on P is a pair $(\boldsymbol{\omega}, \mathcal{D})$, where $(P, \boldsymbol{\omega})$ is a polysymplectic manifold and $\mathcal{D} \subset \mathrm{T}P$ is an integrable distribution on P of rank nk such that

$$\boldsymbol{\omega}|_{\mathcal{D}\times\mathcal{D}} = 0.$$

  In this case, $(P, \boldsymbol{\omega}, \mathcal{D})$ is a *k-symplectic manifold*. We call $\mathcal{D}$ a *polarisation* of $(P, \boldsymbol{\omega})$.

If the two-form $\boldsymbol{\omega}$ is exact, namely $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in \Omega^1(P, \mathbb{R}^k)$, in any of the notions in Definition 2.3, then such concepts are said to be *exact*.

Note that the difference between polysymplectic and k-polysymplectic manifolds relies on the fact that in the polysymplectic case, the dimension of the manifold is proportional to $k+1$ if the polysymplectic form takes values in $\mathbb{R}^k$.

2.3 On $\boldsymbol{\omega}$-Hamiltonian Functions and Vector Fields

Let us survey the basic theory on k-polysymplectic vector fields and functions. Recall that we will not be concerned with the local or global character of the structures to be defined next.

Definition 2.4 Given a k-polysymplectic manifold $(P, \boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha)$, a vector field $Y \in \mathfrak{X}(P)$ is *$\boldsymbol{\omega}$-Hamiltonian* if it is Hamiltonian with respect to all the presymplectic forms $\omega^1, \ldots, \omega^k$, namely $\iota_Y \omega^\alpha$ is closed for $\alpha = 1, \ldots, k$. Let us denote by $\mathfrak{X}_{\boldsymbol{\omega}}(P)$ the space of $\boldsymbol{\omega}$-Hamiltonian vector fields in a k-polysymplectic manifold $(P, \boldsymbol{\omega})$.

Note that if $\iota_Y \omega^\alpha$ is closed, then it generally admits a potential function only locally. Anyhow, this work is mainly concerned with local aspects and the fact that the potential function may not be globally defined will not have any repercussions in what follows.

It is convenient for the study of $\boldsymbol{\omega}$-Hamiltonian vector fields to introduce some generalisation of the Hamiltonian function notion for presymplectic forms to deal simultaneously with all associated $h^1, \ldots, h^k$ (see Awane 1992; de Lucas and Vilariño 2015 for details).

Definition 2.5 Given a k-polysymplectic manifold $(P, \boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha)$, we say that $\boldsymbol{h} = h^\alpha \otimes e_\alpha$ is an *$\boldsymbol{\omega}$-Hamiltonian function* if there exists a vector field $X_{\boldsymbol{h}}$ on P such that $\iota_{X_{\boldsymbol{h}}} \boldsymbol{\omega} = \mathrm{d}\boldsymbol{h}$, namely $\iota_{X_{\boldsymbol{h}}} \omega^\alpha = \mathrm{d}h^\alpha$ for $\alpha = 1, \ldots, k$. In this case, we call $\boldsymbol{h}$ an *$\boldsymbol{\omega}$-Hamiltonian function* for $X_{\boldsymbol{h}}$. We write $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ for the space of $\boldsymbol{\omega}$-Hamiltonian functions of $(P, \boldsymbol{\omega})$.

An $\boldsymbol{\omega}$-Hamiltonian vector field (resp. function) will be simply called k-Hamiltonian at times, if $\boldsymbol{\omega}$ is understood from context or its specific expression is not relevant. In Merino (1997), the author defined the k-Hamiltonian system associated with the $\mathbb{R}^k$-valued Hamiltonian function $\boldsymbol{h}$ as the vector field $X_{\boldsymbol{h}}$ of the above definition. Moreover, Awane (1992) called $\boldsymbol{h}$ a Hamiltonian map of X when X is additionally an infinitesimal automorphism of a certain distribution on which it is assumed that the presymplectic forms of the k-symplectic distribution vanish.

Example 2.6 Consider the two-polysymplectic manifold $(\mathbb{R}^3, \boldsymbol{\omega})$, where $\{u, v, w\}$ are linear coordinates on $\mathbb{R}^3$ and $\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2$, where (see de Lucas and Vilariño 2015 for details)

$$\begin{aligned}\omega^1 &= -\frac{4w}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{1}{v}\mathrm{d}v \wedge \mathrm{d}w + \frac{4w^2}{v^3}\mathrm{d}u \wedge \mathrm{d}v,\\ \omega^2 &= -\frac{4}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{8w}{v^3}\mathrm{d}u \wedge \mathrm{d}v,\end{aligned}$$

is a two-polysymplectic form. The vector fields

$$X_1 = 4u^2\frac{\partial}{\partial u} + 4uv\frac{\partial}{\partial v} + v^2\frac{\partial}{\partial w}, \qquad X_2 = \frac{\partial}{\partial u},$$

are $\boldsymbol{\omega}$-Hamiltonian with $\boldsymbol{\omega}$-Hamiltonian functions

$$\begin{aligned}\boldsymbol{f} &= \left(4uw - 8\frac{u^2w^2}{v^2} - \frac{v^2}{2}\right) \otimes e_1 + \left(4u - 16\frac{u^2w}{v^2}\right) \otimes e_2,\\ \boldsymbol{g} &= -2\frac{w^2}{v^2} \otimes e_1 - 4\frac{w}{v^2} \otimes e_2,\end{aligned}$$

respectively, relative to the two-polysymplectic form $\boldsymbol{\omega}$. △

Every $\boldsymbol{\omega}$-Hamiltonian vector field is associated with at least one $\boldsymbol{\omega}$-Hamiltonian function. Conversely, every $\boldsymbol{\omega}$-Hamiltonian function induces a unique $\boldsymbol{\omega}$-Hamiltonian vector field.

Proposition 2.7 *The space $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ relative to k-polysymplectic manifold $(P, \boldsymbol{\omega})$ becomes a Lie algebra when endowed with the natural operations*

$$\boldsymbol{h} + \boldsymbol{g} := (h^\alpha + g^\alpha) \otimes e_\alpha, \qquad \lambda \cdot \boldsymbol{h} := \lambda h^\alpha \otimes e_\alpha,$$

where $\boldsymbol{h} = h^\alpha \otimes e_\alpha$, $\boldsymbol{g} = g^\alpha \otimes e_\alpha \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, $\lambda \in \mathbb{R}$, and the Lie bracket $\{\cdot,\cdot\}_{\boldsymbol{\omega}} : \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \times \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \to \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ of the form

$$\{\boldsymbol{h}, \boldsymbol{g}\}_{\boldsymbol{\omega}} = \{h^1, g^1\}_{\omega^1} \otimes e_1 + \cdots + \{h^k, g^k\}_{\omega^k} \otimes e_k,$$

where $\{\cdot,\cdot\}_{\omega^\alpha}$ is the Poisson bracket naturally induced by the presymplectic form ω^α, with $\alpha = 1, \ldots, k$.

The product of $\boldsymbol{\omega}$-Hamiltonian functions

$$\boldsymbol{h}\star\boldsymbol{g} = (h^1 g^1) \otimes e_1 + \cdots + (h^k g^k) \otimes e_k,$$

is not in general an $\boldsymbol{\omega}$-Hamiltonian function (de Lucas and Vilariño 2015, pg. 2239). Hence, $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(P), \star, \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ is not in general a Poisson algebra (de Lucas and Vilariño 2015, pg. 2239). Moreover, the map $\{\boldsymbol{h},\cdot\}_{\boldsymbol{\omega}} : \boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \mapsto \{\boldsymbol{g},\boldsymbol{h}\}_{\boldsymbol{\omega}} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, with $\boldsymbol{h} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, is not, in general, a derivation with respect to $\star$ either. Hence, k-polysymplectic geometry is quite different from Poisson and presymplectic geometry. Nevertheless, $\{\boldsymbol{h},\boldsymbol{g}\}_{\boldsymbol{\omega}} = 0$ for every locally constant function $\boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ and any $\boldsymbol{h} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$. This Lie algebra admits other properties, as shown next.

Proposition 2.8 *Consider a k-polysymplectic manifold $(P,\boldsymbol{\omega})$. Every $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ acts as a derivation on the Lie algebra $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(P), \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ in the form*

$$X_{\boldsymbol{h}} \boldsymbol{f} = \{\boldsymbol{f},\boldsymbol{h}\}_{\boldsymbol{\omega}}, \qquad \forall \boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P),$$

where $\boldsymbol{h}$ is an $\boldsymbol{\omega}$-Hamiltonian function for $X_{\boldsymbol{h}}$.

2.4 *k*-Polysymplectic Momentum Maps

Let us survey the theory of k-polysymplectic momentum maps. Note that the presented results are not restricted to Ad^{*k}-equivariant momentum maps (see de Lucas et al. 2023 for further details).

Definition 2.9 A Lie group action $\Phi\colon G \times P \to P$ on a k-polysymplectic manifold $(P,\boldsymbol{\omega})$ is a *k-polysymplectic Lie group action* if $\Phi_g^*\boldsymbol{\omega} = \boldsymbol{\omega}$ for each $g \in G$. In other words,

$$\mathscr{L}_{\xi_P}\boldsymbol{\omega} = 0, \qquad \forall \xi \in \mathfrak{g},$$

where ξ_P is the fundamental vector field of Φ related to $\xi \in \mathfrak{g}$, namely $\xi_P(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Phi(\exp(t\xi), p)$ for any $p \in P$.

Definition 2.10 A *k-polysymplectic momentum map* for a Lie group action $\Phi : G \times P \to P$ with respect to a k-polysymplectic manifold $(P,\boldsymbol{\omega})$ is a mapping $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ such that

$$\iota_{\xi_P}\boldsymbol{\omega} = (\iota_{\xi_P}\omega^\alpha) \otimes e_\alpha = \mathrm{d}\left\langle \mathbf{J}^\Phi, \xi \right\rangle, \qquad \forall \xi \in \mathfrak{g}. \tag{2.2}$$

Equation (2.2) implies that $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ satisfies

$$\iota_{\boldsymbol{\xi}_P}\boldsymbol{\omega} = \mathrm{d}\left\langle \mathbf{J}^\Phi, \boldsymbol{\xi} \right\rangle, \qquad \forall \boldsymbol{\xi} \in \mathfrak{g}^k.$$

and conversely. For simplicity, we will write $\langle \mathbf{J}^\Phi, \boldsymbol{\xi}\rangle =: \mathbf{J}^\Phi_{\boldsymbol{\xi}}$.

Before continuing studying k-polysymplectic momentum maps, recall that every Lie group G gives rise to a Lie group action $I : (g, h) \in G \times G \mapsto I_g(h) = ghg^{-1} \in G$, such that $I_g : h \in G \mapsto I(g, h) \in G$ for every $g \in G$. Then, the *adjoint action* of G on its Lie algebra, $\mathfrak{g}$, reads $\mathrm{Ad} : (g, v) \in G \times \mathfrak{g} \mapsto \mathrm{Ad}_g(v) = \mathrm{T}_e I_g(v) \in \mathfrak{g}$. In turn, the *co-adjoint action* becomes $\mathrm{Ad}^* : (g, \vartheta) \in G \times \mathfrak{g}^* \mapsto \mathrm{Ad}^*_{g^{-1}} \vartheta = \vartheta \circ \mathrm{Ad}_{g^{-1}} \in \mathfrak{g}^*$.

The following definition has been widely used in the literature (Marrero et al. 2015), although we will see that the Ad^{*k}-equivariance condition is no longer necessary (see de Lucas et al. 2023 for details). Moreover, we have changed the standard notation Coad^k to Ad^{*k} to shorten it.

Definition 2.11 A k-polysymplectic momentum map $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ is Ad^{*k}-*equivariant* if

$$\mathbf{J}^\Phi \circ \Phi_g = \mathrm{Ad}^{*k}_{g^{-1}} \circ \mathbf{J}^\Phi, \quad \forall g \in G,$$

where $\mathrm{Ad}^{*k}_{g^{-1}} = \mathrm{Ad}^*_{g^{-1}} \otimes \overset{(k)}{\cdots} \otimes \mathrm{Ad}^*_{g^{-1}}$ and

$$\begin{array}{rccc} \mathrm{Ad}^{*k} : G \times (\mathfrak{g}^*)^k & \longrightarrow & (\mathfrak{g}^*)^k \\ (g, \boldsymbol{\mu}) & \longmapsto & \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu} \end{array}.$$

In other words, the diagram aside is commutative for every $g \in G$.

$$\begin{array}{ccc} P & \xrightarrow{\mathbf{J}^\Phi} & (\mathfrak{g}^*)^k \\ \downarrow{\scriptstyle \Phi_g} & & \downarrow{\scriptstyle \mathrm{Ad}^{*k}_{g^{-1}}} \\ P & \xrightarrow{\mathbf{J}^\Phi} & (\mathfrak{g}^*)^k. \end{array}$$

To simplify the notation, let us introduce the following definition.

Definition 2.12 A *G-invariant $\boldsymbol{\omega}$-Hamiltonian system* is a tuple $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$, where $(P, \boldsymbol{\omega})$ is a k-polysymplectic manifold, $\boldsymbol{h}$ is a $\boldsymbol{\omega}$-Hamiltonian function associated with $X_{\boldsymbol{h}}$, the map $\Phi : G \times P \to P$ is a k-polysymplectic Lie group action satisfying $\Phi_g^* \boldsymbol{h} = \boldsymbol{h}$ for every $g \in G$, and $\mathbf{J}^\Phi$ is a k-polysymplectic momentum map related to Φ. An Ad^{*k}*-equivariant G-invariant $\boldsymbol{\omega}$-polysymplectic Hamiltonian system* is a G-invariant $\boldsymbol{\omega}$-Hamiltonian system $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ such that $\mathbf{J}^\Phi$ is Ad^{*k}-equivariant.

For simplicity, one sometimes calls *$\boldsymbol{\omega}$-Hamiltonian system* a triple $(P, \boldsymbol{\omega}, \boldsymbol{h})$ for a certain $\boldsymbol{\omega}$-Hamiltonian function $\boldsymbol{h}$.

Let us provide the formalism needed to avoid the Ad^{*k}-equivariance.

Proposition 2.13 *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. If*

$$\boldsymbol{\psi}_{g,\boldsymbol{\xi}} : x \in P \longmapsto \mathbf{J}^\Phi_{\boldsymbol{\xi}}(\Phi_g(x)) - \mathbf{J}^\Phi_{\mathrm{Ad}^k_{g^{-1}} \boldsymbol{\xi}}(x) \in \mathbb{R}, \quad \forall g \in G, \quad \forall \boldsymbol{\xi} \in \mathfrak{g}^k,$$

then $\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ is constant on P for every $g \in G$ and $\boldsymbol{\xi} \in \mathfrak{g}^k$. Moreover, $\boldsymbol{\sigma} : g \in G \mapsto \boldsymbol{\sigma}(g) \in (\mathfrak{g}^*)^k$, which is uniquely determined by the condition $\langle \boldsymbol{\sigma}(g), \boldsymbol{\xi} \rangle = \boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ for *every* $\boldsymbol{\xi} \in \mathfrak{g}^k$*, satisfies*

$$\boldsymbol{\sigma}(g_1 g_2) = \boldsymbol{\sigma}(g_1) + \mathrm{Ad}^{*k}_{g_1^{-1}} \boldsymbol{\sigma}(g_2), \quad \forall g_1, g_2 \in G.$$

The map $\boldsymbol{\sigma} : G \to (\mathfrak{g}^*)^k$ of the form

$$\boldsymbol{\sigma}(g) = \mathbf{J}^\Phi \circ \Phi_g - \mathrm{Ad}^{*k}_{g^{-1}} \mathbf{J}^\Phi, \qquad g \in G,$$

is called the *co-adjoint cocycle* associated with the k-polysymplectic momentum map $\mathbf{J}^\Phi$ on P. Moreover, $\mathbf{J}^\Phi$ is an Ad^{*k}-equivariant k-polysymplectic momentum map if and only if $\boldsymbol{\sigma} = 0$.

A map $\boldsymbol{\sigma} : G \to (\mathfrak{g}^*)^k$ is a *coboundary* if there exists $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ such that

$$\boldsymbol{\sigma}(g) = \boldsymbol{\mu} - \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu}, \qquad \forall g \in G.$$

Proposition 2.14 *Let* $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ *be a* k*-polysymplectic momentum map related to a* k*-polysymplectic action* $\Phi : G \times P \to P$ *with co-adjoint cocycle* $\boldsymbol{\sigma}$*. Then,*

(1) *there exists a Lie group action of G on* $(\mathfrak{g}^*)^k$ *of the form*

$$\boldsymbol{\Delta} : (g, \boldsymbol{\mu}) \in G \times (\mathfrak{g}^*)^k \mapsto \boldsymbol{\sigma}(g) + \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu} =: \boldsymbol{\Delta}_g(\boldsymbol{\mu}) \in (\mathfrak{g}^*)^k,$$

(2) *the* k*-polysymplectic momentum map* $\mathbf{J}^\Phi$ *is equivariant with respect to* $\boldsymbol{\Delta}$*, in other words, for every* $g \in G$*, one has the commutative diagram aside.*

$$\begin{array}{ccc} P & \xrightarrow{\mathbf{J}^\Phi} & (\mathfrak{g}^*)^k \\ \big\downarrow{\scriptstyle \Phi_g} & & \big\downarrow{\scriptstyle \boldsymbol{\Delta}_g} \\ P & \xrightarrow{\mathbf{J}^\Phi} & (\mathfrak{g}^*)^k \end{array}.$$

Proposition 2.14 ensures that every k-polysymplectic momentum map $\mathbf{J}^\Phi$ gives rise to an equivariant k-polysymplectic momentum map relative to a new action $\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, called a *k-polysymplectic affine Lie group action*. Note that a k-polysymplectic affine Lie group action can also be expressed by writing $\boldsymbol{\Delta}(g, (\mu^1, \dots, \mu^k)) = (\Delta^1_g \mu^1, \dots, \Delta^k_g \mu^k) \in (\mathfrak{g}^*)^k$, where the mappings $\Delta^1, \dots, \Delta^k$ take the form $\Delta^\alpha : (g, \vartheta) \in G \times \mathfrak{g}^* \mapsto \mathrm{Ad}^*_{g^{-1}} \vartheta + \sigma^\alpha(g) = \Delta^\alpha_g(\vartheta) \in \mathfrak{g}^*$ and $\boldsymbol{\sigma}(g) = (\sigma^1(g), \dots, \sigma^k(g))$, where $\sigma^\alpha(g) = \mathbf{J}^\Phi_\alpha \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}} \mathbf{J}^\Phi_\alpha$ for $\alpha = 1, \dots, k$ and $\mathbf{J}^\Phi_1, \dots, \mathbf{J}^\Phi_k$ are the coordinates of $\mathbf{J}^\Phi$.

3 *k*-Polysymplectic Marsden–Weinstein Reduction

Let us now review previous results in the literature for the k-polysymplectic Marsden–Weinstein reduction to correct previous mistakes and inaccuracies. Furthermore, we introduce the reduction of the dynamical system governed by an $\boldsymbol{\omega}$-Hamiltonian vector field. This concept is novel, as prior research has focused on dynamical systems given by Hamiltonian k-vector fields (Blacker 2019; Marrero et al. 2015). In particular, this section first reviews the previous k-polysymplectic Marsden–Weinstein reduction theory and explains some, only apparently, minor inaccuracies. After that, we focus on resolving a mistake in one of the main results in García-Toraño Andrés and Mestdag (2023), concerning the conditions to obtain a k-polysymplectic reduction. Finally, in Subsection 3.2, we analyse the relations between the conditions for the k-polysymplectic reduction given in Marrero et al. (2015).

3.1 A Review on the *k*-Polysymplectic Marsden–Weinstein Reduction

Let us recall several definitions that are useful for what follows. Some technical assumptions will be first set to improve the applicability of k-polysymplectic Marsden–Weinstein reductions. A *weak regular value* of a mapping $\phi : M \to N$ is a point $x_0 \in N$ such that $\phi^{-1}(x_0)$ is a submanifold of M and $\ker \mathrm{T}_p\phi = \mathrm{T}_p[\phi^{-1}(x_0)]$ for every $p \in \phi^{-1}(x_0)$. In particular, regular values of ϕ are weak regular values too. Moreover, a Lie group action $\Phi : G \times M \to M$ is *quotientable* (Albert 1989) when the space of orbits of the action of G on M, let us say M/G, is a manifold and the projection $\pi : M \to M/G$ is a submersion. In particular, this occurs when Φ is free and proper. To familiarise the reader with weak regular values, which are essential in this work, let us present a simple example of a point that is neither a regular value nor a weak regular value.

Example 3.1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2$. Consider the vector field $X = \frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then, $(\iota_X \mathrm{d} f)(x, y) = 0$ if $x = 0$. However, X is not tangent to $f^{-1}(0) = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$, since $\mathrm{T}_{(0,y)} f^{-1}(0) = \langle \frac{\partial}{\partial y} \rangle$ for every $y \in \mathbb{R}$. Therefore, as $\ker \mathrm{T}_{(0,y)} f \neq \mathrm{T}_{(0,y)} f^{-1}(0)$, it follows that $0 \in \mathbb{R}$ is not a weak regular value of f. Indeed, it is not a regular value either since $Tf = 0$ at points of $f^{-1}(0)$.

More generally, for any function $f : M \to N$, a point $\lambda \in N$ is not a weak regular value of f if $T_p f(v_p) = 0$ for some $v_p \in T_p M$ with $p \in f^{-1}(\lambda)$ that is not tangent to the submanifold $f^{-1}(\lambda)$.

Let us comment on the regular values of k-polysymplectic momentum maps. The codomain of a k-polysymplectic momentum map $\mathbf{J}^\Phi : P \to \mathfrak{g}^{*k}$ may have a large dimension, even larger than the dimension of P, for instance, due to the presence of k copies of $\mathfrak{g}^*$. This implies that it may be impossible for $\mathbf{J}^\Phi$ to be a submersion when k is large enough. Being a submersion is the typical condition used in many types of Marsden–Weinstein reductions (García-Toraño Andrés and Mestdag 2023; Marrero et al. 2015). But this property is harder to satisfy in k-polysymplectic geometry. Note that it is sometimes assumed in the literature that Sard's Theorem ensures that $\mathbf{J}^\Phi$ is frequently a submersion because the set of singular points in P of $\mathbf{J}^\Phi$, i.e. the set of points where $\mathbf{J}^\Phi$ is not a submersion, has an image with zero measure (see Marrero et al. (2010, Lemma 3.4) or Blankenstein and Ratiu (2004, pg. 212)). Nevertheless, the whole image of $\mathbf{J}^\Phi$ may also be a zero measure subset and, in this case, it may happen that $\mathbf{J}^\Phi$ is not a submersion at points in a dense subset of P. Indeed, $\mathbf{J}^\Phi$ is not a submersion at any point in P when $k \dim \mathfrak{g}^* > \dim P$. In such a case, $\mathbf{J}^\Phi$ has no regular points in $\mathfrak{g}^{*k}$. That is one of the reasons why the analysis of weak regular values for k-

polysymplectic momentum maps in de Lucas et al. (2023) is relevant. It also explains why in the symplectic case, when $k = 1$, the assumption of $\mathbf{J}^\Phi$ being a submersion is not so problematic. Note also that one has to assume some regularity conditions on the coordinates of $\mathbf{J}^\Phi_1, \dots, \mathbf{J}^\Phi_k$ to ensure that their level sets are submanifolds, but such mappings do not use to have regular values in k-polysymplectic problems for $k > 1$.

It is also worth stressing that Blacker in Blacker (2019, Theorem 3.22) does not provide any explicit assumption in the structure of $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, although it is implicitly assumed that $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a manifold. In general, Blacker's work (Blacker 2019) does not analyse in detail the technical conditions on the manifold structure of $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Notwithstanding, the structure of spaces of the form $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is investigated.

Lemma 3.2 below will be used to characterise in the next section the so-called k-polysymplectic relative equilibrium points of G-invariant $\boldsymbol{\omega}$-Hamiltonian systems. More importantly, Lemma 3.2 is introduced to prove k-polysymplectic Marsden–Weinstein reduction theorems. The proof of Lemma 3.2 appears in de Lucas et al. (2023). Interestingly, Günther's wrong version of Lemma 3.2 made his k-polysymplectic reduction to be flawed. In fact, Günther states in Günther (1987, Lemma 7.5) a wrong expression for condition (1) in Lemma 3.2. In his work, Günther implicitly claims that, as in the symplectic case, one has

$$\begin{aligned}\ker \iota^*_{\boldsymbol{\mu}}\boldsymbol{\omega} &= \mathrm{T}_p\left(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right)^{\perp,k} \cap \mathrm{T}_p\left(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right) = \mathrm{T}_p(Gp) \cap \mathrm{T}_p\left(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right)\\ &= \mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p),\end{aligned}$$

but the equality between the second and the third expressions is only an inclusion $\supset$ (see Marrero et al. (2015, pg. 12)). It is worth stressing that Günther justifies his Lemma 7.5 by merely saying that its proof is like in the symplectic case (Günther 1987, pg. 48). Moreover, Munteanu et al. (2004) includes a related mistake. Finally, we refer to Marrero et al. (2015, Sections 1 and 2.2) for a comment on these errors.

Lemma 3.2 *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ be a weak regular value of $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$. Then, for every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, one has*

(1) $\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p) = \mathrm{T}_p(Gp) \cap \mathrm{T}_p\left(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right)$,
(2) $\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \mathrm{T}_p(Gp)^{\perp,k}$.

Let us review the conditions of the k-polysymplectic Marsden–Weinstein reduction theorem, which will be crucial in the k-polysymplectic energy–momentum method to correct a mistake in one of the main results in García-Toraño Andrés and Mestdag (2023), in fact, the one, García-Toraño Andrés and Mestdag (2023, Proposition 1), giving the name to the paper.

Recall that the first correct k-polysymplectic Marsden–Weinstein reduction theory can be found in Marrero et al. (2015). The necessary and sufficient conditions to perform a reduction were given by C. Blacker in Blacker (2019), although there is a relevant typo in his theorem, as commented in García-Toraño Andrés and Mestdag (2023). The k-polysymplectic Marsden–Weinstein reduction theorem was proved in Marrero et al. (2015) assuming that the k-polysymplectic momentum map $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$

is Ad^{*k}-equivariant. A version of the k-polysymplectic Marsden–Weinstein reduction theorem without this condition was accomplished in de Lucas et al. (2023). In its correct and most modern form, the reduction theorem reads as in Theorem 3.3 below (see Lucas et al. (2023, Theorem 5.10) for details). Note that when we say that $\boldsymbol{\mu}$ is a weakly regular value of $\mathbf{J}^\Phi$, we also assume that all the components of $\boldsymbol{\mu}$ are weakly regular too. It is worth stressing that even if $\boldsymbol{\mu}$ is a regular value of $\mathbf{J}^\Phi$, then each component μ^α of $\boldsymbol{\mu}$ does not need to be regular for $\mathbf{J}^\Phi_\alpha$ since $\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha) \supset \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$.

Theorem 3.3 (k-polysymplectic Marsden–Weinstein reduction theorem) *Consider a G-invariant $\boldsymbol{\omega}$-Hamiltonian system $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$. Assume that $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in (\mathfrak{g}^*)^k$ is a weak regular value of $\mathbf{J}^\Phi$ and $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ acts in a quotientable manner on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Let $G^{\Delta^\alpha}_{\mu^\alpha}$ denote the isotropy group at μ^α of the Lie group action $\Delta^\alpha : (g, \vartheta) \in G \times \mathfrak{g}^* \mapsto \Delta^\alpha(g, \vartheta) \in \mathfrak{g}^*$ for $\alpha = 1, \ldots, k$. Moreover, let the following (sufficient) conditions hold*

$$\ker(\mathrm{T}_p \mathbf{J}^\Phi_\alpha) = \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) + \ker \omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha} p), \qquad \alpha = 1, \ldots, k, \tag{3.1}$$

$$\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p) = \bigcap_{\alpha=1}^{k} \left(\ker \omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha} p)\right) \cap \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})), \tag{3.2}$$

for every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Then, $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}, \boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a k-polysymplectic manifold, with $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ being uniquely determined by

$$\pi^*_{\boldsymbol{\mu}} \boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}} \boldsymbol{\omega}$$

where $\jmath_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \hookrightarrow P$ is the canonical immersion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is the canonical projection.

The following theorem shows the reduction of the dynamics given by an $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ on P as a consequence of Theorem 3.3, which will be essential for our k-polysymplectic energy–momentum method. Note that in previous works on k-polysymplectic Marsden–Weinstein reductions, the k-polysymplectic Marsden–Weinstein reduction theorem has been applied to reduce the dynamics given by an $\boldsymbol{\omega}$-Hamiltonian k-vector field (Marrero et al. 2015, Theorem 4.4). Nevertheless, since our k-polysymplectic Marsden–Weinstein reduction theorem concerns just $\boldsymbol{\omega}$-Hamiltonian vector fields, the conditions of that theorem can be simplified as follows.

Theorem 3.4 *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $\Phi_{g*}\boldsymbol{h} = \boldsymbol{h}$ for each $g \in G$. Then, the one-parametric group of diffeomorphisms F_t of the vector field $X_{\boldsymbol{h}}$ induces the one-parametric group of diffeomorphisms $\mathcal{F}_t$ of the vector field $X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ such that $\iota_{X_{\boldsymbol{f}_{\boldsymbol{\mu}}}} \boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}}$ and $\jmath^*_{\boldsymbol{\mu}} \boldsymbol{h} = \pi^*_{\boldsymbol{\mu}} \boldsymbol{f}_{\boldsymbol{\mu}}$.*

Proof First, note that $\Phi^*_g \boldsymbol{h} = \boldsymbol{h}$ and our assumptions, in particular $\Phi^*_g \boldsymbol{\omega} = \boldsymbol{\omega}$, yield $\Phi_{g*} X_{\boldsymbol{h}} = X_{\boldsymbol{h}}$ for each $g \in G$. Therefore,

$$\iota_{X_{\boldsymbol{h}}} \mathrm{d}\langle \mathbf{J}^\Phi, \xi \rangle = -\iota_{\xi_P} \iota_{X_{\boldsymbol{h}}} \boldsymbol{\omega} = -\iota_{\xi_P} \mathrm{d}\boldsymbol{h} = 0, \qquad \forall \xi \in \mathfrak{g}.$$

Hence, $X_{\boldsymbol{h}}$ is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Next, for every $\xi \in \mathfrak{g}$, we have

$$\iota_{[\xi_P, X_{\boldsymbol{h}}]}\boldsymbol{\omega} = \mathscr{L}_{\xi_P}\iota_{X_{\boldsymbol{h}}}\boldsymbol{\omega} - \iota_{\xi_P}\mathscr{L}_{X_{\boldsymbol{h}}}\boldsymbol{\omega} = 0,$$

so by the virtue of $\ker \boldsymbol{\omega} = 0$, we obtain that $[\xi_P, X_{\boldsymbol{h}}] = 0$. Thus, the vector field $X_{\boldsymbol{h}}$ projects onto a vector field Y on the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$. In other words, the one-parametric group of diffeomorphisms F_t of $X_{\boldsymbol{h}}$ induces the one-parametric group of diffeomorphisms $\mathcal{F}_t$ of Y so that $\pi_{\boldsymbol{\mu}} \circ F_t = \mathcal{F}_t \circ \pi_{\boldsymbol{\mu}}$ for each $t \in \mathbb{R}$. Then, by Theorem 3.3, one has

$$\jmath^*_{\boldsymbol{\mu}}\mathrm{d}\boldsymbol{h} = \jmath^*_{\boldsymbol{\mu}}(\iota_{X_{\boldsymbol{h}}}\boldsymbol{\omega}) = \iota_{X_{\boldsymbol{h}}}\jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega} = \iota_{X_{\boldsymbol{h}}}\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}}, = \pi^*_{\boldsymbol{\mu}}(\iota_Y\boldsymbol{\omega}_{\boldsymbol{\mu}}), \tag{3.3}$$

where we denoted by $X_{\boldsymbol{h}}$ both the vector field $X_{\boldsymbol{h}}$ on P itself and its restriction to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. The same slight abuse of notation will be hereafter done to simplify the notation.

Due to the invariance of $\boldsymbol{h}$ relative to $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$, there exists a reduced $\mathbb{R}^k$-valued function $\boldsymbol{f}_{\boldsymbol{\mu}}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ such that $\jmath^*_{\boldsymbol{\mu}}\boldsymbol{h} = \pi^*_{\boldsymbol{\mu}}\boldsymbol{f}_{\boldsymbol{\mu}}$. Finally, expression (3.3) gives

$$\pi^*_{\boldsymbol{\mu}}\mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\mathrm{d}\boldsymbol{h} = \pi^*_{\boldsymbol{\mu}}\iota_Y\boldsymbol{\omega}_{\boldsymbol{\mu}}$$

which shows that $Y = X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$ is an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field and $\boldsymbol{f}_{\boldsymbol{\mu}}$ is an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian function associated with $X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$. □

Now, let us recall the sufficient and necessary conditions for a k-polysymplectic reduction given by Blacker in (3.4). His main result is described in Theorem 3.5 with our notation and we have corrected the typo in Blacker (2019, Theorem 3.22) on the k-polysymplectic Marsden–Weinstein reduction. It is worth noting that the typo also appears in the proof of Blacker (2019, Theorem 3.22) and is evident after applying Blacker (2019, Theorem 2.14) to $\boldsymbol{\omega}_x$. Theorem 3.5 also adds certain essential technical conditions that were not explicitly written in Blacker (2019, Theorem 3.22). As remarked by Blacker in Blacker (2019), but apparently not noticed by Mestdag and García-Toraño in García-Toraño Andrés and Mestdag (2023), the sufficient and necessary condition (3.4) also appeared previously to Blacker in a different more implicit manner in Marrero et al. (2015, pg. 12). It is worth seeing also the related work (Martinez 2015) treating the reduction of poly-Poisson structures.

Theorem 3.5 *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be an Ad^{*k}-equivariant G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ be a fixed regular value of $\mathbf{J}^\Phi$. If the stabiliser subgroup $G_{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ under the Ad^{*k} action is connected, and $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is a smooth manifold, then there is a unique $\mathbb{R}^k$-valued two-form $\boldsymbol{\omega}_{\boldsymbol{\mu}} \in \Omega^2(P_{\boldsymbol{\mu}}, \mathbb{R}^k)$ such that $\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega}$ where $\jmath_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \hookrightarrow P$ is the inclusion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to P_{\boldsymbol{\mu}}$ is the canonical projection. The form $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is closed and nondegenerate if and only if*

$$\mathrm{T}_p(G_{\boldsymbol{\mu}}p) = (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}). \tag{3.4}$$

For the sake of completeness, let us now consider the first example of a k-polysymplectic Marsden–Weinstein reduction related to a non-regular value of a

k-polysymplectic momentum map. More examples with potential practical applications will be shown in Sect. 5. Let us analyse the completely integrable, and separable in variables, system in $\mathbb{R}^{2k}$ of the form

$$\frac{\mathrm{d}I_\alpha}{\mathrm{d}t}=0, \qquad \frac{\mathrm{d}\theta_\alpha}{\mathrm{d}t}=F_\alpha(I_\alpha), \qquad \alpha=1,\ldots,k>1, \tag{3.5}$$

for some arbitrary functions $F_1,\ldots,F_k:\mathbb{R}\to\mathbb{R}$. This related to an $\boldsymbol{\omega}$-polysymplectic Hamiltonian system on $\mathbb{R}^{2k}$ relative to the k-polysymplectic form $\boldsymbol{\omega}=\omega^\alpha\otimes e_\alpha$, where $\omega^1,\ldots,\omega^k$ are the presymplectic forms

$$\omega^\alpha=\mathrm{d}\theta^\alpha\wedge\mathrm{d}I^\alpha, \qquad \alpha=1,\ldots,k,$$

where it is important to stress that the right-hand side is not summed over the indices $\alpha=1,\ldots,k$. One has the basis of fundamental vector fields $\partial/\partial\theta^1,\ldots,\partial/\partial\theta^{k-1}$ associated with the Lie group action

$$\begin{aligned}\Phi:&(\lambda_1,\ldots,\lambda_{k-1};\theta_1,\ldots,\theta_k,I)\\ &\in\mathbb{R}^{k-1}\times\mathbb{R}^{2k}\mapsto(\lambda_1+\theta_1,\ldots,\lambda_{k-1}+\theta_{k-1},\theta_k,I)\in\mathbb{R}^{2k},\end{aligned}$$

with $I=(I_1,\ldots,I_k)\in\mathbb{R}^k$. Note that the functions $F_1,\ldots,F_k$ have been chosen to be of the form $F_\alpha=F_\alpha(I_\alpha)$, with $\alpha=1,\ldots,k$, to ensure that (3.5) is $\boldsymbol{\omega}$-Hamiltonian. The latter also explains why (3.5) is called separable. One may now consider a k-polysymplectic momentum map

$$\begin{aligned}\mathbf{J}^\Phi:&(\theta,I)\in\mathbb{R}^{2k}\longmapsto(I_1,\ldots,0)\otimes e_1+\ldots+(0,\ldots,I_{k-1})\otimes e_{k-1}+(0,\ldots,0)\\ &\otimes e_k\in\left(\mathbb{R}^{(k-1)*}\right)^k,\end{aligned}$$

which has no regular points (the codomain of $\mathbf{J}^\Phi$ has dimension larger than its domain for $k>3$) and it is Ad^{*k}-equivariant. Note that (3.5) gives rise to an $\mathbb{R}^{k-1}$-invariant $\boldsymbol{\omega}$-Hamiltonian system.

One may consider the reductions of $\boldsymbol{\omega}$ and (3.5) for any value of $\boldsymbol{\mu}=(\mu^1,\ldots,0)\otimes e_1+\ldots+(0,\ldots,\mu^{k-1})\otimes e_{k-1}\in\left(\mathbb{R}^{(k-1)*}\right)^k$. Then,

$$\begin{aligned}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})&=\{(\theta_\alpha,I_\alpha)\in\mathbb{R}^{2k}\mid I_1=\mu^1,\ \ldots,\ I_{k-1}=\mu^{k-1},\ \theta_1,\ldots,\theta_k,I_k\in\mathbb{R}\}\\ &\simeq\mathbb{R}^k\times\mathbb{R}.\end{aligned}$$

The isotropy subgroup $\mathbb{R}^{k-1}_{\boldsymbol{\mu}}\simeq\mathbb{R}^{k-1}$ acts on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ via Φ and the reduced manifold is diffeomorphic to $\mathbb{R}^2$. The presymplectic forms $\omega^1,\ldots,\omega^{k-1}$ become zero after reducing, but the reduction of ω^k is symplectic. Hence, $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ becomes a k-polysymplectic form with only one symplectic form different from zero. Since the $\boldsymbol{\omega}$-Hamiltonian function of the initial system is a first integral of the $\theta_1,\ldots,\theta_{k-1}$, one can project the initial system onto

$$\frac{\mathrm{d}I_k}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\theta_k}{\mathrm{d}t} = F_k(I_k),$$

which is Hamiltonian relative to $\mathrm{d}\theta_k \wedge \mathrm{d}I_k$, where θ_k, I_k are considered as variables in $\mathbb{R}^2$ in the natural manner.

3.2 On the Conditions for the *k*Polysymplectic Marsden–Weinstein Reduction

It was claimed in García-Toraño Andrés and Mestdag (2023, Proposition 1) that condition (3.2) is enough to ensure that there exists a k-polysymplectic Marsden–Weinstein reduction. In this section, we first show that this is not true and the proposition in García-Toraño Andrés and Mestdag (2023, Proposition 1) is incorrect. This is done by pointing out a mistake in the proof of García-Toraño Andrés and Mestdag (2023, Proposition 1) and then giving a counterexample where (3.2) is satisfied, but there is no k-polysymplectic Marsden–Weinstein reduction and, indeed, (3.1) does not hold. Next, we illustrate that it may happen that (3.1) is satisfied, but (3.2) is not. Finally, we prove an example of a possible k-polysymplectic reduction where (3.1) and (3.2) are not simultaneously satisfied. To keep our exposition simple and highlight our main ideas, we restrict in this subsection all k-polysymplectic momentum maps to the Ad^{*k}-invariant case, as done in García-Toraño Andrés and Mestdag (2023), Marrero et al. (2015).

First, the proof for García-Toraño Andrés and Mestdag (2023, Proposition 1) has a mistake, as there is an inclusion written in the opposite way. In particular, since $\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) \subset \mathrm{T}_p(\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha))$ for $\alpha = 1, \ldots, k$ and every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ for a regular $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$, one has

$$\begin{aligned}&\left\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_1(\mu^1)) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_k(\mu^k)) = 0\right\}\\ &\quad\subset \left\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = 0\right\}\end{aligned}$$

instead of

$$\begin{aligned}&\left\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_1(\mu^1)) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_k(\mu^k)) = 0\right\}\\ &\quad\supset \left\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = 0\right\}\end{aligned}$$

as claimed at the end of page 8 in the proof of García-Toraño Andrés and Mestdag (2023, Proposition 1). In other words, if v is perpendicular to $\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))$ relative to each ω^α, one cannot infer that v is perpendicular to each $\mathrm{T}_p(\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha))$ relative to ω^α for $\alpha = 1, \ldots, k$, since the latter conditions are more restrictive. Then, the proof of García-Toraño Andrés and Mestdag (2023, Proposition 1) only gives

$$\bigcap_{\alpha=1}^{k}(\ker J^*_{\mu^\alpha}\omega^\alpha|_p) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \subset (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}, \quad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}),$$

instead of the claimed

$$\bigcap_{\alpha=1}^{k}(\ker \jmath_{\mu^\alpha}^{*}\omega^\alpha|_p) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \supset (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}, \quad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}),$$

which makes the proof of Proposition 1 fail in proving (3.4), namely the k-polysymplectic Marsden–Weinstein reduction necessary and sufficient condition, and, therefore, the statement of Proposition 1. Indeed, the above mistake is ultimately due to the fact that García-Toraño Andrés and Mestdag (2023, Proposition 1) is false and the comments that follow in García-Toraño Andrés and Mestdag (2023) contain some inaccuracies.

Let us provide a counterexample to show that García-Toraño Andrés and Mestdag (2023, Proposition 1) does not hold. More specifically, we here describe an $\mathbb{R}$-invariant $\boldsymbol{\omega}$-Hamiltonian system relative to a two-symplectic form satisfying condition (3.2) but not giving rise to a k-polysymplectic Marsden–Weinstein reduction. Before that, it is convenient to recall some results from Marrero et al. (2015).

It was proved in Marrero et al. (2015) that $\ker \omega_p^\alpha \subset \ker \mathrm{T}_p\mathbf{J}_\alpha^\Phi$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, which allows one to define the following commutative diagram (see Marrero et al. (2015, pg. 12))

$$\begin{tikzcd}[column sep=large] \mathrm{T}_p(\mathbf{J}^{\Phi\,-1}(\boldsymbol{\mu})) \arrow[r, "\jmath"] \arrow[rr, bend left, "\pi_p^\alpha"] & \ker \mathrm{T}_p\mathbf{J}_\alpha^\Phi \arrow[r, "\pi"] & \dfrac{\ker \mathrm{T}_p\mathbf{J}_\alpha^\Phi}{\ker \omega_p^\alpha} \end{tikzcd}$$

for all $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, where $\jmath$ and π are the canonical injection and projections, respectively. For simplicity, the equivalence class of an element v in a quotient will be denoted by $[v]$. To avoid making the notation too complicated, the specific meaning of $[v]$ will be understood from context. According to Proposition 3.12 in Marrero et al. (2015), the above diagram induces the maps

$$\widetilde{\pi}_p^\alpha : \frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))}{\mathrm{T}_p(G_{\boldsymbol{\mu}}p)} \longrightarrow \frac{\frac{\ker \mathrm{T}_p\mathbf{J}_\alpha^\Phi}{\ker \omega_p^\alpha}}{\{[(\xi_P)_p] \mid \xi \in \mathfrak{g}_{\mu^\alpha}\}}, \qquad \alpha = 1, \dots, k, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}),$$

where $\mathfrak{g}_{\mu^\alpha}$ is the Lie algebra of G_{μ^α} and $\{[(\xi_P)_p] \mid \xi \in \mathfrak{g}_{\mu^\alpha}\} = \mathrm{pr}_\alpha^P(\{(\xi_P)_p \mid \xi \in \mathfrak{g}_{\mu^\alpha}\})$ and $\mathrm{pr}_\alpha^P : \mathrm{T}_pP \to \mathrm{T}_pP/\ker \omega_p^\alpha$ is the canonical projection onto the quotient.

The conditions (3.1) at $p \in P$ are equivalent to each $\widetilde{\pi}_p^\alpha$ being surjective, respectively (Marrero et al. 2015, Lemma 3.15), while (3.2) amounts to $0 = \bigcap_{\alpha=1}^k \ker \widetilde{\pi}_p^\alpha$ (see (Marrero et al. 2015, Lemma 3.16)).

Consider $P = \mathbb{R}^4$ with linear coordinates $\{x, y, z, t\}$ and the presymplectic forms

$$\omega^1 = \mathrm{d}x \wedge \mathrm{d}y, \qquad \omega^2 = \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}y \wedge \mathrm{d}z,$$

which give rise to a two-polysymplectic form $\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2$, because ω^2 is a symplectic form and $\ker \omega^1 \cap \ker \omega^2 = 0$. Consider the Lie group action $\Phi : (\lambda; x, y, z, t) \in \mathbb{R} \times \mathbb{R}^4 \mapsto (x + \lambda, y, z, t) \in \mathbb{R}^4$. The Lie algebra of fundamental vector fields of Φ is $V = \langle \partial_x \rangle \simeq \mathbb{R}$. Moreover, Φ admits a two-polysymplectic momentum map relative to $(\mathbb{R}^4, \boldsymbol{\omega})$ given by

$$\mathbf{J}^{\Phi} : (x, y, z, t) \in \mathbb{R}^4 \longmapsto \boldsymbol{\mu} = (y, t) \in (\mathbb{R}^*)^2,$$

which is clearly Ad^{*2}-equivariant. Additionally, $\mathbf{J}^{\Phi}$ is regular for every value of $(\mathbb{R}^*)^2$. Hence, $\mathbf{J}^{\Phi-1}(y, t) = \{(x, y, z, t) \in \mathbb{R}^4 : x, z \in \mathbb{R}\} \simeq \mathbb{R}^2$ is a submanifold for every $(y, t) \in (\mathbb{R}^*)^2$ and

$$\mathrm{T}_p(\mathbf{J}^{\Phi-1}(y, t)) = \langle \partial_x, \partial_z \rangle, \qquad \forall p \in \mathbf{J}^{\Phi-1}(y, t).$$

Moreover, $G_{\boldsymbol{\mu}} = \mathbb{R}$ for each $\boldsymbol{\mu} = (y, t) \in (\mathbb{R}^*)^2$ and $G_{\boldsymbol{\mu}}$ acts freely and properly on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Let us prove that condition (3.2) does not imply nor the reduction of $\boldsymbol{\omega}$, namely (3.4), neither (3.1).

In our example, one has $\boldsymbol{\mu} = (y, t)$ with $\mu^1 = y$ and $\mu^2 = t$, while

$$\ker \mathrm{T}_p\mathbf{J}^{\Phi}_1 = \langle \partial_x, \partial_z, \partial_t \rangle, \quad \ker \omega^1 = \langle \partial_t, \partial_z \rangle, \quad \ker \mathrm{T}_p\mathbf{J}^{\Phi}_2 = \langle \partial_x, \partial_y, \partial_z \rangle, \quad \ker \omega^2 = 0,$$

and

$$\{[(\xi_P)_p] : \xi \in \mathfrak{g}_{\mu^1}\} = \langle [\partial_x] \rangle, \qquad \{[(\xi_P)_p] : \xi \in \mathfrak{g}_{\mu^2}\} = \langle [\partial_x] \rangle$$

on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Then, we have the mappings

$$\widetilde{\pi}^1_p : \langle [\partial_z] \rangle = \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))/\mathrm{T}_p(G_{\boldsymbol{\mu}} p) \longmapsto \langle 0 \rangle = (\ker \mathrm{T}_p\mathbf{J}^{\Phi}_1 / \ker \omega^1_p)/\langle [\partial_x] \rangle$$

and

$$\widetilde{\pi}^2_p : \langle [\partial_z] \rangle = \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))/\mathrm{T}_p(G_{\boldsymbol{\mu}} p) \longmapsto \langle [\partial_y], [\partial_z] \rangle = (\ker \mathrm{T}_p\mathbf{J}^{\Phi}_2 / \ker \omega^2_p)/\langle [\partial_x] \rangle.$$

As $\widetilde{\pi}^2_p([\partial_z]) = [\partial_z]$, we have

$$\ker \widetilde{\pi}^1_p = \langle [\partial_z] \rangle, \qquad \ker \widetilde{\pi}^2_p = \langle 0 \rangle.$$

Hence, $\ker \widetilde{\pi}^1_p \cap \ker \widetilde{\pi}^2_p = 0$ and condition (3.2) is satisfied. But $\mathrm{Im}\, \widetilde{\pi}^2_p = \langle [\partial_z] \rangle$ and $\widetilde{\pi}^2_p$ is not surjective. Thus, condition (3.1) does not hold for $\alpha = 2$ in our example. In fact, ω^1, ω^2 become isotropic when restricted to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and give rise to two zero differential two-forms on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$, which is a one-dimensional manifold. Hence, no two-symplectic manifold is induced on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ despite that condition (3.2) is satisfied.

One can directly prove that condition (3.2) is satisfied in the previous example, but condition (3.1) is not. This shows more easily that Proposition 1 in García-Toraño

Andrés and Mestdag (2023) is false and that (3.2) does not imply (3.1), but our previous approach illustrates how we obtained our counterexample. Indeed, in our present counterexample, the fact that $\widetilde{\pi}^2_p$ is not surjective implies that (3.1) does not hold. Recall that

$$\ker \mathrm{T}_p \mathbf{J}^\Phi_2 = \langle \partial_x, \partial_y, \partial_z \rangle, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}),$$

while

$$\begin{aligned}&\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) + \ker \omega^2_p + \mathrm{T}_p(G_{\mu^2} p) \\ &\quad = \langle \partial_x, \partial_z \rangle + \{0\} + \langle \partial_x \rangle = \langle \partial_x, \partial_z \rangle, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}).\end{aligned}$$

On the other hand, condition (3.2) is satisfied since

$$\mathrm{T}_p(G_{\boldsymbol{\mu}} p) = \langle \partial_x \rangle$$

and

$$(\ker \omega^1_p + \mathrm{T}_p(G_{\mu^1} p)) \cap (\ker \omega^2_p + \mathrm{T}_p(G_{\mu^2} p)) \cap \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \langle \partial_x \rangle,$$

reads

$$(\langle \partial_t, \partial_z \rangle + \langle \partial_x \rangle) \cap (\langle 0 \rangle + \langle \partial_x \rangle) \cap \langle \partial_x, \partial_z \rangle = \langle \partial_x \rangle.$$

Since this example is constructed so as to obtain a one-dimensional reduced manifold, it is known that the reduction of the two-polysymplectic form is not a k-polysymplectic form.

The following examples illustrate some relations between the conditions (3.1), (3.2) and the existence of k-polysymplectic Marsden–Weinstein reductions.

Example 3.6 This example shows that if condition (3.1) is satisfied, then condition (3.2) does not need to hold. Consider a two-polysymplectic manifold $(\mathbb{R}^6, \boldsymbol{\omega})$. Let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be global linear coordinates on $\mathbb{R}^6$ and define

$$\begin{aligned}\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 &= (\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_1 \\ &\quad + (\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_2.\end{aligned}$$

Then, $\ker \omega^1_p = \langle \partial_3, \partial_4 \rangle$, $\ker \omega^2_p = \langle \partial_1, \partial_2 \rangle$, and $\ker \omega^1_p \cap \ker \omega^2_p = 0$ for every $p \in \mathbb{R}^6$. This turns $\boldsymbol{\omega}$ into a two-polysymplectic form.

Let us provide now a Lie group action proving our initial claim. Given the Lie group action $\Phi : (\lambda; x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R} \times \mathbb{R}^6 \mapsto (x_1 + \lambda, x_2, x_3 + \lambda, x_4, x_5, x_6) \in \mathbb{R}^6$, its Lie algebra of fundamental vector fields reads $\langle \partial_1 + \partial_3 \rangle$. The two-polysymplectic momentum map associated with Φ is given by

$$\mathbf{J}^\Phi : (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \longmapsto (x_2, x_4) = \boldsymbol{\mu} \in (\mathbb{R}^*)^2,$$

which is Ad^{*2}-equivariant. Moreover, every $\boldsymbol{\mu} = (x_2, x_4) \in (\mathbb{R}^*)^2$ is a regular value of $\mathbf{J}^\Phi$. Therefore, $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \,:\, x_1, x_3, x_5, x_6 \in \mathbb{R}\} \simeq \mathbb{R}^4$ is a submanifold of $\mathbb{R}^6$ for every $\boldsymbol{\mu} \in (\mathbb{R}^*)^2$ and

$$\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \langle \partial_1, \partial_3, \partial_5, \partial_6 \rangle, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}).$$

Hence, $\ker \mathrm{T}_p\mathbf{J}^\Phi_1 = \langle \partial_1, \partial_3, \partial_4, \partial_5, \partial_6 \rangle$ while $\ker \mathrm{T}_p\mathbf{J}^\Phi_2 = \langle \partial_1, \partial_2, \partial_3, \partial_5, \partial_6 \rangle$. Condition (3.1) holds because both sides of the condition are equal to

$$\begin{aligned}
\langle \partial_1, \partial_3, \partial_4, \partial_5, \partial_6 \rangle &= \langle \partial_1, \partial_3, \partial_5, \partial_6 \rangle + \langle \partial_3, \partial_4 \rangle + \langle \partial_1 + \partial_3 \rangle, \\
\langle \partial_1, \partial_2, \partial_3, \partial_5, \partial_6 \rangle &= \langle \partial_1, \partial_3, \partial_5, \partial_6 \rangle + \langle \partial_1, \partial_2 \rangle + \langle \partial_1 + \partial_3 \rangle,
\end{aligned}$$

for $\mathbf{J}^\Phi_1, \mathbf{J}^\Phi_2$, respectively. However, condition (3.2) is not satisfied, namely

$$\begin{aligned}
&\bigcap_{\alpha=1}^{2} \left(\ker \omega^\alpha_p + \mathrm{T}_p(G_{\mu^\alpha} p) \right) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \\
&= (\langle \partial_3, \partial_4 \rangle + \langle \partial_1 + \partial_3 \rangle) \cap (\langle \partial_1, \partial_2 \rangle + \langle \partial_1 + \partial_3 \rangle) \cap \langle \partial_1, \partial_3, \partial_5, \partial_6 \rangle \\
&= \langle \partial_1, \partial_3 \rangle \neq \langle \partial_1 + \partial_3 \rangle = \mathrm{T}_p(G_{\boldsymbol{\mu}} p),
\end{aligned}$$

for any $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. By Marrero et al. (2015, Lemmas 3.15 and 3.16), one has that $\widetilde{\pi}^1_p$ and $\widetilde{\pi}^2_p$ are surjective but $\ker \widetilde{\pi}^1_p \cap \ker \widetilde{\pi}^2_p \neq 0$. One can also verify this fact by computing $\widetilde{\pi}^\alpha_p$ for $\alpha = 1, 2$. Namely, this follows from

$$\begin{aligned}
\widetilde{\pi}^1_p &: \langle [\partial_1], [\partial_5], [\partial_6] \rangle \in \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))/\mathrm{T}_p(G_{\boldsymbol{\mu}} p) \\
&\longmapsto \langle [\partial_5], [\partial_6] \rangle = (\ker \mathrm{T}_p\mathbf{J}^\Phi_1 / \ker \omega^1_p)/\langle [\partial_1 + \partial_3] \rangle. \\
\widetilde{\pi}^2_p &: \langle [\partial_1], [\partial_5], [\partial_6] \rangle \in \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))/\mathrm{T}_p(G_{\boldsymbol{\mu}} p) \\
&\longmapsto \langle [\partial_5], [\partial_6] \rangle = (\ker \mathrm{T}_p\mathbf{J}^\Phi_2 / \ker \omega^2_p)/\langle [\partial_1 + \partial_3] \rangle.
\end{aligned}$$

for all $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Note that $[\partial_1 + \partial_3] = [\partial_1]$ in the first line, while $[\partial_1 + \partial_3] = [\partial_3]$ in the second. $\triangle$

Example 3.7 Let us prove that the k-polysymplectic Marsden–Weinstein reduction theorem in Marrero et al. (2015) gives sufficient, but not necessary conditions for the reduction to hold. In this respect, there are cases where the reduction is possible, condition (3.2) holds, while condition (3.1) does not. To illustrate this, let us consider a two-polysymplectic manifold $(\mathbb{R}^7, \boldsymbol{\omega})$, where $\{x_1, \ldots, x_7\}$ are global linear coordinates and

$$\begin{aligned}
\boldsymbol{\omega} &= \omega^1 \otimes e_1 + \omega^2 \otimes e_2 \\
&= (\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_5 \wedge \mathrm{d}x_7 + \mathrm{d}x_3 \wedge \mathrm{d}x_6) \otimes e_1 + (\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_2 \,.
\end{aligned}$$

This give rise to a two-polysymplectic structure on $\mathbb{R}^7$ since $\ker \omega^1 = \langle \partial_4 \rangle$, $\ker \omega^2 = \langle \partial_1, \partial_2, \partial_7 \rangle$ and $\ker \omega^1 \cap \ker \omega^2 = 0$. Consider the Lie group action $\Phi : \mathbb{R} \times \mathbb{R}^7 \to$

$\mathbb{R}^7$ corresponding to translations along the x_5 coordinate. Then, its Lie algebra of fundamental vector fields is $\langle\partial_5\rangle$. A two-polysymplectic momentum map associated with Φ reads

$$\mathbf{J}^{\Phi} : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7 \longmapsto (x_7, x_6) = \boldsymbol{\mu} \in (\mathbb{R}^2)^*.$$

Note that $\mathbf{J}^{\Phi}$ is Ad^{*2}-equivariant and every $\boldsymbol{\mu} \in \mathbb{R}^{2*}$ is a regular value of $\mathbf{J}^{\Phi}$. Then,

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7 \mid x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\} \simeq \mathbb{R}^5$$

is a submanifold of $\mathbb{R}^7$ for every $\boldsymbol{\mu} = (x_7, x_6) \in \mathbb{R}^2$ and

$$\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \langle\partial_1, \partial_2, \partial_3, \partial_4, \partial_5\rangle, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}).$$

Condition (3.2) is satisfied, while (3.1) for $\mathbf{J}_1^{\Phi}$ is not since

$$\begin{aligned}\widetilde{\pi}_p^1 &: \langle[\partial_1], [\partial_2], [\partial_3], [\partial_4]\rangle \in \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \mapsto \langle[\partial_1], [\partial_2], [\partial_3], [\partial_6]\rangle \\ &= (\ker \mathrm{T}_p\mathbf{J}_1^{\Phi} / \ker \omega_p^1)/\langle[\partial_5]\rangle.\end{aligned}$$

Therefore, $\widetilde{\pi}_p^1$ is not surjective. However, the reduced manifold $P_{\boldsymbol{\mu}} = \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \simeq \mathbb{R}^4$ inherits a two-polysymplectic form, namely

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes e_1 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \otimes e_2,$$

in the variables x_1, x_2, x_3, x_4 naturally defined in $P_{\boldsymbol{\mu}}$. In summary, both conditions (3.1) and (3.2) ensure a k-polysymplectic Marsden–Weinstein reduction. But they are not necessary, they are only sufficient. △

3.3 On the *k*-Polysymplectic Manifold Given by the Product of *k* Symplectic Manifolds

Let us review a relevant example of k-polysymplectic manifold and apply Theorem 3.3 to it (see Marrero et al. (2015) for details). This will illustrate how the k-polysymplectic reduction theorem works. Remarkably, many practical examples have a related k-polysymplectic manifold similar to the one in this section. Moreover, this structure will be used in one of the physical examples studied in Sect. 5.

Let $P = P_1 \times \cdots \times P_k$ for some symplectic manifolds $(P_\alpha, \omega^\alpha)$ with $\alpha = 1, \ldots, k$. If $\mathrm{pr}_\alpha : P \to P_\alpha$ is the canonical projection onto the α-th component, P_α, in P, then $(P, \boldsymbol{\omega} = \sum_{\alpha=1}^k \mathrm{pr}_\alpha^* \omega^\alpha \otimes e_\alpha)$ is a k-polysymplectic manifold. To simplify the notation, we will write $\mathrm{pr}_\alpha^* \omega^\alpha$ as ω^α. Moreover, assume that a Lie group action $\Phi^\alpha : G_\alpha \times P_\alpha \to P_\alpha$ admits a symplectic momentum map $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ and each Φ^α acts in a quotientable manner on the level sets given by weak regular values of $\mathbf{J}^{\Phi^\alpha}$ for each $\alpha = 1, \ldots, k$.

Define the Lie group action

$$\Phi : (g_1, \dots, g_k, x_1, \dots, x_k) \in G \times P \longmapsto (\Phi^1_{g_1}(x_1), \dots, \Phi^k_{g_k}(x_k)) \in P. \quad (3.6)$$

Then, $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ is the Lie algebra of G and we have the k-polysymplectic momentum map

$$\mathbf{J} : (x_1, \dots, x_k) \in P \longmapsto (0, \dots, \mathbf{J}^\alpha, \dots, 0) \otimes e_\alpha \in \mathfrak{g}^{*k},$$

where $\mathbf{J}^\alpha(x_1, \dots, x_k) = \mathbf{J}^{\Phi^\alpha}(x_\alpha)$ for $\alpha = 1, \dots, k$ and $\mathfrak{g}^* = \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_k^*$ is the dual space to $\mathfrak{g}$. Suppose, that $\mu^\alpha \in \mathfrak{g}_\alpha^*$ is a weak regular value of $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \dots, k$. Hence, $\boldsymbol{\mu} = (0, \dots, \mu^\alpha, \dots, 0) \otimes e_\alpha \in (\mathfrak{g}^*)^k$ is a weak regular value of $\mathbf{J}$. Then, Φ acts in a quotientable on the level sets of $\mathbf{J}$.

Therefore, if $p = (x_1, \dots, x_k) \in \mathbf{J}^{-1}(\boldsymbol{\mu})$, it follows that

$$\begin{aligned}
\ker \mathrm{T}_p \mathbf{J}^{\Phi^\alpha} &= \mathrm{T}_{x_1} P_1 \oplus \cdots \oplus \ker \mathrm{T}_{x_\alpha} \mathbf{J}^{\Phi^\alpha} \oplus \cdots \oplus \mathrm{T}_{x_k} P_k \,, \\
\mathrm{T}_p \left(\mathbf{J}^{-1}(\boldsymbol{\mu}) \right) &= \ker \mathrm{T}_{x_1} \mathbf{J}^{\Phi^1} \oplus \cdots \oplus \ker \mathrm{T}_{x_k} \mathbf{J}^{\Phi^k} \,, \\
\ker \omega_p^\alpha &= \mathrm{T}_{x_1} P_1 \oplus \cdots \oplus \mathrm{T}_{x_{\alpha-1}} P_{\alpha-1} \oplus \{0\} \oplus \mathrm{T}_{x_{\alpha+1}} P_{\alpha+1} \oplus \cdots \oplus \mathrm{T}_{x_k} P_k \,, \\
\mathrm{T}_p \left(G^{\Delta^\alpha}_{\mu^\alpha} p \right) &= \mathrm{T}_{x_1} (G_1 x_1) \oplus \cdots \oplus \mathrm{T}_{x_\alpha} \left(G^{\Delta^\alpha}_{\alpha \mu^\alpha} x_\alpha \right) \oplus \cdots \oplus \mathrm{T}_{x_k} (G_k x_k) \,, \\
\mathrm{T}_p \left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p \right) &= \mathrm{T}_{x_1} \left(G^{\Delta^1}_{1\mu^1} x_1 \right) \oplus \cdots \oplus \mathrm{T}_{x_k} \left(G^{\Delta^k}_{k\mu^k} x_k \right) .
\end{aligned}$$

Then, it follows immediately that

$$\ker \mathrm{T}_p \mathbf{J}^{\Phi^\alpha} = \mathrm{T}_p \left(\mathbf{J}^{-1}(\mu) \right) + \ker \omega_p^\alpha + \mathrm{T}_p \left(G^{\Delta^\alpha}_{\mu^\alpha} p \right), \qquad \alpha = 1, \dots, k,$$

and

$$\mathrm{T}_p \left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p \right) = \bigcap_{\beta=1}^{k} \left(\ker \omega_p^\beta + \mathrm{T}_p \left(G^{\Delta^\beta}_{\mu^\beta} p \right) \right) \cap \mathrm{T}_p \left(\mathbf{J}^{-1}(\boldsymbol{\mu}) \right),$$

for every weakly regular $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ and $p \in \mathbf{J}^{-1}(\boldsymbol{\mu})$. Recall that, by Theorem 3.3, these equations guarantee that the reduced space $\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ can be endowed with a k-polysymplectic structure, while

$$\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \simeq \mathbf{J}^{\Phi^1 -1}(\mu^1)/G^{\Delta^1}_{1\mu^1} \times \cdots \times \mathbf{J}^{\Phi^k -1}(\mu^k)/G^{\Delta^k}_{k\mu^k}.$$

△

4 The *k*-Polysymplectic Energy Momentum-Method

4.1 *k*-Polysymplectic Relative Equilibrium Points

This section introduces the notion of k-polysymplectic relative equilibrium point relative to an $\boldsymbol{\omega}$-Hamiltonian vector field X. This notion is devised to analyse the relative stability of $\boldsymbol{\omega}$-Hamiltonian vector fields and extends the relative equilibrium point notion for symplectic manifolds to the k-symplectic realm (see Abraham and Marsden 1978 for details on the symplectic case). In brief, a relative equilibrium point for a dynamical system given by a vector field is a point whose evolution is given by Lie group symmetries of the vector field. If the vector field is additionally Hamiltonian relative to some geometric structure, then it is usual to demand the Lie group symmetries to leave invariant the same geometric structure (Lucas et al. 2024). It is generally interesting to analyse the behaviour of solutions close to relative equilibrium points, i.e. if they get closer or move away from those points.

Definition 4.1 Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. A point $z_e \in P$ is a *k-polysymplectic relative equilibrium point* of the $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ if there exists $\xi \in \mathfrak{g}$ so that

$$(X_{\boldsymbol{h}})(z_e) = (\xi_P)(z_e).$$

The above definition retrieves, for $k = 1$, the standard relative equilibrium notion for symplectic Hamiltonian systems (Abraham and Marsden 1978). Furthermore, Lemma 3.2 and the fact that $X_{\boldsymbol{h}}$ is tangent to the level sets of $\mathbf{J}^\Phi$ show that $\xi \in \mathfrak{g}$ in Definition 4.1 is, in fact, an element of $\mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$, which is a Lie subalgebra of $\mathfrak{g}$, and $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$.

Note that a k-polysymplectic relative equilibrium point $z_e \in P$ projects onto $\pi_{\boldsymbol{\mu}_e}(z_e)$, with $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$, which becomes an equilibrium point of the vector field $X_{f_{\boldsymbol{\mu}_e}}$, obtained by projection of $X_{\boldsymbol{h}}$ onto the reduced space $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$. This explains the term *relative* used in the *relative equilibrium point* term.

The following theorem provides the characterisation of k-polysymplectic relative equilibrium points of an $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ by studying the critical points of a modified $\mathbb{R}^k$-valued function $\boldsymbol{h}_\xi$ on P. This is an application of the Lagrange multiplier theorem, where the role of the multiplier is played by $\xi \in \mathfrak{g}$.

Theorem 4.2 *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. Then, $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if there exists $\xi \in \mathfrak{g}$ such that z_e is a critical point of the following $\mathbb{R}^k$-valued function*

$$\boldsymbol{h}_\xi := \boldsymbol{h} - \langle \mathbf{J}^\Phi - \boldsymbol{\mu}_e, \xi \rangle, \tag{4.1}$$

where $\boldsymbol{\mu}_e := \mathbf{J}^\Phi(z_e) \in (\mathfrak{g}^*)^k$.

Proof Let z_e be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$, i.e. $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$ for some $\xi \in \mathfrak{g}$. Then,

$$\mathrm{d}\boldsymbol{h}_\xi(z_e) = \mathrm{d}(\boldsymbol{h} - \langle \mathbf{J}^\Phi, \xi \rangle)(z_e) = (\iota_{X_{\boldsymbol{h}} - \xi_P}\boldsymbol{\omega})(z_e) = 0.$$

Hence, $z_e \in P$ is a critical point of the $\mathbb{R}^k$-valued function $\boldsymbol{h}_\xi$.

Conversely, assume that z_e is a critical point of some $\boldsymbol{h}_\xi$ with $\xi \in \mathfrak{g}$. Then, $0 = \mathrm{d}\boldsymbol{h}_\xi(z_e) = (\iota_{X_{\boldsymbol{h}} - \xi_P}\boldsymbol{\omega})(z_e) = 0$ and $(X_{\boldsymbol{h}} - \xi_P)(z_e) \in \ker \boldsymbol{\omega}_{z_e}$. Since $\ker \boldsymbol{\omega} = 0$, one has that $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$. Hence, z_e is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. □

Example 4.3 (*The cotangent bundle of two-covelocities of* $\mathbb{R}^2$) Let Q be an n-dimensional manifold and let $\pi_Q : \mathrm{T}^*Q \to Q$ be the cotangent bundle projection. Consider the Whitney sum $\bigoplus^k \mathrm{T}^*Q = \mathrm{T}^*Q \oplus_Q \overset{(k)}{\cdots} \oplus_Q \mathrm{T}^*Q$ of k copies of T^*Q and the projection $\pi^k_Q : \bigoplus^k \mathrm{T}^*Q \to Q$. It is well-known that $\bigoplus^k \mathrm{T}^*Q$ can be identified with the space of one-jets, $J^1(Q, \mathbb{R}^k)_0$, of maps $\gamma : Q \to \mathbb{R}^k$ with $\gamma(q) = 0$, via the diffeomorphism $j^1_q\gamma \in J^1(Q, \mathbb{R}^k)_0 \mapsto \big(\mathrm{d}\gamma^1(q), \ldots, \mathrm{d}\gamma^k(q)\big) \in \bigoplus^k \mathrm{T}^*Q$, where γ^α is the α-th component of γ (de León et al. 2015). Then, $\bigoplus^k \mathrm{T}^*Q$ is called *the cotangent bundle of k-covelocities of Q*. Moreover, $J^1(Q, \mathbb{R}^k)_0$ is a k-polysymplectic manifold (see de León et al. 2015 for details).

The following example will illustrate our k-polysymplectic energy–momentum method. Consider that $(\mathbb{R}^6, \boldsymbol{\omega})$ is a two-polysymplectic manifold with the two-polysymplectic form

$$
\begin{aligned}
\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 &= (\mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_4) \otimes e_1 \\
&\quad + (\mathrm{d}x_1 \wedge \mathrm{d}x_5 + \mathrm{d}x_2 \wedge \mathrm{d}x_6) \otimes e_2
\end{aligned}
$$

since

$$
\ker \omega^1 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle, \qquad \ker \omega^2 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle,
$$

and $\ker \omega^1 \cap \ker \omega^2 = 0$. Let us consider the Lie group action $\Phi : \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}^6$ given by

$$
\begin{aligned}
\Phi : &(\lambda; x_1, x_2, x_3, x_4, x_5, x_6) \\
&\in \mathbb{R} \times \mathbb{R}^6 \longmapsto (x_1 + \lambda, x_2 + \lambda, x_3 + \lambda, x_4 + \lambda, x_5 + \lambda, x_6 + \lambda) \in \mathbb{R}^6.
\end{aligned}
$$

The fundamental vector fields associated with the Lie group action Φ are spanned by

$$
\xi_P = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}.
$$

Note that the Lie group action Φ is two-polysymplectic since it leaves $\boldsymbol{\omega}$ invariant, namely $\mathscr{L}_{\xi_P}\boldsymbol{\omega} = 0$. Then, Φ gives rise to a two-polysymplectic momentum map $\mathbf{J}^\Phi$ for $\boldsymbol{\mu} = (\mu^1, \mu^2)$ given by

$$
\begin{aligned}
\mathbf{J}^\Phi : (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 &\longmapsto (x_3 + x_4 - x_1 - x_2, x_5 + x_6 - x_1 - x_2) \\
&= \boldsymbol{\mu} \in (\mathbb{R}^*)^2.
\end{aligned}
$$

Therefore, the level set of the two-polysymplectic momentum map $\mathbf{J}^{\Phi}$ has the following form

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \left\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \mid x_3 + x_4 - x_1 - x_2 = \mu^1, \;\; x_5 + x_6 - x_1 - x_2 = \mu^2\right\}. \tag{4.2}$$

Note that every $\boldsymbol{\mu} \in (\mathbb{R}^*)^2$ is a regular value of a two-polysymplectic momentum map $\mathbf{J}^{\Phi}$ and $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \simeq \mathbb{R}^4$. Since Φ is defined on a connected one-dimensional Lie group $\mathbb{R} = G$, one has that $\mathbf{J}^{\Phi}$ is an Ad^{*2}-equivariant two-polysymplectic momentum map. Then,

$$\mathrm{T}_p(G_{\boldsymbol{\mu}} p) = \mathrm{T}_p(G_{\mu^1} p) = \mathrm{T}_p(G_{\mu^2} p) = \left\langle \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle,$$

$$\ker \mathrm{T}_p \mathbf{J}_1^{\Phi} = \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle,$$

$$\ker \mathrm{T}_p \mathbf{J}_2^{\Phi} = \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_6} \right\rangle,$$

$$\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \left\langle \sum_{i=1}^{6} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right\rangle,$$

and one can verify that conditions (3.1) and (3.2) are fulfilled.

Recall that $\iota_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \hookrightarrow P$ is the natural immersion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is the canonical projection. Then, remembering that the elements of the Lie group $\mathbb{R}$ act by translations on $\mathbb{R}^6$ via Φ, Theorem 3.3 yields that the reduced manifold $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}} \simeq \mathbb{R}^3, \boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a two-polysymplectic manifold with coordinates $(y_1, y_2, y_3) \in \mathbb{R}^3$, satisfying that

$$\begin{aligned} &y_1 = x_1 - x_2\,, && y_2 = x_3 - x_1\,, && y_3 = x_5 - x_1\,,\\ &y_4 = x_1 + x_2 - x_3 - x_4\,, && y_5 = x_1 + x_2 - x_5 - x_6\,, && y_6 = x_1\,, \end{aligned}$$

with

$$\boldsymbol{\omega}_{\mu} = \omega^1_{\boldsymbol{\mu}} \otimes e_1 + \omega^2_{\boldsymbol{\mu}} \otimes e_2 = \mathrm{d}y_1 \wedge \mathrm{d}y_2 \otimes e_1 + \mathrm{d}y_1 \wedge \mathrm{d}y_3 \otimes e_2.$$

Next, let us consider an $\boldsymbol{\omega}$-Hamiltonian vector field, $X_{\boldsymbol{h}}$, on $P = \mathbb{R}^6$ whose $\boldsymbol{\omega}$-Hamiltonian function is $\mathbb{R}$-invariant. Then, $X_{\boldsymbol{h}}$ is tangent to each $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, and it will have the following form

$$X_{\boldsymbol{h}} = F_1 \sum_{i=1}^{6} \frac{\partial}{\partial x_i} + F_2 \left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right) + F_3 \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5} \right) \\ + F_4 \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right),$$

for certain uniquely defined G-invariant $F_1, \ldots, F_4 \in \mathscr{C}^\infty(P)$. Then, a point $z_e \in P$ is a two-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$, which holds, if and only if, $F_1(z_e) = 1$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. However, let us verify that we obtain the same result using Theorem 4.2.

First, $\mathrm{d}h^1$ and $\mathrm{d}h^2$ read

$$\begin{aligned} \mathrm{d}h^1 = \iota_{X_{\boldsymbol{h}}} \omega^1 = &-(F_1 + F_2 + F_3)\, \mathrm{d}x_1 \\ &- (F_1 - F_2)\, \mathrm{d}x_2 + (F_1 + F_4)\, \mathrm{d}x_3 + (F_1 + F_3 - F_4)\, \mathrm{d}x_4\,, \\ \mathrm{d}h^2 = \iota_{X_{\boldsymbol{h}}} \omega^2 = &-(F_1 + F_3)\, \mathrm{d}x_1 - F_1 \mathrm{d}x_2 + (F_1 + F_4)\, \mathrm{d}x_5 + (F_1 + F_3 - F_4)\, \mathrm{d}x_6\,. \end{aligned}$$

Then, Theorem 4.2 yields that $z_e \in P$ is a two-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if $\mathrm{d}h^1_\xi(z_e) = 0$ and $\mathrm{d}h^2_\xi(z_e) = 0$ for some $\xi \in \mathbb{R}$. Indeed, using (4.2), one has

$$\begin{aligned} \mathrm{d}h^1_\xi = \mathrm{d}h^1 - \mathrm{d}J^1_\xi = &-(F_1 + F_2 + F_3 - \xi)\, \mathrm{d}x_1 - (F_1 - F_2 - \xi)\, \mathrm{d}x_2 \\ &+ (F_1 + F_4 - \xi)\, \mathrm{d}x_3 + (F_1 + F_3 - F_4 - \xi)\, \mathrm{d}x_4, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathrm{d}h^2_\xi = \mathrm{d}h^2 - \mathrm{d}J^2_\xi = &-(F_1 + F_3 - \xi)\, \mathrm{d}x_1 - (F_1 - \xi)\, \mathrm{d}x_2 \\ &+ (F_1 + F_4 - \xi)\, \mathrm{d}x_5 + (F_1 + F_3 - F_4 - \xi)\, \mathrm{d}x_6, \end{aligned} \tag{4.4}$$

for $\xi \in \mathbb{R}$. Since at z_e both (4.3) and (4.4) must vanish, one gets that this happens if and only if $F_1(z_e) = \xi$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. Therefore, $z_e \in P$ is a two-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ under the above-mentioned conditions.

Finally, let us verify that $\pi_{\boldsymbol{\mu}_e}(z_e)$ is a critical point of the $f^\alpha_{\boldsymbol{\mu}_e} \in \mathscr{C}^\infty(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G_{\boldsymbol{\mu}_e})$. The reduced vector field $X_{f_{\boldsymbol{\mu}_e}}$ has the form

$$X_{f_{\boldsymbol{\mu}_e}} = (2\widetilde{F}_4 - \widetilde{F}_3) \frac{\partial}{\partial y_1} + (\widetilde{F}_2 + \widetilde{F}_3 - \widetilde{F}_4) \frac{\partial}{\partial y_2} - \widetilde{F}_4 \frac{\partial}{\partial y_3},$$

where $F_i = \pi^*_{\boldsymbol{\mu}_e} \widetilde{F}_i$ for $i = 2, 3, 4$. Note that the projection exists because F_2, F_3, F_4 are G-invariant. Then,

$$\begin{aligned} \mathrm{d}f^1_{\boldsymbol{\mu}_e}(\pi_{\boldsymbol{\mu}_e}(z_e)) &= \left(\iota_{X_{f_{\boldsymbol{\mu}_e}}} \omega^1_{\boldsymbol{\mu}_e} \right)_{\pi_{\boldsymbol{\mu}_e}(z_e)} \\ &= \left(2\widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_3(\pi_{\boldsymbol{\mu}_e}(z_e)) \right) \mathrm{d}y_2 \\ &\quad + \left(\widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_2(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_3(\pi_{\boldsymbol{\mu}_e}(z_e)) \right) \mathrm{d}y_1 = 0, \end{aligned}$$

and

$$
\begin{aligned}
\mathrm{d} f^2_{\boldsymbol{\mu}_e}(\pi_{\boldsymbol{\mu}_e}(z_e)) &= \left(\iota_{X_{f_{\boldsymbol{\mu}_e}}} \omega^2_{\boldsymbol{\mu}_e}\right)_{\pi_{\boldsymbol{\mu}_e}(z_e)} \\
&= \left(2\widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_3(\pi_{\boldsymbol{\mu}_e}(z_e))\right) \mathrm{d}y_3 + \widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e))\mathrm{d}y_1 = 0.
\end{aligned}
$$

Indeed, $\pi_{\boldsymbol{\mu}_e}(z_e)$ is a critical point of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$, hence $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$.

4.2 Stability in the *k*-Polysymplectic Energy Momentum-Method

Let us develop the stability analysis related to the k-polysymplectic energy–momentum method relative to a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. Recall that Theorem 4.2 characterises k-polysymplectic relative equilibrium points as critical points of the $\mathbb{R}^k$-valued function (4.1). However, when studying the stability of k-polysymplectic relative equilibrium points, due to the symmetry of our problems, we need to investigate how the second variation of $\boldsymbol{h}_\xi$ in the directions tangent to the isotropy group $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ affects the positive definiteness of $\boldsymbol{h}_\xi$. Note also that the results of this section are concerned with cases when a k-polysymplectic reduction is possible and (3.4) is satisfied.

Let us define the second variation of $\boldsymbol{h}_\xi$ at a k-polysymplectic relative equilibrium point $z_e \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ as the mapping $(\delta^2 \boldsymbol{h}_\xi)_{z_e} : \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)) \times \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)) \to \mathbb{R}$, with $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$, of the form

$$
\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e}(v_1, v_2) = \sum_{\alpha=1}^{k} \iota_Y \left(\mathrm{d}\left(\iota_X \mathrm{d} h^\alpha_\xi\right)\right)_{z_e} \otimes e_\alpha, \tag{4.5}
$$

for some vector fields X, Y on P defined on a neighbourhood of $z_e \in P$ and such that $v_1 = X_{z_e}$, $v_2 = Y_{z_e}$. The following proposition shows that, since z_e is a k-polysymplectic relative equilibrium point, the above definition does not depend on the value of the particular chosen vector fields X, Y out of z_e and $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ is well-defined.

Proposition 4.4 *Let $z_e \in P$ be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ on a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. If $\{x_1, \ldots, x_n\}$ are coordinates on a neighbourhood of $z_e \in P$, then*

$$
(\delta^2 h^\alpha_\xi)_{z_e}(w, v) = \sum_{i,j=1}^{n} \frac{\partial^2 h^\alpha_\xi}{\partial x_i \partial x_j}(z_e) w_i v_j, \quad \forall w, v \in \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)), \quad \alpha = 1, \ldots, k,
$$

where $w = \sum_{i=1}^n w_i \partial/\partial x_i$ and $v = \sum_{i=1}^n v_i \partial/\partial x_i$.

Proof From (4.5) for $\alpha = 1, \dots, k$, we have

$$
\begin{aligned}
(\delta^2 h^\alpha_\xi)_{z_e}(w, v) &= \iota_Y(\mathrm{d}\iota_X \mathrm{d}h^\alpha_\xi)_{z_e} \\
&= \sum_{i,j=1}^{n} \frac{\partial^2 h^\alpha_\xi}{\partial x_i \partial x_j}(z_e) w_i v_j + \sum_{i,j=1}^{n} \frac{\partial h^\alpha_\xi}{\partial x_i}(z_e) \frac{\partial X_i}{\partial x_j}(z_e) v_j \\
&= \sum_{i,j=1}^{n} \frac{\partial^2 h^\alpha_\xi}{\partial x_i \partial x_j}(z_e) w_i v_j \,,
\end{aligned}
$$

where $X = \sum_{i=1}^n X_i \partial/\partial x_i$ with $X(z_e) = w$, and we have used that z_e is a k-polysymplectic relative equilibrium point and, therefore, $\boldsymbol{h}_\xi$ has a critical point at z_e, namely $(Zh^\alpha_\xi)(z_e) = 0$ for every vector field Z on P and $\alpha = 1, \dots, k$. □

Note that the maps $(\delta^2 h^\alpha_\xi)_{z_e}$ are symmetric for $\alpha = 1, \dots, k$. Therefore, $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ is symmetric. Let us study (4.5) in more detail.

Proposition 4.5 *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $z_e \in P$ be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. Then,*

$$
(\delta^2 \boldsymbol{h}_\xi)_{z_e}((\zeta_P)_{z_e}, v_{z_e}) = 0, \qquad \forall \zeta \in \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}, \qquad \forall v_{z_e} \in \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)),
$$

with $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$. Moreover,

$$
(\delta^2 h^\alpha_\xi)_{z_e}(Y_{z_e}, \cdot) = 0, \quad \forall\, Y_{z_e} \in \ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)), \quad \alpha = 1, \dots, k. \tag{4.6}
$$

Proof First, since $\boldsymbol{h} \in \mathscr{C}^\infty(P, \mathbb{R}^k)$ is G-invariant and $\mathbf{J}^\Phi$ is equivariant with respect to the k-polysymplectic affine Lie group action $\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, then for every $g \in G$ and $p \in P$, one has

$$
\begin{aligned}
\boldsymbol{h}_\xi(\Phi_g(p)) &= \boldsymbol{h}(\Phi_g(p)) - \langle \mathbf{J}^\Phi(\Phi_g(p)), \xi \rangle + \langle \boldsymbol{\mu}_e, \xi \rangle \\
&= \boldsymbol{h}(p) - \langle \boldsymbol{\Delta}_g \mathbf{J}^\Phi(p), \xi \rangle + \langle \boldsymbol{\mu}_e, \xi \rangle = \boldsymbol{h}(p) - \sum_{\alpha=1}^{k} \langle \mathbf{J}^\Phi_\alpha(p), \Delta^T_{g\alpha}\xi \rangle \otimes e_\alpha + \langle \boldsymbol{\mu}_e, \xi \rangle,
\end{aligned}
$$

where $\boldsymbol{\Delta}^T_g : \mathfrak{g}^k \to \mathfrak{g}^k$ is the transpose of $\boldsymbol{\Delta}_g$ for $g \in G$ and $\Delta_{g1}, \dots, \Delta_{gk}$ are its components. Let us substitute $g = \exp(t\zeta)$, with $\zeta \in \mathfrak{g}$, and differentiate with respect to t. Then,

$$
\begin{aligned}
\left(\iota_{\zeta_P} \mathrm{d}\boldsymbol{h}_\xi\right)_{z_e} &= -\sum_{\alpha=1}^{k} \left\langle \mathbf{J}^\Phi_\alpha(p), \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Delta^T_{\exp(t\zeta)\alpha}\xi \right\rangle \otimes e_\alpha \\
&= -\sum_{\alpha=1}^{k} \left\langle \mathbf{J}^\Phi_\alpha(p), (\zeta^{\Delta_\alpha}_{\mathfrak{g}})_\xi \right\rangle \otimes e_\alpha,
\end{aligned} \tag{4.7}
$$

where $(\zeta_{\mathfrak{g}}^{\Delta_\alpha})_\xi$ is the fundamental vector field of $\Delta_\alpha^T : G \times \mathfrak{g} \to \mathfrak{g}$ at $\xi \in \mathfrak{g}$ for $\alpha = 1, \ldots, k$. Taking the second variation of (4.7) relative to $p \in P$, evaluating at $z_e \in P$, and contracting with v_{z_e}, one has

$$\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e} ((\zeta_P)_{z_e}, v_{z_e}) = -\sum_{\alpha=1}^{k} \left\langle \mathrm{T}_{z_e} \mathbf{J}_\alpha^\Phi \left(v_{z_e}\right), (\zeta_{\mathfrak{g}}^{\Delta_\alpha})_\xi \right\rangle \otimes e_\alpha.$$

Therefore, the second variation $\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e} ((\zeta_P)_{z_e}, v_{z_e})$ vanishes since $v_{z_e} \in \mathrm{T}_{z_e} (\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)) \subset \ker \mathrm{T}_{z_e} \mathbf{J}_\alpha^\Phi$.

Concerning (4.6), it is a consequence of (4.5) and the fact that, for every vector field Y on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ taking values in $\ker \omega^\alpha \cap \mathrm{T}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$, it follows that

$$\iota_Y \mathrm{d}h^\alpha = \omega^\alpha(X_{\boldsymbol{h}}, Y) = 0, \qquad \iota_Y \mathrm{d}\langle \mathbf{J}_\alpha^\Phi, \xi\rangle = \omega^\alpha(\xi_P, Y) = 0,$$

for $\alpha = 1, \ldots, k$ and every $\xi \in \mathfrak{g}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$. $\square$

Proposition 4.5 and Proposition 3.2 state that $\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e}$ is degenerate in the directions tangent to $\mathrm{T}_{z_e}\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e\right)$, while each $(\delta^2 h^\alpha_\xi)_{z_e}$ is degenerate in the directions of $\ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$. On the other hand, since $\ker(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ contains $\ker \mathrm{T}_{z_e}\pi_{\boldsymbol{\mu}_e}$, one can define a bilinear two-form on $\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)} P_{\boldsymbol{\mu}_e}$, with $P_{\boldsymbol{\mu}_e} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$, by reducing to that space the bilinear two-form $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$. By using an adapted coordinate system, one can prove that the reduction of $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ to $\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)} P_{\boldsymbol{\mu}_e}$ gives the behaviour of the Hessian of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ on $P_{\boldsymbol{\mu}_e}$. It is worth noting that the reduction $\boldsymbol{f}_{\mu_e}$ to $P_{\boldsymbol{\mu}}$ of $\boldsymbol{h}_\xi$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ does not depend on ξ, as the value of $\boldsymbol{h}_\xi$ on points of $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ does not really depend on ξ: it is only the restriction of $\boldsymbol{h}$ to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$. Note also that only directions transverse to the orbit of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ are significant for determining, via the variation of $\boldsymbol{h}_\xi$, the stability character of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ at one of its equilibrium points.

There are many manners to ensure the stability of a k-polysymplectic reduced Hamiltonian system. This suggests us to give the following definition of formal stability. For the case of a symplectic manifold, it retrieves the standard condition for the stability of a reduced symplectic problem (Marsden and Simo 1988).

Definition 4.6 Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $z_e \in P$ be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. Then, z_e is called a *formally stable k-polysymplectic relative equilibrium point* if, for a family of supplementary spaces $\mathcal{S}^\alpha$ such that $\mathcal{S}^\alpha \oplus (\mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) + \ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))) = \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$ and $\mathcal{S}^1 + \cdots + \mathcal{S}^k + \mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) = \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$, one has

$$\left(\delta^2 h^\alpha_\xi\right)_{z_e} (v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \mathcal{S}^\alpha \backslash \{0\}, \qquad \alpha = 1, \ldots, k. \tag{4.8}$$

Note that, given a family of subspaces $W_1, \ldots, W_k$ of a vector space E such that $\cap_{\alpha=1}^k W_\alpha = 0$, one cannot infer that any supplementary spaces $V_\alpha \oplus W_\alpha = E$ will satisfy $V_1 + \cdots + V_k = E$. This is, essentially, why the condition $\mathcal{S}^1 + \cdots + \mathcal{S}^k +$

$\mathrm{T}_{z_e}(G^{\Delta}_{\boldsymbol{\mu}_e})$ was added. Indeed, to ensure the stability on the reduced manifold, we will use the fact that the projection of $\mathcal{S}^1 + \cdots + \mathcal{S}^k$ to the tangent space to an equilibrium point in a reduced manifold spans the total tangent space at that point.

If a system satisfies our formal stability, then $\sum_{\alpha=1}^{k} f^{\alpha}_{\boldsymbol{\mu}_e}$ has a strict minimum at $\pi_{\boldsymbol{\mu}_e}(z_e)$ and the function is invariant relative to the evolution of the reduced $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian system. Hence, that system is stable. The converse is not true, as in the symplectic case (Abraham and Marsden 1978). We will not study all methods to prove stability in the reduced k-symplectic Hamiltonian system in this paper, and we will leave this for further work.

The proof of the above-mentioned fact relies on using a coordinate system on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ adapted to its fibration onto $P_{\boldsymbol{\mu}_e}$ and the fact that the obtained results involve geometric objects that are independent of the coordinate system (see de Lucas and Zawora 2021; Zawora 2021 for a symplectic analogue). In the adapted coordinate system, the Hessian of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ on the reduced space $P_{\boldsymbol{\mu}_e}$ at $\pi_{\boldsymbol{\mu}_e}(z_e)$ is retrieved by the Hessian of $\boldsymbol{h}_{\xi}$ on directions of $\mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$ that are not tangent to $\ker \mathrm{T}_{z_e}\pi_{\boldsymbol{\mu}_e}$. The Hessian of the reduced function $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ can be decomposed into k components. The vector subspaces $\mathcal{S}^1, \ldots, \mathcal{S}^k$ project onto a series of spaces spanning $\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)}P_{\boldsymbol{\mu}_e}$. Condition (4.8) implies that

$$
\begin{aligned}
&\frac{\partial^2 f^{\alpha}_{\boldsymbol{\mu}_e}}{\partial z_i \partial z_j}(\pi_{\boldsymbol{\mu}_e}(z_e))v^i v^j > 0\,, \qquad \forall v \in \mathrm{Im}\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)}\pi_{\boldsymbol{\mu}_e}(\mathcal{S}^{\alpha})\backslash\{0\}\,,\\
&\frac{\partial^2 f^{\alpha}_{\boldsymbol{\mu}_e}}{\partial z_i \partial z_j}(\pi_{\boldsymbol{\mu}_e}(z_e))v^i v^j \geq 0\,, \qquad \forall v \in \mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)}P_{\boldsymbol{\mu}_e}\,,
\end{aligned}
$$

for $\alpha = 1, \ldots, k$. Then,

$$
\sum_{\alpha=1}^{k} \frac{\partial^2 f^{\alpha}_{\boldsymbol{\mu}_e}}{\partial z_i \partial z_j}(\pi_{\boldsymbol{\mu}_e}(z_e))v^i v^j > 0, \qquad \forall v \in \mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)}P_{\boldsymbol{\mu}_e}\backslash\{0\}.
$$

Consequently, the second-order Taylor part of $\sum_{\alpha=1}^{k} f^{\alpha}_{\boldsymbol{\mu}_e}$ is definite-positive and we have a strict minimum. The components $f^{\alpha}_{\boldsymbol{\mu}_e}$ are constants of motion for $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$, and hence the flow of $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$, for an initial condition close enough to $\pi_{\boldsymbol{\mu}_e}(z_e)$ can be restricted to an open neighbourhood of $\pi_{\boldsymbol{\mu}_e}(z_e)$.

It is worth noting that we will also call *formally stable k-polysymplectic relative equilibrium points* points for which each (4.8) is negative-definite, as similar results can be obtained. In particular, their projections will be stable equilibrium points. It is simple to obtain many more stability criteria.

5 Applications and Examples

This section illustrates how the theory and applications of the previous sections can be applied to relevant examples with physical and mathematical applications.

5.1 Complex Schwarz Equations

The first example illustrates how locally automorphic Lie systems (Gràcia et al. 2019) can be seen as $\boldsymbol{\omega}$-Hamiltonian systems relative to a k-polysymplectic structure.

Consider the differential equation the t-dependent complex differential equation given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = a, \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{3}{2}\frac{a^2}{v} + 2b(t)v, \qquad z, v, a \in \mathbb{C}, \tag{5.1}$$

for a certain complex t-dependent function $b(t)$ defined on $\mathcal{O} = \{(z, v, a) \in \mathrm{T}^2\mathbb{C} : v \neq 0\}$, which can be considered as a system of real differential equations in a natural manner.

The system (5.1) can be understood as the complex analogue of the Lie system on $\mathcal{O}_{\mathbb{R}} = \{(z, v, a) \in \mathrm{T}^2\mathbb{R} : v \neq 0\}$ studied in de Lucas and Vilariño (2015). More specifically, (5.1) is a first-order representation for the third-order complex differential equation

$$\frac{\mathrm{d}^3 z}{\mathrm{d}t^3}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{-1} - \frac{3}{2}\left(\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\right)^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{-2} = 2b(t).$$

The left-hand side of the above expression retrieves, for $z \in \mathbb{R}$, exactly the real version of the *Schwarzian derivative* (also called *Schwarz equation*) of a function $z(t)$ of t, usually represented by $\{z(t), t\}_{sc}$, which appears in many research problems. The ideas in our work and (5.1) can be used to potentially extend to the complex realm results for the real third-order Kummer–Schwarz equation and Schwarz derivatives obtained via Lie systems (see Bozhkov and da Conceição 2020; de Lucas and Sardón 2020 and references therein). It is worth noting that the Schwarz derivative plays a significant role in studying linearisation in time-dependent systems, projective systems, mathematical functions theory, and more (cf. Guieu and Roger 2007; Hille 1997; Lehto 1979).

In real coordinates

$$v_1 = \mathfrak{Re}(z), \quad v_2 = \mathfrak{Im}(z), \quad v_3 = \mathfrak{Re}(v),$$
$$v_4 = \mathfrak{Im}(v), \quad v_5 = \mathfrak{Re}(a), \quad v_6 = \mathfrak{Im}(a),$$

the system (5.1) is associated with the t-dependent vector field

$$X = X_1 + 2b_R(t)X_2 + 2b_I(t)X_3,$$

where $b_R(t) = \mathfrak{Re}(b(t))$, $b_I(t) = \mathfrak{Im}(b(t))$, and

$$X_1 = v_3\frac{\partial}{\partial v_1} + v_4\frac{\partial}{\partial v_2} + v_5\frac{\partial}{\partial v_3} + v_6\frac{\partial}{\partial v_4} + \frac{3}{2}\frac{2v_4v_5v_6 + (v_5^2 - v_6^2)v_3}{v_3^2 + v_4^2}\frac{\partial}{\partial v_5}$$
$$+\frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2 - v_6^2)}{v_3^2 + v_4^2}\frac{\partial}{\partial v_6},$$

$$
\begin{aligned}
X_2 &= v_3\frac{\partial}{\partial v_5} + v_4\frac{\partial}{\partial v_6}, \qquad X_3 = -v_4\frac{\partial}{\partial v_5} + v_3\frac{\partial}{\partial v_6},\\
X_4 &= -v_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_4} - 2v_5\frac{\partial}{\partial v_5} - 2v_6\frac{\partial}{\partial v_6}, \qquad X_5 = v_4\frac{\partial}{\partial v_3} - v_3\frac{\partial}{\partial v_4}\\
&\quad +2v_6\frac{\partial}{\partial v_5} - 2v_5\frac{\partial}{\partial v_6},\\
X_6 &= -v_4\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_2} - v_6\frac{\partial}{\partial v_3} + v_5\frac{\partial}{\partial v_4} - \frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2 - v_6^2)}{(v_3^2+v_4^2)}\frac{\partial}{\partial v_5}\\
&\quad +\frac{3}{2}\frac{2v_4v_5v_6 + v_3(v_5^2 - v_6^2)}{(v_3^2+v_4^2)}\frac{\partial}{\partial v_6}.
\end{aligned}
$$

These vector fields satisfy the following commutation relations

$$
\begin{array}{lllll}
[X_1, X_2] = X_4\,, & [X_1, X_3] = X_5\,, & [X_1, X_4] = X_1\,, & [X_1, X_5] = X_6\,, & [X_1, X_6] = 0\,,\\
 & [X_2, X_3] = 0\,, & [X_2, X_4] = -X_2\,, & [X_2, X_5] = -X_3\,, & [X_2, X_6] = -X_5\,,\\
 & & [X_3, X_4] = -X_3\,, & [X_3, X_5] = X_2\,, & [X_3, X_6] = X_4\,,\\
 & & & [X_4, X_5] = 0\,, & [X_4, X_6] = -X_6\,,\\
 & & & & [X_5, X_6] = X_1\,,
\end{array}
$$

Then, $X_1, \ldots, X_k$ give rise to a Lie algebra of vector fields V_{sc} that is isomorphic to $\mathbb{C} \otimes \mathfrak{sl}_2$ as a real vector space. Indeed, $\langle X_1, X_2, X_4\rangle \simeq \mathfrak{sl}(2, \mathbb{R}) \simeq \langle X_3, X_4, X_6\rangle$. Additionally, $\mathbb{C}\otimes\mathfrak{sl}_2$ decomposes as $\langle X_1, X_4, X_2\rangle \oplus \langle X_6, X_5, X_3\rangle$. Then V_{sc} is graded as $V_{sc} = E_{-1} \oplus E_0 \oplus E_1$, where $E_{-1} = \langle X_6, X_1\rangle$, $E_0 = \langle X_4, X_5\rangle$, and $E_1 = \langle X_3, X_2\rangle$, with $[E_i, E_j] = E_{i+j}$, where the sum is in the additive group $\{-1, 0, 1\}$. A long calculation shows that $X_1 \wedge \cdots \wedge X_6 \neq 0$ almost everywhere. The latter linear independence and the fact that $X_1, \ldots, X_6$ span a Lie algebra of vector fields spanning $\mathrm{T}\mathcal{O}$ explains why it is said that (5.1) can be related to a locally automorphic Lie system (cf. Gràcia et al. 2019).

Meanwhile, the Lie algebra of Lie symmetries of the system (5.1) related to the Lie algebra V_{sc} reads

$$
\begin{aligned}
2Y_1 &= \left(v_1^2 - v_2^2\right)\frac{\partial}{\partial v_1} + 2v_1v_2\frac{\partial}{\partial v_2} + 2(v_1v_3 - v_2v_4)\frac{\partial}{\partial v_3} + 2(v_3v_2 + v_1v_4)\frac{\partial}{\partial v_4}\\
&\quad +2\left(v_3^2 + v_1v_5 - v_4^2 - v_2v_6\right)\frac{\partial}{\partial v_5} + 2(v_5v_2 + 2v_3v_4 + v_2v_6)\frac{\partial}{\partial v_6},\\
Y_2 &= \frac{\partial}{\partial v_1}, \qquad Y_3 = \frac{\partial}{\partial v_2},\\
Y_4 &= -v_1\frac{\partial}{\partial v_1} - v_2\frac{\partial}{\partial v_2} - v_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_4} - v_5\frac{\partial}{\partial v_5} - v_6\frac{\partial}{\partial v_6},\\
Y_5 &= v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2} + v_4\frac{\partial}{\partial v_3} - v_3\frac{\partial}{\partial v_4} + v_6\frac{\partial}{\partial v_5} - v_5\frac{\partial}{\partial v_6}.\\
2Y_6 &= -2v_1v_2\frac{\partial}{\partial v_1} + \left(v_1^2 - v_2^2\right)\frac{\partial}{\partial v_2} - 2(v_2v_3 + v_1v_4)\frac{\partial}{\partial v_3} + 2(v_1v_3 - v_2v_4)\frac{\partial}{\partial v_4}\\
&\quad -2(2v_3v_4 + v_2v_5 + v_1v_6)\frac{\partial}{\partial v_5} + 2\left(v_3^2 - v_4^2 + v_1v_5 - v_2v_6\right)\frac{\partial}{\partial v_6}.
\end{aligned}
$$

In other words, $[X_i, Y_j] = 0$ for every $i, j = 1, \ldots, 6$. The commutation relations for the vector fields $Y_1, \ldots, Y_6$ are

$$
\begin{array}{lllll}
[Y_1, Y_2] = Y_4\,, & [Y_1, Y_3] = Y_5\,, & [Y_1, Y_4] = Y_1\,, & [Y_1, Y_5] = Y_6\,, & [Y_1, Y_6] = 0\,,\\
 & [Y_2, Y_3] = 0\,, & [Y_2, Y_4] = -Y_2\,, & [Y_2, Y_5] = -Y_3\,, & [Y_2, Y_6] = -Y_5\,,\\
 & & [Y_3, Y_4] = -Y_3\,, & [Y_3, Y_5] = Y_2\,, & [Y_3, Y_6] = Y_4\,,\\
 & & & [Y_4, Y_5] = 0\,, & [Y_4, Y_6] = -Y_6\,,\\
 & & & & [Y_5, Y_6] = Y_1\,.
\end{array}
$$

Note that $Y_1, \ldots, Y_6$ admit identical structure constants as $X_1, \ldots, X_6$. One can choose one-forms $\eta^1, \ldots, \eta^6$ to be the dual to $Y_1, \ldots, Y_6$. The existence of these dual forms is ensured by the condition $Y_1 \wedge \cdots \wedge Y_6 \neq 0$ and the fact that $Y_1, \ldots, Y_6$ span $\mathrm{T}\mathcal{O}$. These dual one-forms remain invariant relative to the Lie derivatives with respect to the vector fields $X_1 \ldots, X_6$, i.e. $\mathscr{L}_{X_i}\eta^j = 0$ for $i, j = 1, \ldots, 6$.

Moreover, the differential forms $\mathrm{d}\eta^1, \ldots, \mathrm{d}\eta^6$, or their linear combinations, are closed differential forms that are invariant relative to the Lie derivatives along $X_1, \ldots, X_6$. These properties make them Hamiltonian vector fields relative to the presymplectic forms $\mathrm{d}\eta^1, \ldots, \mathrm{d}\eta^6$.

The appropriate linear combinations of these forms yield a set of presymplectic forms with the zero intersection of their kernels, resulting in $X_1, \ldots, X_6$ being $\boldsymbol{\omega}$-Hamiltonian vector fields.

In particular,

$$
\begin{aligned}
\mathrm{d}\eta^1 &= -\eta^5 \wedge \eta^6 - \eta^1 \wedge \eta^4\,, & \mathrm{d}\eta^2 &= -\eta^3 \wedge \eta^5 - \eta^4 \wedge \eta^2\,,\\
\mathrm{d}\eta^3 &= -\eta^4 \wedge \eta^3 - \eta^5 \wedge \eta^2\,, & \mathrm{d}\eta^4 &= -\eta^1 \wedge \eta^2 - \eta^3 \wedge \eta^6\,,\\
\mathrm{d}\eta^5 &= -\eta^1 \wedge \eta^3 - \eta^6 \wedge \eta^2\,, & \mathrm{d}\eta^6 &= -\eta^1 \wedge \eta^5 - \eta^6 \wedge \eta^4\,.
\end{aligned}
$$

Every vector field in $\langle X_1, \ldots, X_6\rangle$ becomes an $\boldsymbol{\omega}$-Hamiltonian vector field relative to the two-polysymplectic form $\mathrm{d}\eta^1 \otimes e_1 + \mathrm{d}\eta^2 \otimes e_2$. The same applies to $\mathrm{d}\eta^5 \otimes e_1 + \mathrm{d}\eta^6 \otimes e_2$, and many other two-polysymplectic forms. This also extends to three-polysymplectic forms such as $\mathrm{d}\eta^1 \otimes e_1 + \mathrm{d}\eta^2 \otimes e_2 + \mathrm{d}\eta^3 \otimes e_3$, provided that the kernels of their presymplectic components have zero intersection.

Let us focus on the three-polysymplectic form defined by

$$
\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 + \omega^3 \otimes e_3 = \mathrm{d}\eta^1 \otimes e_1 + \mathrm{d}\eta^2 \otimes e_2 + \mathrm{d}\eta^4 \otimes e_3.
$$

A two-polysymplectic Marsden–Weinstein reduction can be performed by taking, for instance, the $\boldsymbol{\omega}$-Hamiltonian vector field X_1 and the Lie symmetry X_6, which satisfies that $[X_1, X_6] = 0$. Then, a two-polysymplectic momentum map $\mathbf{J}^\Phi : \mathcal{O} \to (\mathbb{R}^*)^3$ is given by

$$
\iota_{X_6}\mathrm{d}\mathbf{J}^\Phi = \iota_{X_6}\omega^1 \otimes e_1 + \iota_{X_6}\omega^2 \otimes e_2 + \iota_{X_6}\omega^3 \otimes e_3 = \mathrm{d}J_1 \otimes e_1 + \mathrm{d}J_2 \otimes e_2 + \mathrm{d}J_3 \otimes e_3.
$$

It is a matter of a long calculation to prove that $\mathrm{d}J_1 \wedge \mathrm{d}J_2 \wedge \mathrm{d}J_3 \neq 0$ based on the fact that $\partial(J_1, J_2, J_2)/\partial(v_1, v_2, v_3) \neq 0$ almost everywhere. Therefore, $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ has dimension three. Moreover, due to $\iota_{X_6}\mathrm{d}\mathbf{J}^\Phi = 0$, the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/X_6$ is two-dimensional.

Note that the vector field X_1 is tangent to the level set $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ since

$$\iota_{X_1}\iota_{X_6}\mathrm{d}\eta^\alpha = X_1 J_\alpha = 0, \qquad \alpha = 1, 2, 3.$$

Therefore, by Theorem 3.4 the vector field X_1 reduces onto the manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/X_6$.

Then, after some calculations, we obtain that condition (3.1) is fulfilled. To verify condition (3.2), which has the form

$$\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p) = \bigcap_{\alpha=1}^{k} \left(\ker \omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha} p)\right) \cap \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})),$$

one can note that

$$\mathrm{T}_p(G_{\boldsymbol{\mu}} p) = \langle X_6 \rangle \subset \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) \subset \mathrm{T}_p P.$$

Moreover, we have

$$\ker \omega^1 = \langle Y_2, Y_3 \rangle, \quad \ker \omega^2 = \langle Y_1, Y_6 \rangle, \quad \ker \omega^3 = \langle Y_4, Y_5 \rangle.$$

In turn, this amounts to obtaining three determinants, each being nonzero, implying that no element of $\ker \omega^1, \ker \omega^2, \ker \omega^3$ belongs to $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. In particular, at a generic point,

$$\det \begin{pmatrix} Y_2 J_2 & Y_2 J_3 \\ Y_3 J_2 & Y_3 J_3 \end{pmatrix} \neq 0, \qquad \det \begin{pmatrix} Y_1 J_1 & Y_1 J_3 \\ Y_6 J_1 & Y_6 J_3 \end{pmatrix} \neq 0, \qquad \det \begin{pmatrix} Y_4 J_1 & Y_4 J_2 \\ Y_5 J_1 & Y_5 J_2 \end{pmatrix} \neq 0.$$

Finally, the condition (3.2) is satisfied, namely

$$\left(\langle Y_2, Y_3 \rangle + \langle X_6 \rangle\right) \cap \left(\langle Y_1, Y_6 \rangle + \langle X_6 \rangle\right) \cap \left(\langle Y_4, Y_5 \rangle + \langle X_6 \rangle\right) \cap \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \langle X_6 \rangle.$$

Hence, Theorem 3.3 can be applied.

5.2 The *k*-Polysymplectic Manifold Given by the Product of *k* Symplectic Manifolds

This section presents an illustrative example of the k-polysymplectic Marsden–Weinstein reduction of a product of k symplectic manifolds (see Sect. 3.3). This example shows different types of systems of differential equations that can be understood as Hamiltonian systems relative to a k-polysymplectic manifold and describes their reductions. In particular, the so-called diagonal prolongations of Lie–Hamilton systems, which appear also in the multidimensional generalisations of some integral systems, like in the case of the Winternitz–Smorodinsky oscillator on $\mathrm{T}^*\mathbb{R}$ (see de Lucas and Sardón 2020), can be considered as Hamiltonian systems relative to a

k-polysymplectic manifold. One can also consider higher-dimensional Winternitz–Smorodinsky oscillators.

Let us provide some new details to the formalism in Sect. 3.3. Define $P = P_1 \times \cdots \times P_k$ for some k symplectic manifolds $(P_\alpha, \omega^\alpha)$, where $\alpha = 1, \ldots, k$. This gives rise to a k-polysymplectic manifold $(P, \mathrm{pr}_\alpha^* \omega^\alpha \otimes e_\alpha)$. Assume that each Lie group action $\Phi^\alpha : G_\alpha \times P_\alpha \to P_\alpha$ admits a symplectic momentum map $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for $\alpha = 1, \ldots, k$. Define the Lie group action of $G = G_1 \times \ldots \times G_k$ on P as (3.6). If one defines $\mathfrak{g} = \bigoplus_{\alpha=1}^k \mathfrak{g}_\alpha$, then there exists a k-polysymplectic momentum map

$$\mathbf{J} : (x_1, \ldots, x_k) \in P \longmapsto (0, \ldots, \mathbf{J}^\alpha, \ldots, 0) \otimes e_\alpha = \begin{pmatrix} \mathbf{J}^1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}^k \end{pmatrix} \in \mathfrak{g}^{*k},$$

where there is a summation over α and we assume $\mathbf{J}^\alpha(x_1, \ldots, x_k) = \mathbf{J}^{\Phi^\alpha}(x_\alpha)$ for $\alpha = 1, \ldots, k$ and the matrix array is a practical representation of the image of $\mathbf{J}$. Note that $\boldsymbol{\mu} = (0, \ldots, \mu^\alpha, \ldots, 0) \otimes e_\alpha \in \mathfrak{g}^{*k}$ is a weak regular value of $\mathbf{J}$ if and only if each $\mu^\alpha \in \mathfrak{g}_\alpha^*$ is a weak regular point of its corresponding $\mathbf{J}^{\Phi^\alpha}$. Assume that some $G_{\boldsymbol{\mu}}^\Delta$ acts in a quotientable manner on the associated level $\mathbf{J}^{-1}(\boldsymbol{\mu})$. This happens if and only if every $G_{\mu^\alpha}^{\Delta^\alpha}$ acts on a quotientable manner on each $\mathbf{J}^{\Phi^\alpha -1}(\mu^\alpha)$ for $\alpha = 1, \ldots, k$.

We already showed that the conditions (3.1) and (3.2) are satisfied. By Theorem 3.3, these equations guarantee that, on the reduced manifold $\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$, there exists a uniquely induced k-polysymplectic manifold,

$$\left(\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} \simeq \mathbf{J}^{\Phi^1 -1}(\mu^1)/G_{1\boldsymbol{\mu}}^{\Delta^1} \times \cdots \times \mathbf{J}^{\Phi^k -1}(\mu^k)/G_{k\boldsymbol{\mu}}^{\Delta^k} \;\; \boldsymbol{\omega}_{\boldsymbol{\mu}} = \sum_{\alpha=1}^k \omega^{\mu^\alpha} \otimes e_\alpha \right)$$

for some reduced presymplectic forms $\omega_{\mu^1}, \ldots, \omega_{\mu^k}$.

Next, let us consider a vector field X on P that is $\boldsymbol{\omega}$-Hamiltonian and G-invariant. By Theorem 3.4, the vector field X can be written in the following way

$$X = \sum_{\alpha=1}^k X_\alpha,$$

where each X_α can be considered as a vector field on P_α that is tangent to $\mathbf{J}^{\Phi^\alpha -1}(\mu^\alpha)$ for $\alpha = 1, \ldots, k$. Recall that $\iota_{X_\alpha} \omega^\beta = \delta_\alpha^\beta \mathrm{d}h^\alpha$ for $\alpha, \beta = 1, \ldots, k$. Moreover, this frequently happens in diagonal prolongations of Lie–Hamilton systems, where we have a vector field $X^{[m]}$ defined on a manifold of the form N^m that can be considered as a copy of a Hamiltonian system on each N relative to a symplectic manifold on that N (cf. de Lucas and Sardón 2020). Then,

$$\mathrm{d}\boldsymbol{h} = \sum_{\alpha=1}^k \mathrm{d}h^\alpha \otimes e_\alpha = \sum_{\alpha=1}^k \iota_X \omega^\alpha \otimes e_\alpha.$$

Next, $\boldsymbol{h}_\xi = \boldsymbol{h} - \langle \mathbf{J} - \boldsymbol{\mu}_e, \xi \rangle$ for $\xi \in \mathfrak{g}$, and Theorem 4.2 yields that $z_e = (z_{1e}, \ldots, z_{ke}) \in P$ is a k-polysymplectic relative equilibrium point if and only if each $z_{\alpha e}$ is a symplectic relative equilibrium point of a Hamiltonian vector field X_α on the symplectic manifold $(P_\alpha, \omega^\alpha)$ relative to some $\xi_\alpha \in \mathfrak{g}_\alpha$. Then, a k-polysymplectic relative equilibrium point z_e is formally stable if there exists a series of supplementary spaces $\mathcal{S}^\alpha$ to $\mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) \oplus (\ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}(\mathbf{J}^{-1}(\boldsymbol{\mu}_e)))$ in $\mathrm{T}_{z_e}\mathbf{J}^{-1}(\boldsymbol{\mu}_e)$, with $\alpha = 1, \ldots, k$, such that

$$\left(\delta^2 h^\alpha_\xi\right)_{z_e}(v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \mathcal{S}^\alpha \setminus \{0\}, \qquad \alpha = 1, \ldots, k$$

and $\mathcal{S}^1 + \ldots + \mathcal{S}^k + \mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) = \mathrm{T}_{z_e}(\mathbf{J}^{-1}(\boldsymbol{\mu}_e))$.

5.2.1 Product of Oscillators

Let us detail a practical application of the formalism above. Consider the product of k isotropic three-dimensional oscillators given by the equations

$$\frac{\mathrm{d}^2 x^i_\alpha}{\mathrm{d}t^2} = -b^2_\alpha x^i_\alpha, \qquad \alpha = 1, \ldots, k, \qquad i = 1, 2, 3,$$

where the $b_\alpha > 0$, with $\alpha = 1, \ldots, k$, are a series of constants. The above system of second-order differential equations can be written as a first-order system of differential equations

$$\begin{cases} \frac{\mathrm{d}x^i_\alpha}{\mathrm{d}t} = p^i_\alpha, \\ \frac{\mathrm{d}p^i_\alpha}{\mathrm{d}t} = -b^2_\alpha x^i_\alpha, \end{cases} \qquad \alpha = 1, \ldots, k, \qquad i = 1, 2, 3, \tag{5.2}$$

on the product manifold $P = (\mathrm{T}^*\mathbb{R}^3)^k$. The α-th factor $\mathrm{T}^*\mathbb{R}^3$ in P is a symplectic manifold equipped with the symplectic form

$$\omega^\alpha = \sum_{i=1}^3 \mathrm{d}x^i_\alpha \wedge \mathrm{d}p^i_\alpha,$$

where we stress that there is no sum over the index α. Then, P is a k-polysymplectic manifold when endowed with the $\mathbb{R}^k$-valued form $\boldsymbol{\omega} = \sum_{\alpha=1}^k \omega^\alpha \otimes e_\alpha$, where $\omega^1, \ldots, \omega^k$ are considered as pulled back to P in the natural way. Moreover, (5.2) describes the integral curves of the vector field

$$X_{\boldsymbol{h}} = \sum_{\alpha=1}^k \sum_{i=1}^3 \left(p^i_\alpha \frac{\partial}{\partial x^i_\alpha} - b^2_\alpha x^i_\alpha \frac{\partial}{\partial p^i_\alpha} \right),$$

which is $\boldsymbol{\omega}$-Hamiltonian admitting an $\boldsymbol{\omega}$-Hamiltonian function

$$\boldsymbol{h} = \frac{1}{2}\sum_{\alpha=1}^{k}\left(p_\alpha^2 + b_\alpha^2 x_\alpha^2\right)\otimes e_\alpha, \qquad p_\alpha^2 = \sum_{i=1}^{3}(p_\alpha^i)^2, \qquad x_\alpha^2 = \sum_{i=1}^{3}(x_\alpha^i)^2.$$

Let us consider a Lie group action $\Phi^\alpha : \mathrm{SO}(3)\times(\mathrm{T}^*\mathbb{R}^3)_\alpha \to (\mathrm{T}^*\mathbb{R}^3)_\alpha$, where each Φ^α is the lift of the natural Lie group action $\Psi : \mathrm{SO}(3)\times\mathbb{R}^3 \to \mathbb{R}^3$ induced by rotations on $\mathbb{R}^3$ to the α-th copy of $\mathrm{T}^*\mathbb{R}^3$ in P. Then, the resulting Lie group action Φ on $(\mathrm{T}^*\mathbb{R}^3)^k$ given by (3.6) reads

$$\Phi : \mathrm{SO}(3)^k \times (\mathrm{T}^*\mathbb{R}^3)^k \longrightarrow (\mathrm{T}^*\mathbb{R}^3)^k.$$

The Lie algebra of fundamental vector fields of Φ is spanned by the basis of vector fields on P of the form

$$\begin{aligned}
\xi^1_{\alpha P} &= \left(x_\alpha^1\frac{\partial}{\partial x_\alpha^2} - x_\alpha^2\frac{\partial}{\partial x_\alpha^1} + p_\alpha^1\frac{\partial}{\partial p_\alpha^2} - p_\alpha^2\frac{\partial}{\partial p_\alpha^1}\right),\\
\xi^2_{\alpha P} &= \left(x_\alpha^2\frac{\partial}{\partial x_\alpha^3} - x_\alpha^3\frac{\partial}{\partial x_\alpha^2} + p_\alpha^2\frac{\partial}{\partial p_\alpha^3} - p_\alpha^3\frac{\partial}{\partial p_\alpha^2}\right),\\
\xi^3_{\alpha P} &= \left(x_\alpha^3\frac{\partial}{\partial x_\alpha^1} - x_\alpha^1\frac{\partial}{\partial x_\alpha^3} + p_\alpha^3\frac{\partial}{\partial p_\alpha^1} - p_\alpha^1\frac{\partial}{\partial p_\alpha^3}\right),
\end{aligned}$$

with $\alpha = 1,\ldots,k$. These vector fields are Lie symmetries of $\boldsymbol{\omega}$ and $\boldsymbol{h}$. Moreover, a k-polysymplectic momentum map associated with Φ is given by $\mathbf{J} : (\mathrm{T}^*\mathbb{R}^3)^k \to [(\mathfrak{so}_3^k)^*]^k$ such that

$$\mathbf{J}(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k) = (0,0,0;\ldots;J_\alpha^1,J_\alpha^2,J_\alpha^3;\ldots;0,0,0)\otimes e_\alpha$$

where $\boldsymbol{q}_\alpha = (x_\alpha^1,x_\alpha^2,x_\alpha^3,p_\alpha^1,p_\alpha^2,p_\alpha^3)\in\mathrm{T}^*\mathbb{R}^3$ for $\alpha = 1,\ldots,k$, while

$$(J_\alpha^1,J_\alpha^2,J_\alpha^3) = (x_\alpha^1 p_\alpha^2 - x_\alpha^2 p_\alpha^1, x_\alpha^2 p_\alpha^3 - x_\alpha^3 p_\alpha^2, x_\alpha^3 p_\alpha^1 - x_\alpha^1 p_\alpha^3),$$

and $\alpha = 1,\ldots,k$. Note that the elements of $\mathfrak{so}_3^*$ are represented by the coordinates given in an appropriate basis. The function $x_\alpha^1 p_\alpha^2 - x_\alpha^2 p_\alpha^1$ is the angular momentum, $p_{\alpha\varphi}$, of the α-th particle in the corresponding spherical coordinates $\{r_\alpha,\theta_\alpha,\varphi_\alpha\}$. Meanwhile, $L_\alpha^2 = (J_\alpha^1)^2 + (J_\alpha^2)^2 + (J_\alpha^3)^2$ is the square of the total angular momentum of the α-th particle. Both quantities are conserved by the evolution of $X_{\boldsymbol{h}}$.

The momentum map $\mathbf{J}$ is $(\mathrm{Ad}^*)^k$-equivariant. Recall that $\mathbf{J} = (0,\ldots,\mathbf{J}^\alpha,\ldots,0)\otimes e_\alpha$. Then, $\boldsymbol{\mu} = (0,0,0;\ldots;J_\alpha^1,J_\alpha^2,J_\alpha^3;\ldots;0,0,0)\otimes e_\alpha$ is weakly regular value of $\mathbf{J}$ if and only if each triple $\mu^\alpha = (J_\alpha^1,J_\alpha^2,J_\alpha^3)\in\mathfrak{so}_3^*$ is a weakly regular value of $\mathbf{J}^{\Phi^\alpha}$, where $\alpha = 1,\ldots,k$. Let us fix some weakly regular $\boldsymbol{\mu}$. Then,

$$\mathrm{T}_{\boldsymbol{q}}(\mathbf{J}^{-1}(\boldsymbol{\mu})) = \bigoplus_{\alpha=1}^{k}\mathrm{T}_{\boldsymbol{q}_\alpha}(\mathbf{J}_\alpha^{\Phi^\alpha -1}(\mu^\alpha)), \qquad \forall\mathbf{q} = (\mathbf{q}_1,\ldots,\mathbf{q}_k)\in P.$$

Moreover,

$$\xi^i_{\alpha P} J^j_\beta = -\delta_{\alpha\beta}\epsilon_{ijk} J^k_\beta, \qquad i, j = 1, 2, 3, \qquad \alpha, \beta = 1, \ldots, k.$$

The isotropy subgroup of Φ at $\boldsymbol{\mu}$ is given by the Cartesian product of all the isotropy subgroups corresponding to each μ^α relative to Φ^α and $\alpha = 1, \ldots, k$. To obtain G_{μ^α}, one may verify when $\sum_{i=1}^3 \lambda_i(\xi^i_{\alpha P})$ belongs to $\mathrm{T}_{\boldsymbol{q}_\alpha}(\mathbf{J}^{\Phi^\alpha -1}(\mu_\alpha))$, namely $\sum_{i=1}^3 \lambda_i(\xi^i_P)_\alpha J^j_\alpha = 0$ for $j = 1, 2, 3$ (with no summation over α), which occurs if and only if

$$\begin{pmatrix} 0 & -J^3_\alpha & J^2_\alpha \\ J^3_\alpha & 0 & -J^1_\alpha \\ -J^2_\alpha & J^1_\alpha & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix of coefficients has rank two for $L^2_\alpha \neq 0$. Moreover, $\mathbf{J}^{\Phi^\alpha}$ has a regular value at μ_α when $L^2_\alpha \neq 0$ for every $\alpha = 1, \ldots, k$. Let us restrict to that case. Each isotropy subgroup G_{μ^α} has always dimension one. Hence, the reduced manifold $\mathbf{J}^{\Phi -1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ has dimension $6k - 3k - k = 2k$. The conditions for the k-polysymplectic Marsden–Weinstein reduction results, as already commented, from the ones for the symplectic reduction on each component, which are satisfied. Hence, the k-polysymplectic Marsden–Weinstein reduction exists.

Note that $\boldsymbol{h} = \frac{1}{2}\sum_{\alpha=1}^k (p^2_{\alpha r} + p^2_{\alpha\varphi}/(r^2_\alpha \sin^2\theta_\alpha) + p^2_{\alpha\theta}/r^2_\alpha + b^2_\alpha r^2_\alpha) \otimes e_\alpha$ and $\omega^\alpha = \mathrm{d}r_\alpha \wedge \mathrm{d}p_{\alpha r} + \mathrm{d}\theta_\alpha \wedge \mathrm{d}p_{\alpha\theta} + \mathrm{d}\varphi_\alpha \wedge \mathrm{d}p_{\alpha\varphi}$, for each $\alpha = 1, \ldots, k$, in spherical coordinates for the k component manifolds of $(\mathrm{T}^*\mathbb{R}^3)^k$. Then, the differential equations for the integral curves of $X_{\boldsymbol{h}}$ read

$$\frac{\mathrm{d}p_{\alpha r}}{\mathrm{d}t} = \frac{p^2_{\alpha\varphi}}{r^3_\alpha \sin^2\theta_\alpha} + \frac{p^2_{\alpha\theta}}{r^3_\alpha} - b^2_\alpha r_\alpha, \qquad \frac{\mathrm{d}p_{\alpha\varphi}}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}p_{\alpha\theta}}{\mathrm{d}t} = \frac{p^2_{\alpha\varphi}\cos\theta_\alpha}{r^2_\alpha \sin^3\theta_\alpha},$$
$$\frac{\mathrm{d}r_\alpha}{\mathrm{d}t} = p_{\alpha r}, \qquad \frac{\mathrm{d}\theta_\alpha}{\mathrm{d}t} = \frac{p_{\alpha\theta}}{r^2_\alpha}, \qquad \frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}t} = \frac{p_{\alpha\varphi}}{r^2_\alpha \sin^2\theta_\alpha}.$$

k-Polysymplectic relative equilibrium points are given by those points for which the vector field $X_{\boldsymbol{h}}$ on $(\mathrm{T}^*\mathbb{R}^3)^k$ corresponding to the dynamics is proportional to one of the fundamental vector fields of Φ. In particular, let us take $z_e \in P$ such that $z_e = (r_\alpha, \theta_\alpha = \frac{\pi}{2}, \varphi_\alpha, p_{\alpha r} = 0, p_{\alpha\theta} = 0, p_{\alpha\varphi})$ and $L_\alpha = b_\alpha r^2_\alpha = p_{\alpha\varphi}$ for every $\alpha = 1, \ldots, k$ on analysed points. Then, the $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ at such points is

$$X_{\boldsymbol{h}} = \sum_{\alpha=1}^k \frac{p_{\alpha\varphi}}{r^2_\alpha}\frac{\partial}{\partial\varphi_\alpha}.$$

This implies that $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. Let us demonstrate this by applying Theorem 4.2. This theorem ensures that z_e is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if there exists $\xi \in \mathfrak{so}^k_3$

such that $\boldsymbol{h}_\xi = \boldsymbol{h} - \langle \mathbf{J}^\Phi - \boldsymbol{\mu}_e, \xi \rangle$ has a critical point at z_e. Indeed, for

$$\xi = (p_{1\varphi}/r_1^2, 0, 0; \ldots; p_{k\varphi}/r_k^2, 0, 0) \in \mathfrak{so}_3^k,$$

the $\mathbb{R}^k$-valued function

$$\boldsymbol{h}_\xi = \boldsymbol{h} - \langle \mathbf{J}^\Phi - \boldsymbol{\mu}_e, \xi \rangle = \sum_{\alpha=1}^{k} (h^\alpha - \langle (0, \ldots, \mathbf{J}^{\Phi_\alpha} - (L_\alpha, 0, 0), \ldots, 0), \xi \rangle) \otimes e_\alpha,$$

has a critical point at z_e. Therefore, z_e is a k-polysymplectic relative equilibrium point.

By Theorem 3.3, the reduced manifolds is $(\mathrm{T}^*\mathbb{R})^k$ with coordinates $\{r_\alpha, p_{\alpha r}\}$ for $\alpha = 1, \ldots, k$. The reduced k-polysymplectic form on the reduced manifold is given by

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \sum_{\alpha=1}^{k} \mathrm{d}r_\alpha \wedge \mathrm{d}p_{\alpha r} \otimes e_\alpha,$$

and the reduced $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian reads

$$\boldsymbol{f}_{\boldsymbol{\mu}} = \frac{1}{2} \sum_{\alpha=1}^{k} \left(p_{\alpha r}^2 + \frac{L_\alpha^2}{r_\alpha^2} + b_\alpha^2 r_\alpha^2 \right) \otimes e_\alpha.$$

Furthermore, one has that

$$\frac{\mathrm{d}p_{\alpha r}}{\mathrm{d}t} = -b_\alpha^2 r_\alpha + \frac{L_\alpha^2}{r_\alpha^3}, \qquad \frac{\mathrm{d}r_\alpha}{\mathrm{d}t} = p_{\alpha r}, \qquad \alpha = 1, \ldots, k.$$

Thus, the equilibrium points of $X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$ have $p_{\alpha r} = 0$ and

$$-b_\alpha^2 r_\alpha + \frac{L_\alpha^2}{r_\alpha^3} = 0$$

for $\alpha = 1, \ldots, k$. Note that this point is the projection of a k-polysymplectic relative equilibrium point $z_e \in P$.

The Hessian of the functions $f_{\boldsymbol{\mu}}^\alpha$ is positive-definite in a supplementary to the kernel of $\boldsymbol{\omega}_{\mu^\alpha}$ at the equilibrium point. Indeed,

$$\mathrm{Hess}(f_{\boldsymbol{\mu}}^\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 4b_\alpha^2 \end{pmatrix}.$$

Moreover, the function $\sum_{\alpha=1}^{k} f_{\boldsymbol{\mu}}^\alpha$ has a positive-definite Hessian and the equilibrium point becomes a strict minimum. This means that the reduced k-polysymplectic relative equilibrium point is stable. In the original manifold, the orbits around k-polysymplectic relative equilibrium points remain in the anti-image in $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ of an open neighbourhood of the projection of the k-polysymplectic relative equilibrium points.

5.3 *k*-Polysymplectic Affine Lie Systems

Let us apply our techniques to a family of affine inhomogeneous systems of first-order differential equations. It is worth noting that all such systems are Lie systems (Cariñena and de Lucas 2011). We will hereafter call such differential equations *affine Lie systems*. Many such systems appear in control theory and other relevant disciplines (Cariñena and Ramos 2003). In particular, we are here concerned with affine Lie systems admitting a Lie algebra of Hamiltonian vector fields relative to a k-polysymplectic form. We call them *k-polysymplectic affine Lie systems*.

Although our techniques could be extended to other affine Lie systems, let us restrict ourselves to the particular case

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \\ b_4(t) \\ b_5(t) \end{pmatrix} + b_6(t)\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \tag{5.3}$$

where $b_1(t), \ldots, b_6(t)$ are arbitrary t-dependent functions. The above system is the system of differential equations describing the integral curves of the t-dependent vector field

$$X = \sum_{\alpha=1}^{6} b_\alpha(t) X_\alpha,$$

where

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial x_4}, \quad X_5 = \frac{\partial}{\partial x_5},$$
$$X_6 = x_5\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_5}.$$

These vector fields span a six-dimensional Lie algebra of vector fields V, with the following non-vanishing commutation relations

$$[X_3, X_6] = -X_5, \qquad [X_5, X_6] = X_3.$$

Consider the case where $b_1(t), \ldots, b_6(t)$ are constants, denoted as $c_1, \ldots, c_6 \in \mathbb{R}$, respectively. Since the vector fields $X_1 \wedge \cdots \wedge X_6 = 0$, the methods presented in Sect. 5.1 for describing k-polysymplectic forms compatible with Lie systems do not apply to (5.3). Nevertheless, there exists a two-polysymplectic form on $\mathbb{R}^5$ given by

$$\boldsymbol{\omega} = (\mathrm{d}x_3 \wedge \mathrm{d}x_5 + \mathrm{d}x_4 \wedge \mathrm{d}x_1) \otimes e_1 + (\mathrm{d}x_3 \wedge \mathrm{d}x_5 + \mathrm{d}x_4 \wedge \mathrm{d}x_2) \otimes e_2$$

turning all the vector fields of V into $\boldsymbol{\omega}$-Hamiltonian vector fields. Indeed, $\boldsymbol{\omega}$-Hamiltonian functions for $X_1, \ldots, X_6$ have the form

$$
\begin{aligned}
&\boldsymbol{h}_1 = -x_4 \otimes e_1\,, && \boldsymbol{h}_2 = -x_4 \otimes e_2\,, && \boldsymbol{h}_3 = x_5 \otimes e_1 + x_5 \otimes e_2\,,\\
&\boldsymbol{h}_4 = x_1 \otimes e_1 + x_2 \otimes e_2\,, && \boldsymbol{h}_5 = -x_3 \otimes e_1 - x_3 \otimes e_2\,, && \boldsymbol{h}_6 = \frac{1}{2}(x_3^2 + x_5^2) \otimes e_1 + \frac{1}{2}(x_3^2 + x_5^2) \otimes e_2\,.
\end{aligned}
$$

The flow of the vector field X_4 gives rise to a two-polysymplectic Lie group action $\Phi : \mathbb{R} \times \mathbb{R}^5 \to \mathbb{R}^5$. Moreover, X_4, which spans the space of fundamental vector fields of Φ, is a Lie symmetry of system (5.3). Then, a two-polysymplectic momentum map associated with Φ reads

$$
\mathbf{J}^{\Phi} : (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mapsto (x_1, x_2) = \boldsymbol{\mu} \in \mathbb{R}^{*2}.
$$

Note that $\boldsymbol{\mu} \in \mathbb{R}^{*2}$ is a regular value of $\mathbf{J}^{\Phi}$, and $\mathbf{J}^{\Phi}$ is Ad^{*2}-equivariant two-polysymplectic momentum map. It can be proved that the example satisfies the conditions (3.1) and (3.2). Hence, Theorem 3.3 can be applied. The vector field X_4 is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and $\mathrm{T}_x(G_{\boldsymbol{\mu}} x) = \langle \frac{\partial}{\partial x_4} \rangle$ for $x \in \mathbb{R}^5$. Therefore, $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathbb{R}$ is a two-dimensional manifold and the variables $\{x_3, x_5\}$ can be considered in a natural manner as variables on $P_{\boldsymbol{\mu}}$. The reduced two-polysymplectic form reads

$$
\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega^1_{\boldsymbol{\mu}} \otimes e_1 + \omega^2_{\boldsymbol{\mu}} \otimes e_2 = \mathrm{d}x_3 \wedge \mathrm{d}x_5 \otimes e_1 + \mathrm{d}x_3 \wedge \mathrm{d}x_5 \otimes e_2.
$$

To apply Theorem 3.4, the affine Lie system must be tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, which can be ensured by assuming that its associated $\boldsymbol{\omega}$-Hamiltonian function has to be invariant relative to X_4. These conditions are satisfied by imposing $c_1 = c_2 = 0$. The resulting vector field, $X_{\boldsymbol{\omega}} = c_3 X_3 + c_4 X_4 + c_5 X_5 + c_6 X_6$, projects onto $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathbb{R}$ giving rise to an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field of the form

$$
X_{\boldsymbol{\mu}} = c_6 \left(x_5 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_5} \right) + c_3 \frac{\partial}{\partial x_3} + c_5 \frac{\partial}{\partial x_5}.
$$

The $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian function of $X_{\boldsymbol{\mu}}$ reads

$$
\begin{aligned}
\boldsymbol{f}_{\boldsymbol{\mu}} = &\left(c_3 x_5 - c_5 x_3 + c_6 \left(\frac{x_3^2}{2} + \frac{x_5^2}{2} \right) \right) \otimes e_1 \\
&+ \left(c_3 x_5 - c_5 x_3 + c_6 \left(\frac{x_3^2}{2} + \frac{x_5^2}{2} \right) \right) \otimes e_2.
\end{aligned}
$$

Next, the methods introduced in Sect. 4 will be employed to find the two-polysymplectic relative equilibrium points of the $\boldsymbol{\omega}$-Hamiltonian vector field

$$
Y = X_4 + X_6
$$

and study their stability. According to Theorem 4.2, a two-polysymplectic relative equilibrium point $z_e \in P$ is a point for which there exists $\xi \in \mathfrak{g} \simeq \mathbb{R}$ such that z_e is a critical point of each component of the $\mathbb{R}^2$-valued function

$$\boldsymbol{h}_\xi = \left(x_1 - \xi(x_1 - \mu^1) + \frac{1}{2}(x_3^2 + x_5^2)\right) \otimes e_1 + \left(x_2 - \xi(x_2 - \mu^2) + \frac{1}{2}(x_3^2 + x_5^2)\right) \otimes e_1.$$

This happens for $\xi = 1$ and $z_e = (x_1, x_2, x_3 = 0, x_4, x_5 = 0) \in \mathbb{R}^5$, where x_1, x_2, x_4 are arbitrary.

To examine the relative stability of $z_e \in \mathbb{R}^5$, note that the supplementary spaces to $\mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e} z_e) + \ker \omega^1_{z_e}$ and $\mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e} z_e) + \ker \omega^2_{z_e}$ in $\mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$ have the form

$$\mathcal{S}^1 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} \right\rangle, \qquad \mathcal{S}^2 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} \right\rangle,$$

respectively. Then, $\mathcal{S}^1 + \mathcal{S}^2 + \mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e} z_e) = \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$ and the Hessian of $(\delta^2 h^1_\xi)_{z_e}$ at z_e in the subspace $\mathcal{S}^1$ and the Hessian of $(\delta^2 h^2_\xi)_{z_e}$ in the subspace $\mathcal{S}^2$ are definite-positive. Therefore, by our criterion, a two-polysymplectic relative equilibrium point $z_e \in \mathbb{R}^5$ is relatively stable, namely its projection to $P_{\boldsymbol{\mu}_e}$ is stable. More specifically, the reduced system has an $\boldsymbol{\omega}_{\boldsymbol{\mu}_e}$-Hamiltonian function whose components, $f^1_{\boldsymbol{\mu}_e}, f^2_{\boldsymbol{\mu}_e}$ are such that their Hessians at equilibrium points $\pi_{\boldsymbol{\mu}_e}(z_e) = (x_3 = 0, x_5 = 0)$ are definite-positive in the directions of $\ker(\omega^1_{\boldsymbol{\mu}_e})_{z_e}$ and $\ker(\omega^2_{\boldsymbol{\mu}_e})_{z_e}$, respectively. Indeed, the reduced $\boldsymbol{\omega}_{\boldsymbol{\mu}_e}$-Hamiltonian function reads

$$\boldsymbol{f}_{\boldsymbol{\mu}_e} = \frac{1}{2}\left(x_3^2 + x_5^2\right) \otimes e_1 + \frac{1}{2}\left(x_3^2 + x_5^2\right) \otimes e_2$$

and the function

$$f^1_{\boldsymbol{\mu}_e} + f^2_{\boldsymbol{\mu}_e} = x_3^2 + x_5^2,$$

is invariant under the dynamics of $Y_{\boldsymbol{\mu}_e}$ and has a strict minimum at $\pi_{\boldsymbol{\mu}_e}(z_e) = (x_3 = 0, x_5 = 0)$. Hence, the reduced two-polysymplectic Hamiltonian system is stable.

5.4 Quantum Quadratic Hamiltonian Operators

Next, let us analyse an example based upon the Wei–Norman equations for the automorphic Lie system related to quantum mechanical systems described by quadratic Hamiltonian operators, which describe as particular cases quantum harmonic oscillators with/without dissipation (Cariñena and de Lucas 2011; Wei and Norman 1963). In this case, the system of differential equations under study is the one determining the integral curves of the time-dependent vector field

$$X = \sum_{\alpha=1}^{6} b_\alpha(t) X_\alpha^R, \tag{5.4}$$

for certain t-dependent functions $b_1(t), \ldots, b_6(t)$ and the vector fields

$$X_1^R = \frac{\partial}{\partial v_1} + v_5 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5^2 \frac{\partial}{\partial v_6}, \qquad X_2^R = v_1 \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} + \frac{1}{2} v_4 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5 \frac{\partial}{\partial v_5},$$
$$X_3^R = v_1^2 \frac{\partial}{\partial v_1} + 2v_1 \frac{\partial}{\partial v_2} + e^{v_2} \frac{\partial}{\partial v_3} - v_4 \frac{\partial}{\partial v_5} + \frac{1}{2} v_4^2 \frac{\partial}{\partial v_6}, \qquad X_4^R = \frac{\partial}{\partial v_4},$$
$$X_5^R = \frac{\partial}{\partial v_5} - v_4 \frac{\partial}{\partial v_6}, \qquad X_6^R = \frac{\partial}{\partial v_6}.$$

The commutation relations between the above vector fields read

$$\left[X_1^R, X_2^R\right] = X_1^R,$$
$$\left[X_1^R, X_3^R\right] = 2\, X_2^R, \quad \left[X_2^R, X_3^R\right] = X_3^R,$$
$$\left[X_1^R, X_4^R\right] = 0, \quad \left[X_2^R, X_4^R\right] = -\frac{1}{2} X_4^R, \quad \left[X_3^R, X_4^R\right] = X_5^R,$$
$$\left[X_1^R, X_5^R\right] = -X_4^R, \quad \left[X_2^R, X_5^R\right] = \frac{1}{2} X_5^R, \quad \left[X_3^R, X_5^R\right] = 0, \quad \left[X_4^R, X_5^R\right] = -X_6^R,$$
$$\left[X_1^R, X_6^R\right] = 0, \quad \left[X_2^R, X_6^R\right] = 0, \quad \left[X_3^R, X_6^R\right] = 0, \quad \left[X_4^R, X_6^R\right] = 0, \quad \left[X_5^R, X_6^R\right] = 0.$$

It is known that the Lie algebra of Lie symmetries of $\langle X_1^R, \ldots, X_6^R \rangle$ is spanned by

$$X_1^L = e^{v_2} \frac{\partial}{\partial v_1} + 2v_3 \frac{\partial}{\partial v_2} + v_3^2 \frac{\partial}{\partial v_3}, \qquad X_2^L = \frac{\partial}{\partial v_2} + v_3 \frac{\partial}{\partial v_3}, \qquad X_3^L = \frac{\partial}{\partial v_3},$$
$$X_4^L = e^{-v_2/2}(e^{v_2} - v_1 v_3) \frac{\partial}{\partial v_4} - e^{-v_2/2} v_3 \frac{\partial}{\partial v_5} - e^{-v_2/2}(e^{v_2} - v_1 v_3) v_5 \frac{\partial}{\partial v_6},$$
$$X_5^L = v_1 e^{-v_2/2} \frac{\partial}{\partial v_4} + e^{-v_2/2} \frac{\partial}{\partial v_5} - v_1 v_5 e^{-v_2/2} \frac{\partial}{\partial v_6}, \quad X_6^L = \frac{\partial}{\partial v_6}.$$

In particular, let us focus on systems (5.4) with constant coefficients and, in particular,

$$X_5^R = \frac{\partial}{\partial v_5} - v_4 \frac{\partial}{\partial v_6}.$$

Then, a Lie symmetry of our system is given by

$$Y = \frac{\partial}{\partial v_5}.$$

A two-polysymplectic form on $\mathbb{R}^6$ can be defined in the following way

$$
\begin{aligned}
\boldsymbol{\omega} &= \omega^1 \otimes e_1 + \omega^2 \otimes e_2 \\
&= (\mathrm{d}v_1 \wedge \mathrm{d}v_3 + \mathrm{d}v_2 \wedge \mathrm{d}v_4 + \mathrm{d}v_5 \wedge \mathrm{d}v_1 \\
&\quad + \mathrm{d}v_4 \wedge \mathrm{d}v_6) \otimes e_1 + (\mathrm{d}v_4 \wedge \mathrm{d}v_6 - \mathrm{d}v_3 \wedge \mathrm{d}v_5) \otimes e_2 \,.
\end{aligned}
$$

Note that

$$
\ker \omega^1 = \left\langle \frac{\partial}{\partial v_3} + \frac{\partial}{\partial v_5}, \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_6} \right\rangle, \qquad \ker \omega^2 = \left\langle \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\rangle,
$$

and $\ker \omega^1 \cap \ker \omega^2 = 0$ and $(\mathbb{R}^6, \boldsymbol{\omega})$ becomes a two-polysymplectic manifold. The vector field Y is a Lie symmetry of the two-polysymplectic form, i.e. $\mathscr{L}_Y \boldsymbol{\omega} = 0$. Then,

$$
\iota_{Y_3} \omega^1 = \mathrm{d}v_1, \qquad \iota_{Y_3} \omega^2 = \mathrm{d}v_3,
$$

and a two-polysymplectic momentum map $\mathbf{J}^\Phi$ associated with the Lie group action given by the flow of Y reads

$$
\mathbf{J}^\Phi : x \in \mathbb{R}^6 \longmapsto (v_1, v_3) = (\mathfrak{g}^*)^2 \in \boldsymbol{\mu} \simeq \mathbb{R}^{*2}.
$$

Note that every $\boldsymbol{\mu} = (\mu^1, \mu^2) \in \mathbb{R}^{*2}$ is a weakly regular value of $\mathbf{J}^\Phi$, which is Ad^{*2}-equivariant. The isotropy group for every $\boldsymbol{\mu} \in \mathbb{R}^{*2}$ reads $G_{\boldsymbol{\mu}} = \mathbb{R}$. Hence, $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a submanifold, as well as $\mathbf{J}_1^{\Phi-1}(\mu^1)$ and $\mathbf{J}_2^{\Phi-1}(\mu^2)$. Since Y_5 is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, then $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ can be locally coordinated by the functions $\{v_2, v_4, v_6\}$.

The vector field X_5^R is $\boldsymbol{\omega}$-Hamiltonian with

$$
\begin{aligned}
\iota_{X_5^R} \boldsymbol{\omega} &= \iota_{X_5^R} \omega^1 \otimes e_1 + \iota_{X_5^R} \omega^2 \otimes e_2 \\
&= \mathrm{d}\left(v_1 + \frac{v_4^2}{2} \right) \otimes e_1 + \mathrm{d}\left(v_3 + \frac{v_4^2}{2} \right) \otimes e_2 = \mathrm{d}\boldsymbol{h}_5^R.
\end{aligned} \tag{5.5}
$$

Then, the reduced two-forms read

$$
\omega_{\boldsymbol{\mu}}^1 = \mathrm{d}v_2 \wedge \mathrm{d}v_4 + \mathrm{d}v_4 \wedge \mathrm{d}v_6, \qquad \omega_{\boldsymbol{\mu}}^2 = \mathrm{d}v_4 \wedge \mathrm{d}v_6.
$$

Furthermore, one has

$$
\ker \omega_{\boldsymbol{\mu}}^1 = \left\langle \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_6} \right\rangle, \qquad \ker \omega_{\boldsymbol{\mu}}^2 = \left\langle \frac{\partial}{\partial v_2} \right\rangle,
$$

and $\omega_{\boldsymbol{\mu}}^1, \omega_{\boldsymbol{\mu}}^2$ define a two-polysymplectic form on $P_{\boldsymbol{\mu}}$. Moreover, the $\boldsymbol{\omega}$-Hamiltonian function of X_5^R is invariant relative to Y. Then, Theorem 3.4 ensures that the projection of X_5^R onto $P_{\boldsymbol{\mu}}$ exists and is given by

$$X^R = -v_4 \frac{\partial}{\partial v_6},$$

which is the $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field of the $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian function

$$\boldsymbol{f}_{\boldsymbol{\mu}} = \left(\mu^1 + \frac{v_4^2}{2}\right) \otimes e_1 + \left(\mu^2 + \frac{v_4^2}{2}\right) \otimes e_2,$$

which has a critical point at every point $(v_4 = 0, v_6)$, where v_6 is arbitrary. Such points are not stable equilibrium points. In particular, this $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian function does not satisfy that $f_{\boldsymbol{\mu}}^1 + f_{\boldsymbol{\mu}}^2$ has a strict minimum at the equilibrium point: it has only a minimum. Note that the points in $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ that project onto the above-mentioned equilibrium points are two-polysymplectic relative equilibrium points. The analysis of (5.5) with our two-polysymplectic energy–momentum methods at the mentioned two-polysymplectic relative equilibrium points suggests the same results.

5.5 Equilibrium Points and Vector Fields with Polynomial Coefficients

Let us illustrate certain aspects of our k-polysymplectic energy–momentum method by studying vector fields with a polynomial behaviour. Moreover, our example will illustrate some features of weakly regular points of k-polysymplectic momentum maps and the character of their associated k-polysymplectic Marsden–Weinstein reductions.

Consider coordinates $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ on $\mathbb{R}^8$ and the vector field X on $\mathbb{R}^8$ given by

$$X = x_6^a \frac{\partial}{\partial x_2} + x_4^b \frac{\partial}{\partial x_3} - x_3^c \frac{\partial}{\partial x_4} + x_8^d \frac{\partial}{\partial x_7} - x_7^e \frac{\partial}{\partial x_8},$$

where $a, b, c, d, e \in \mathbb{N}$. Define the two-polysymplectic for $\boldsymbol{\omega}$ on $\mathbb{R}^8$ of the form

$$\begin{aligned}\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 &= (\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_1 \wedge \mathrm{d}x_5) \otimes e_1 \\ &+ (\mathrm{d}x_2 \wedge \mathrm{d}x_6 + \mathrm{d}x_7 \wedge \mathrm{d}x_8) \otimes e_2.\end{aligned}$$

Then,

$$\begin{aligned}&\ker \omega_x^1 = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle, \qquad \ker \omega_x^2 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \right\rangle, \\ &\ker \omega_x^1 \cap \ker \omega_x^2 = 0\end{aligned}$$

for any $x \in \mathbb{R}^8$, and thus $\boldsymbol{\omega}$ becomes a two-polysymplectic form.

The vector field X admits the Lie symmetries, $Y_1 = \frac{\partial}{\partial x_2}$, $Y_2 = \frac{\partial}{\partial x_1}$, $Y_3 = \frac{\partial}{\partial x_5}$, which span a three-dimensional abelian Lie algebra of vector fields. These Lie symmetries are the infinitesimal generators of the translations along the x_2, x_1, and x_5 coordinates, and they also leave the two-polysymplectic form invariant, i.e. $\mathscr{L}_{Y_i}\omega^\alpha = 0$ for $i = 1, 2, 3$ and $\alpha = 1, 2$. They give rise to a Lie group action $\Phi : \mathbb{R}^3 \times \mathbb{R}^8 \to \mathbb{R}^8$ that leaves invariant $\boldsymbol{\omega}$.

Since

$$\iota_{Y_1}\omega^1 = 0\,, \qquad \iota_{Y_2}\omega^1 = \mathrm{d}x_5\,, \qquad \iota_{Y_3}\omega^1 = -\mathrm{d}x_1\,,$$
$$\iota_{Y_1}\omega^2 = \mathrm{d}x_6\,, \qquad \iota_{Y_2}\omega^2 = 0\,, \qquad \iota_{Y_3}\omega^2 = 0\,,$$

a two-polysymplectic momentum map $\mathbf{J}^\Phi$ can be defined by setting

$$\mathbf{J}^\Phi : x \in \mathbb{R}^8 \longmapsto \mathbf{J}^\Phi(x) = (0, x_5, -x_1; x_6, 0, 0) \in (\mathbb{R}^{3*})^2 \simeq (\mathbb{R}^3)^2.$$

Then, $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle$ for each $x \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and $\boldsymbol{\mu} \in (\mathbb{R}^3)^2$. The two-polysymplectic momentum map $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant and every $\boldsymbol{\mu} \in (\mathbb{R}^3)^2$ is a weakly regular value of $\mathbf{J}^\Phi$ since each $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a five-dimensional submanifold of $\mathbb{R}^8$ and its tangent space at each point coincides with the kernel of $\mathbf{J}^\Phi$ at that point. Moreover, $\mathbf{J}^\Phi$ has no regular points.

Note that Y_2 and Y_3 do not take values at x in $\mathrm{T}_x(G_{\boldsymbol{\mu}}x)$ but Y_1 does. The assumptions of Theorem 3.3 are satisfied, and the quotient space $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x\left(G_{\boldsymbol{\mu}}x\right)$ is a two-dimensional subspace, where

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x\left(G_{\boldsymbol{\mu}}x\right) = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle, \qquad \forall x \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$$

and

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega^1_{\boldsymbol{\mu}} \otimes e_1 + \omega^2_{\boldsymbol{\mu}} \otimes e_2 = (\mathrm{d}x_3 \wedge \mathrm{d}x_4) \otimes e_1 + (\mathrm{d}x_7 \wedge \mathrm{d}x_8) \otimes e_2.$$

The vector field X is $\boldsymbol{\omega}$-Hamiltonian relative to

$$\begin{aligned} \mathrm{d}\boldsymbol{h} = \iota_X\boldsymbol{\omega} &= \iota_X\omega^1 \otimes e_1 + \iota_X\omega^2 \otimes e_2 \\ &= \mathrm{d}\left(\frac{1}{1+b}x_4^{b+1} + \frac{1}{c+1}x_3^{c+1}\right) \otimes e_1 \\ &\quad + \mathrm{d}\left(\frac{1}{1+a}x_6^{a+1} + \frac{1}{d+1}x_8^{d+1} + \frac{1}{1+e}x_7^{e+1}\right) \otimes e_2\,. \end{aligned}$$

Moreover, $\boldsymbol{h}$ is invariant relative to the Lie symmetries Y_1, Y_2, Y_3. By Theorem 3.4, the vector field X projects onto the quotient manifold and its projection $X_{\boldsymbol{\mu}}$ is given by

$$X_{\boldsymbol{\mu}} = x_4^b\frac{\partial}{\partial x_3} - x_3^c\frac{\partial}{\partial x_4} + x_8^d\frac{\partial}{\partial x_7} - x_7^e\frac{\partial}{\partial x_8},$$

which is an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field since

$$
\begin{aligned}
\mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}} = \iota_{X_\mu}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}\left(\frac{1}{1+b}x_4^{b+1} + \frac{1}{c+1}x_3^{c+1}\right)\otimes e_1 \\
+\mathrm{d}\left(\frac{1}{d+1}x_8^{d+1} + \frac{1}{1+e}x_7^{e+1}\right)\otimes e_2.
\end{aligned}
$$

According to Theorem 4.2, a point z_e is a two-polysymplectic relative equilibrium point if it is a critical point of $\boldsymbol{h}_\xi$ for some $\xi = (\xi_1, \xi_2, \xi_3) \in \mathfrak{g} \simeq \mathbb{R}^3$. Then, one has that

$$
\begin{aligned}
\mathrm{d}\boldsymbol{h}_\xi &= \mathrm{d}f_\xi^1 \otimes e_1 + \mathrm{d}f_\xi^2 \otimes e_2 \\
&= \left(x_4^b\mathrm{d}x_4 + x_3^c\mathrm{d}x_3 - \xi_2\mathrm{d}x_5 + \xi_3\mathrm{d}x_1\right)\otimes e_1 \\
&\quad + \left((x_6^a - \xi_1)\mathrm{d}x_6 + x_8^d\mathrm{d}x_8 + x_7^e\mathrm{d}x_7\right)\otimes e_2\,.
\end{aligned}
$$

It follows that $\xi_2 = \xi_3 = 0$ and we have two-polysymplectic relative equilibrium points of X of the form $z_e = (x_1, x_2, 0, 0, x_5, x_6, 0, 0)$ for $x_6^a = \xi_1$. In fact, x_1, x_2, x_5, x_6 are arbitrary. Indeed, $(X_{\boldsymbol{\mu}_e})_{[z_e]} = 0$ for $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$.

To analyse the stability of the above-mentioned two-polysymplectic relative equilibrium points, let us analyse the second derivatives of $\boldsymbol{h}_\xi$ at such points z_e. Then,

$$
\begin{aligned}
(\delta^2\boldsymbol{h}_\xi)_{z_e} &= (\delta^2 h_\xi^1)_{z_e}\otimes e_1 + (\delta^2 h_\xi^2)_{z_e}\otimes e_2 \\
&= \left(cx_3^{c-1}\mathrm{d}x_3\otimes\mathrm{d}x_3 + bx_4^{b-1}\mathrm{d}x_4\otimes\mathrm{d}x_4\right)\otimes e_1 \\
&\quad + \left(ex_7^{e-1}\mathrm{d}x_7\otimes\mathrm{d}x_7 + dx_8^{d-1}\mathrm{d}x_8\otimes\mathrm{d}x_8\right)\otimes e_2\,.
\end{aligned}
$$

Taking into account that the supplementary spaces to $\mathrm{T}_{z_e}G_{\boldsymbol{\mu}_e} + \ker\omega^1_{z_e} \cap \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$ and $\mathrm{T}_{z_e}G_{\boldsymbol{\mu}_e} + \ker\omega^2_{z_e} \cap \mathrm{T}_{z_e}(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e))$ can be chosen $\mathcal{S}^1_{z_e} = \langle\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\rangle$ and $\mathcal{S}^2_{z_e} = \langle\frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}\rangle$, respectively, Definition 4.6 and the posterior explanation give that the z_e are stable two-polysymplectic relative equilibrium points if

$$
(\delta^2 h_\xi^1)_{z_e}(v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \mathcal{S}^1_{z_e}\setminus\{0\},
$$

and

$$
\left(\delta^2 h_\xi^2\right)_{z_e}(v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \mathcal{S}^2_{z_e}\setminus\{0\}.
$$

These inequalities hold if and only if $b, c, d, e = 1$. Hence, the z_e are formally stable two-polysymplectic relative equilibrium points of X for $b, c, d, e = 1$. Indeed, it is immediate that previous conditions of stability imply that in the reduced space close to the equilibrium point, the coordinates x_3, x_4, x_7, x_8 are bounded for every motion close enough to the equilibrium point, which ensures real stability.

6 Conclusions and Outlook

In this work, we have devised a new energy–momentum method for systems of ordinary differential equations given by Hamiltonian vector fields with respect to a k-polysymplectic form. This led to define and charactere k-polysymplectic relative equilibrium points and introducing new techniques to study stability through k-polysymplectic geometry. In this respect, we have also reviewed several aspects and mistakes in previous Marsden–Weinstein reductions for k-polysymplectic forms and Hamiltonian systems (Blacker 2019; García-Toraño Andrés and Mestdag 2023; Marrero et al. 2015; Munteanu et al. 2004), which is a relevant part in energy–momentum methods. To illustrate our new energy–momentum method and theoretical results, we have studied several relevant examples in detail: complex Schwarz equations, the product of several symplectic manifolds, with a family of particles subjected to the effect of an isotropic potential for each of them, affine homogeneous differential equations (with potential applications to control theory and Lie systems theory), quantum harmonic oscillators with dissipation, integrable symplectic systems, and some dynamical systems with polynomial coefficients.

A non-autonomous analogue of the methods devised in this paper could be accomplished by using the Lyapunov theory depicted in de Lucas and Zawora (2021). Note that the stability with respect to k-polysymplectic forms is a topic that requires further development. The criteria here used are enough for the family of examples to be studied, but a deeper study with an analysis of all possibilities is in order. Recall also that, even in the one-polysymplectic case, the criterion for the stability of the energy–momentum method, which we here recover as a particular case and is classical (Abraham and Marsden 1978), is not a necessary condition for the stability of the reduced system.

The study of complex Schwarz equations and the Schwarzian derivative could be more appropriately studied through a complex Lie system formalism. We aim to study this possibility in further works.

Acknowledgements J. de Lucas, X. Rivas and B.M. Zawora acknowledge partial financial support from the Nowe Idee 2B-POB II project PSP: 501-D111-20-2004310 funded by the “Inicjatywa Doskonałości - Uczelnia Badawcza” (IDUB) program. J. de Lucas acknowledges a CRM-Simons professorship funded by the Simons Foundation and the Centre de Recherches Mathématiques (CRM) of the Université Montréal. X. Rivas acknowledges funding from J. de Lucas’s CRM-Simons professorship to accomplish a stay at the CRM which allowed us to obtain several relevant findings of the present work. X. Rivas also acknowledges funding from the Spanish Ministry of Science and Innovation, grants PID2021-125515NB-C21, and RED2022-134301-T of AEI, and Ministry of Research and Universities of the Catalan Government, project 2021 SGR 00603. B.M. Zawora acknowledges funding from the IDUB mikrogrant program to accomplish a research stay at the CRM. L. Colombo acknowledges financial support from Grant PID2022-137909NB-C21 funded by MCIN/AEI/ 10.13039/501100011033. B.M. Zawora acknowledge partial financial support from. B.M. Zawora and X. Rivas acknowledge partial funding from the Simons Foundation to take part in a conference in the Simons Center for Geometry and Physics of Stony Brook University, where the last parts of this paper were accomplished and several relevant results concerning the stability of reduced k-polysymplectic Hamiltonian systems and their applications were devised. We extend our gratitude to the referees for their thorough review of the manuscript, their insightful comments, and their recommendation to examine Example 3.1. B.M. Zawora acknowledges funding from the Norwegian Financial Mechanism 2014-2021 with project registration number 2019/34/H/ST1/00636.

Author Contributions All authors taking part in the work have contributed equally to all tasks.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

Abarbanel, H., Holm, D.: Nonlinear stability analysis of inviscid flows in three dimensions: incompressible fluids and barotropic fluids. Phys. Fluids **30**(11), 3369–3382 (1987). https://doi.org/10.1063/1.866469

Abraham, R., Marsden, J.E.: Foundations of mechanics. In: AMS Chelsea publishing, vol. 364. 2nd edn. Benjamin/Cummings Pub. Co., New York (1978). https://doi.org/10.1090/chel/364

Albert, C.: Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact. J. Geom. Phys. **6**(4), 627–649 (1989). https://doi.org/10.1016/0393-0440(89)90029-6

Awane, A.: k-symplectic structures. J. Math. Phys. **33**(12), 4046 (1992). https://doi.org/10.1063/1.529855

Awane, A., Goze, M.: Pfaffian systems, k-symplectic systems, 1st edn. Springer, Dordrecht (2000). https://doi.org/10.1007/978-94-015-9526-1

Bai, C., Zhang, H.: A weak energy-momentum method for stochastic instability induced by dissipation and random excitations. Probab. Eng. Mech. **37**, 35–40 (2014). https://doi.org/10.1016/j.probengmech.2014.03.009

Blacker, C.: Polysymplectic reduction and the moduli space of flat connections. J. Phys. A Math. Theor. **52**(33), 335201 (2019). https://doi.org/10.1088/1751-8121/ab2eed

Blankenstein, G., Ratiu, T.S.: Singular reduction of implicit Hamiltonian systems. Rep. Math. Phys. **53**(2), 211–260 (2004). https://doi.org/10.1016/S0034-4877(04)90013-4

Bloch, A.M.: Nonholonomic mechanics and control. In: Interdisciplinary Applied Mathematics, vol. 24. Springer, New York (2003). https://doi.org/10.1007/978-1-4939-3017-3

Bozhkov, Y.D., da Conceição, P.R.: On the generalizations of the Kummer-Schwarz equation. Nonlinear Anal. **192**, 111691 (2020). https://doi.org/10.1016/j.na.2019.111691

Búa, L., Bucataru, I., de León, M., Salgado, M., Vilariño, S.: Symmetries in Lagrangian field theory. Rep. Math. Phys. **75**(3), 333–357 (2015). https://doi.org/10.1016/S0034-4877(15)30010-0

Cariñena, J.F., de Lucas, J.: Lie systems: theory, generalisations, and applications. Diss. Math. **479**, 1–162 (2011). https://doi.org/10.4064/dm479-0-1

Cariñena, J.F., Grabowski, J., Marmo, G.: Lie-Scheffers systems: a geometric approach. Napoli Series on Physics and Astrophysics. Bibliopolis, Naples (2000)

Cariñena, J.F., Grabowski, J., Marmo, G.: Superposition rules, lie theorem, and partial differential equations. Rep. Math. Phys. **60**(2), 237–258 (2007). https://doi.org/10.1016/S0034-4877(07)80137-6

Cariñena, J.F., Ramos, A.: Applications of Lie systems in quantum mechanics and control theory. Banach Center Publ. **59**, 143–162 (2003). https://doi.org/10.4064/bc59-0-7

de León, M., Merino, E., Oubiña, J.A., Salgado, M.: Stable almost cotangent structures. Bolletino Unione Mat. Ital. B (7) **11**(3), 509–529 (1997)

de León, M., Méndez, I., Salgado, M.: p-almost tangent structures. Rend. Circ. Mat. Palermo **37**(2), 282–294 (1988). https://doi.org/10.1007/BF02844526

de León, M., Méndez, I., Salgado, M.: Regular p-almost cotangent structures. J. Korean Math. Soc. **25**(2), 273–287 (1988)

de León, M., Salgado, M., Vilariño, S.: Methods of differential geometry in classical field theories. World Sci. (2015). https://doi.org/10.1142/9693

de Lucas, J., Rivas, X., Vilariño, S., Zawora, B.M.: On k-polycosymplectic Marsden–Weinstein reductions. J. Geom. Phys. **191**, 104899 (2023). https://doi.org/10.1016/j.geomphys.2023.104899

de Lucas, J., Sardón, C.: A Guide to Lie Systems with Compatible Geometric Structures. World Scientific Publishing Co. Pte. Ltd., Singapore (2020). https://doi.org/10.1142/q0208

de Lucas, J., Vilariño, S.: k-symplectic Lie systems: theory and applications. J. Differ. Equ. **258**(6), 2221–2255 (2015). https://doi.org/10.1016/j.jde.2014.12.005

de Lucas, J., Zawora, B.M.: A time-dependent energy-momentum method. J. Geom. Phys. **170**, 104364 (2021). https://doi.org/10.1016/j.geomphys.2021.104364

Echeverría-Enríquez, A., Muñoz-Lecanda, M.C., Román-Roy, N.: Geometry of Lagrangian first-order classical field theories. Fortschr. Phys. **44**(3), 235–280 (1996). https://doi.org/10.1002/prop.2190440304

García-Toraño Andrés, E., Mestdag, T.: Conditions for symmetry reduction of polysymplectic and polycosymplectic structures. J. Phys. A Math. Theor. **56**(33), 335202 (2023). https://doi.org/10.1088/1751-8121/ace74c

Giachetta, G., Mangiarotti, L., Sardanashvily, G.A.: New Lagrangian and Hamiltonian Methods in Field Theory. World Scientific, River Edge (1997). https://doi.org/10.1142/2199

Gràcia, X., de Lucas, J., Muñoz-Lecanda, M.C., Vilariño, S.: Multisymplectic structures and invariant tensors for Lie systems. J. Phys. A Math. Theor. **52**(21), 215201 (2019). https://doi.org/10.1088/1751-8121/ab15f2

Guieu, L., Roger, C.: L'algèbre et le Groupe de Virasoro. Aspects géométriques et algébriques, généralisations. Les Publications CRM, Montréal (2007)

Günther, C.: The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: the local case. J. Differ. Geom. **25**(1), 23–53 (1987). https://doi.org/10.4310/jdg/1214440723

Hille, E.: Ordinary Differential Equations in the Complex Domain. Dover Publications, Mineola (1997)

Kanatchikov, I.V.: Canonical structure of classical field theory in the polymomentum phase space. Rep. Math. Phys. **41**(1), 49–90 (1998). https://doi.org/10.1016/S0034-4877(98)80182-1

Lee, J.M.: Manifolds and differential geometry. In: Graduate Studies in Mathematics, vol. 107. American Mathematical Society, Providence (2009)

Lee, J.M.: Introduction to smooth manifolds. In: Graduate Texts in Mathematics. vol. 218, 2nd edn. Springer New York Heidelberg Dordrecht London (2012). https://doi.org/10.1007/978-1-4419-9982-5

Lehto, O.: Remarks on Nehari's theorem about the Schwarzian derivative and Schlicht functions. J. Anal. Math. **36**, 184–190 (1979). https://doi.org/10.1007/BF02798778

Lucas, J. d, Maskalaniec, A., Zawora, B.M.: Cosymplectic geometry, reductions, and energy-momentum methods with applications. J. Nonlinear Math. Phys. **31**(1), 64 (2024). https://doi.org/10.1007/s44198-024-00225-w

Marrero, J.C., Román-Roy, N., Salgado, M., Vilariño, S.: Reduction of polysymplectic manifolds. J. Phys. A Math. Theor. **48**(5), 055206 (2015). https://doi.org/10.1088/1751-8113/48/5/055206

Marrero, J.C., Román-Roy, N., Salgado, M., Vilariño, S.: Symmetries, Noether's theorem and reduction in k-cosymplectic field theories. In: AIP Conference Proceedings, vol. 1260, no. 1, pp. 173–179 (2010). https://doi.org/10.1063/1.3479319

Marsden, J.E., Posbergh, T.A., Simo, J.C.: Stability of coupled rigid body and geometrically exact rods: block diagonalization and the energy-momentum method. Phys. Rep. **193**, 280–360 (1990). https://doi.org/10.1016/0370-1573(90)90125-L

Marsden, J.E., Ratiu, T.: Introduction to mechanics and symmetry. A basic exposition of classical mechanical systems. In: Texts in Applied Mathematics, vol. 17. Springer-Verlag, New York (1999). https://doi.org/10.1007/978-0-387-21792-5

Marsden, J.E., Simo, J.C.: The energy momentum method. Act. Acad. Sci. Tau. **1**(124), 245–268 (1988)

Marsden, J.E., Simo, J.C., Lewis, D., Posbergh, T.: A block diagonalization theorem in the energy-momentum method. In: Dynamics and control of multibody systems. Contemp. Math., vol. 97, pp. 297–313. Amer. Math. Soc., Providence (1989). https://doi.org/10.1090/conm/097/1021043

Martinez, N.: Poly-symplectic groupoids and poly-Poisson structures. Lett. Math. Phys. **105**(5), 693–721 (2015)

McLean, M., Norris, L.K.: Covariant field theory on frame bundles of fibered manifolds. J. Math. Phys. **41**(10), 6808–6823 (2000). https://doi.org/10.1063/1.1288797

Merino, E.E.: Geometría k-simpléctica y k-cosimpléctica. Aplicaciones a las teorías clásicas de campos. PhD thesis, Universidade de Santiago de Compostela (1997)

Munteanu, F., Rey, A.M., Salgado, M.: The Günther's formalism in classical field theory: momentum map and reduction. J. Math. Phys. **45**(5), 1730–1751 (2004). https://doi.org/10.1063/1.1688433

Norris, L.K.: Generalized symplectic geometry on the frame bundle of a manifold. In: *Proceedings of Symposia in Pure Mathematics*, vol. 54, no. 2, pp. 435–465. Amer. Math. Soc., Providence (1993). https://doi.org/10.1090/pspum/054.2

Ortega, J., Planas-Bielsa, V., Ratiu, T.: Asymptotic and Lyapunov stability of constrained and Poisson equilibria. J. Differ. Equ. **214**(1), 92–127 (2005). https://doi.org/10.1016/j.jde.2004.09.016

Ortega, J.P., Ratiu, T.S.: Momentum maps and Hamiltonian reduction, In: Progress in Mathematics, vol. 222. Birkhäuser Boston, Inc., Boston (2004). https://doi.org/10.1007/978-1-4757-3811-7

Román-Roy, N., Salgado, M., Vilariño, S.: Symmetries and conservation laws in the Günther k-symplectic formalism of field theory. Rev. Math. Phys. **19**(10), 1117–1147 (2007). https://doi.org/10.1142/S0129055X07003188

Sardanashvily, G.A.: Generalized Hamiltonian formalism for field theory. Constraint systems. World Scientific Publishing Company, River Edge (1995). https://doi.org/10.1142/2550

Simo, J., Lewis, D., Marsden, J.: Stability of relative equilibria. Part I: The reduced energy-momentum method. Arch. Rat. Mech. Anal. **115**, 15–59 (1991). https://doi.org/10.1007/BF01881678

Simo, J., Tarnow, N.: The discrete energy-momentum method. Conserving algorithms for nonlinear elastodynamics. Z. Angew. Math. Phys. **43**, 757–792 (1992). https://doi.org/10.1007/BF00913408

Vidyasagar, M.: Nonlinear systems analysis, In: Classics in Applied Mathematics, vol. 42. Society for Industrial and Applied Mathematics (SIAM), Philadelphia. (2002). https://doi.org/10.1137/1.9780898719185

Wang, L., Krishnaprasad, P.: Gyroscopic control and stabilization. J. Nonlinear Sci. **2**, 367–415 (1992). https://doi.org/10.1007/BF01209527

Wei, J., Norman, E.: Lie algebraic solution of linear differential equations. J. Math. Phys. **4**(4), 575–581 (1963). https://doi.org/10.1063/1.1703993

Winternitz, P.: Lie groups and solutions of nonlinear differential equations. In: Wolf, K.B. (ed.) Nonlinear Phenomena. Lecture Notes in Physics, vol. 189, pp. 263–305. Springer, Berlin, Heidelberg (1983). https://doi.org/10.1007/3-540-12730-5_12

Zawora, B.M.: A time-dependent energy-momentum method. Master's thesis, University of Warsaw, Faculty of Physics (2021)

Zenkov, D.V., Bloch, A.M., Marsden, J.E.: The energy-momentum method for the stability of non-holonomic systems. Dyn. Stab. Syst. **13**(2), 123–165 (1998). https://doi.org/10.1080/02681119808806257