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Contact Lie systems: theory and applications

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Abstract
A Lie system is a time-dependent system of differential equations describing the integral curves of a time-dependent vector field that can be considered as a curve in a finite-dimensional Lie algebra of vector fields V. We call V a Vessiot–Guldberg Lie algebra. We define and analyse contact Lie systems, namely Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a contact manifold. We also study contact Lie systems of Liouville type, which are invariant relative to the flow of a Reeb vector field. Liouville theorems, contact Marsden–Weinstein reductions, and Gromov non-squeezing theorems are developed and applied to contact Lie systems. Contact Lie systems on three-dimensional Lie groups with Vessiot–Guldberg Lie algebras of right-invariant vector fields and associated with left-invariant contact forms are classified. Our results are illustrated with examples having relevant physical and mathematical applications, e.g. Schwarz equations, Brockett systems, quantum mechanical systems, etc. Finally, a Poisson coalgebra method to derive superposition rules for contact Lie systems of Liouville type is developed.

Keywords: Lie system, superposition rule, contact manifold, coalgebra method, contact Marsden–Weinstein reduction

(Some figures may appear in colour only in the online journal)

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1. Introduction

A *Lie system* is a time-dependent system of first-order ordinary differential equations whose general solution can be expressed as a time-independent function, a *superposition rule*, depending on a generic finite family of particular solutions and some constants to be related to initial conditions [17, 19, 82].

The Lie theorem states that a Lie system is equivalent to a time-dependent vector field that can be considered as a curve in a finite-dimensional Lie algebra of vector fields, a so-called *Vessiot–Guldberg Lie algebra* (VG Lie algebra, hereafter). Despite that being a Lie system is more the exception than the rule, Lie systems appear in many relevant mathematical and physical applications, which strongly motivates their analysis. For instance, [17] contains around 240 references about Lie systems and related topics. It is worth stressing the relevance of the research on Lie systems accomplished by Winternitz and his collaborators in the *Centre de Recherches Mathématiques* in the University of Montréal (Canada), the so-called *CRM School* (see [34] and references therein). Examples of Lie systems are Riccati equations and most of their generalisations [17, 51, 82]. Lie systems also appear in the study of Winternitz–Smorodinsky oscillators [12], Milne–Pinney equations [12], Kummer–Schwarz equations [17], projective Schrödinger equations [21], Bäcklund transformations and systems of partial differential equations (PDEs) [63], conditional symmetries [53], superequations [9], et cetera (see [17, 34] and references therein).

Different types of differential geometric structures, e.g. symplectic or Dirac ones, allow one to attach a certain function, a so-called *Hamiltonian*, to a vector field. Such vector fields are said to be *Hamiltonian* with respect to the geometric structure used to define them [2, 34]. At first, Lie systems admitting a VG Lie algebra of Hamiltonian vector fields relative to symplectic or Poisson manifolds, the so-called *Lie–Hamilton systems*, were analysed [18]. New powerful geometric techniques were applied to Lie–Hamilton systems, e.g. the Poisson coalgebra method to derive superposition rules [7]. Unfortunately, [23] provides a no-go theorem showing that not every Lie system is a Lie–Hamilton one. This suggested the analysis of Lie systems with VG Lie algebras of Hamiltonian vector fields relative to new geometric manifolds [34]. Nowadays, this has originated the study of Lie systems with VG Lie algebras of Hamiltonian vector fields relative to different types of geometric structures and their related problems. This also led to new theoretical results in differential geometry [35, 54]. In particular, [6, 7, 18, 19, 21, 39] analyse Lie–Hamilton systems (see [18] for Lie–Hamilton systems relative to a symplectic form). *Dirac–Lie systems* are Lie systems admitting a VG Lie algebra of Hamiltonian vector fields relative to a Dirac structure. This allows one to use Dirac geometry to study Dirac–Lie systems. Additionally, *k-symplectic Lie systems*, i.e. Lie systems admitting a VG Lie algebra of Hamiltonian vector fields relative to a k-symplectic manifold, were analysed in [35]. It was proved in [35] that k-symplectic manifolds allow for defining certain Poisson algebras of functions. These Poisson algebras enable us to study of integrable systems and superposition rules more efficiently than previous methods. Meanwhile, *multisymplectic Lie systems*, along with a particular type of multisymplectic reduction, were studied in [52, 54]. Relevantly, the development of a general multisymplectic reduction has been an open problem for several decades now [11, 37]. The works [35, 54] show how Lie systems lead to interesting advances in pure differential geometry too.

It is interesting that finding Lie systems with VG Lie algebras of Hamiltonian vector fields relative to a geometric structure has led to many more new applications than mere Lie systems, which satisfy less restrictive conditions [7, 12, 21, 34]. It is remarkable that geometric structures allow for the construction of superposition rules, constants of motion, and the analysis of relevant properties of Lie systems without relying on the analysis/solution of systems of partial or ordinary differential equations as the most classical and old methods [19, 20, 82], which may extraordinarily simplify the study of Lie systems [34]. Geometric techniques also provide new viewpoints to the nature and properties of superposition rules [7] and mathematical/physical problems [21, 59].

In recent years, the interest in dissipative systems has grown significantly. In part, this is due to the incorporation of contact geometry [8, 45, 57] to the study of non-conservative Lagrangian and Hamiltonian mechanical systems [13, 15, 29, 31, 41]. A *contact form* is a non-vanishing differential one-form, η, whose differential, $\mathrm{d}\eta$, is such that $\mathrm{d}\eta|_{\ker\eta\times\ker\eta}\neq 0$ is non-degenerate. Contact forms, in particular, and contact geometry, in general, have proved to be very useful in many different problems in areas such as thermodynamics [14, 75], circuit theory [46], non-holonomic systems [28], quantum mechanics [24], gravitation and general relativity [44, 65], control theory [66], among others [32, 57, 77]. Moreover, contact geometry has drawn, by itself, much attention in recent times [47–49]. Recently, the notion of cocontact manifold has also been developed to introduce explicit dependence on time [25, 43, 70].

In this context, this work investigates Lie systems possessing a VG Lie algebra of Hamiltonian vector fields relative to a contact form [29], the referred to as *contact Lie systems*. Contact Lie systems can be considered as a particular case of *Jacobi–Lie systems* (see [4, 5, 56]), which were first introduced in [56]. Nevertheless, [56] just contained one non-trivial example of Jacobi–Lie system giving rise to a contact Lie system, and it did not analyse the properties that are characteristic for contact Lie systems. In fact, [56] was mostly dealing with Jacobi–Lie systems on one- and two-dimensional manifolds, which do not retrieve contact Lie systems. Although we prove here that contact Lie systems can be analysed via Lie systems with other compatible geometric structures, we show that contact geometry is the appropriate setting to analyse contact Lie systems and their natural features: associated volume forms or reductions.

As a particular case, this work analyses the hereafter called *contact Lie systems of Liouville type*, namely contact Lie systems that are invariant relative to the Reeb vector field of their associated contact manifolds. For these systems, we introduce certain Liouville theorems and Gromov non-squeezing theorems, whose application can be considered as pioneering in the literature of Lie systems (see [34, 54]). Moreover, it is remarkable that the literature on contact Hamiltonian systems is mostly focused on dissipative systems [14, 15, 24, 26, 29, 41]. Meanwhile, this work also treats contact Hamiltonian systems not related to dissipation while having physical applications, which fulfils a gap in the literature.

Willett's reduction of contact manifolds [81] is here applied to the reduction of contact Lie systems. This is more general than some other reductions appearing in the literature [30]. As far as we know, types of Marsden–Weinstein reductions have only been applied to Lie systems in [54] for multisymplectic Lie systems.

Although contact Lie systems are naturally related to Lie–Hamilton Lie systems on symplectic manifolds of larger dimension, this relation is shown to have no practical applications to our purposes. This fact can be illustrated via our classification of *automorphic Lie systems* (see [34] for a definition) on three-dimensional Lie groups with a non-abelian VG Lie algebra of right-invariant vector fields and related to left-invariant contact forms. Note that automorphic Lie systems are relevant as the solution of every Lie system can be obtained via a particular solution of an automorphic Lie system and the integration of a VG Lie algebra to a Lie group action [17, 19].

Finally, an adaptation of the coalgebra method to obtain superposition rules, which was firstly aimed at Lie–Hamilton and Dirac–Lie systems [18, 23], has been devised for a class of Jacobi–Lie systems, and indirectly, for contact Lie systems of Liouville type. To illustrate our methods, an application to derive a superposition rule for an automorphic Lie system on the Lie group $\mathrm{SL}(2,\mathbb{R})$ has been developed.

The structure of the work goes as follows. In section 2, a review on contact geometry and contact Hamiltonian systems is provided and Willet's reduction on contact manifolds is sketched. Section 3 is the theoretical core of the article, introducing the notion of contact Lie system and of contact Lie system of Liouville type. Moreover, a Gromov's non-squeezing theorem for contact Lie systems of Liouville type is stated and proved. In section 3.1, we analyse the existence of underlying geometric structures for contact Lie systems and why contact geometry is more appropriate to analyse them. Section 4 classifies a class of contact automorphic Lie systems on three-dimensional Lie groups with a non-abelian Lie algebra and a left-invariant associated contact form. Section 5 is devoted to presenting four examples: the Brockett control system, the Schwarz equation, a family of quantum contact Lie systems, and a contact Lie system that is not of Liouville type. In particular, an example of contact Marsden–Weinstein reduction of contact Lie systems is discussed. Finally, section 6 devises a coalgebra method for obtaining superposition rules for a class of Jacobi–Lie systems, which gives, in particular, techniques to obtain superposition rules for contact Lie systems of Liouville type. As an application, the superposition rule for an automorphic Lie system on $\mathrm{SL}(2,\mathbb{R})$ is retrieved.

2. Review on contact mechanics

From now on, all manifolds and mappings are assumed to be smooth and connected, unless otherwise stated. This will simplify our presentation while stressing its main points. The space of vector fields on a manifold M is denoted by $\mathfrak{X}(M)$, while $\Omega^s(M)$ stands for the space of differential s-forms on M. Einstein notation will be hereafter used. Moreover, n will stand for a natural number, namely $n \in \{1,2,3,\ldots\}$.

2.1. Contact Hamiltonian systems

Let us provide a brief introduction to contact geometry (see [8, 45, 57] for details). Let us recall that a distribution $\mathcal{D}$ of corank one on M is *maximally non-integrable* if every mapping $\rho_x : \mathcal{D}_x \times \mathcal{D}_x \ni (v,w) \mapsto [X^v, Y^w]_x + \mathcal{D}_x \in \mathrm{T}_xM/\mathcal{D}_x$, with $x \in M$, where X^v, Y^w are any vector fields on M such that $X^v_x = v, Y^w_x = w$, is a surjection. Note that for $\mathcal{D}$ to be non-integrable, it is enough to ensure that there exist two vector fields X, Y on M taking values in $\mathcal{D}$ so that $[X,Y]$ does not take values in $\mathcal{D}$. Nevertheless, to say that $\mathcal{D}$ is maximally non-integrable, every $x \in M$ has to be related to some X, Y taking values in $\mathcal{D}$, and whose election may depend on the point x, so that $[X,Y]_x$ does not belong to $\mathcal{D}_x$.

A *contact manifold* is a pair (M,ξ) such that M is a $(2n+1)$-dimensional manifold M and ξ is a corank one maximally non-integrable distribution on M. We call ξ a *contact distribution* on M. Note that ξ can locally be, on an open neighbourhood U of each point $x \in M$, described as the kernel of a one-form $\eta \in \Omega^1(U)$ such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form on U. Note that if ξ were a non-integrable distribution but not a contact distribution, then one would just have that $\eta_x \wedge (\mathrm{d}\eta)^n_x \neq 0$ at some point $x \in M$, but $\eta \wedge (\mathrm{d}\eta)^n$ would not be a volume form.

A *co-orientable contact manifold* is a pair (M,η), where η is a one-form on M such that $(M, \ker \eta)$ is a contact manifold. Then, η is called a *contact form*. Since this work focus on local properties of contact manifolds and related structures, we will hereafter restrict ourselves to co-oriented contact manifolds. To simplify the terminology, co-oriented contact manifolds will be called contact manifolds as in the standard modern literature on contact geometry [13, 29, 41]. Moreover, if not otherwise stated, (M,η) will hereafter stand for a contact manifold.

Note that if η is a contact form on M, then $f\eta$ is also a contact form on M for every non-vanishing function $f \in \mathscr{C}^\infty(M)$. Moreover, a one-form η on M is such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form on M if and only if η induces a decomposition of the tangent bundle to M of the form $\mathrm{T}M = \ker\eta \oplus \ker\mathrm{d}\eta$. Recall that we assume $\dim M = 2n+1 > 1$.

A contact manifold (M,η) determines a unique vector field $R \in \mathfrak{X}(M)$, called the *Reeb vector field*, such that $\iota_R \mathrm{d}\eta = 0$ and $\iota_R\eta = 1$. Then, $\mathscr{L}_R\eta = 0$ and, therefore, $\mathscr{L}_R\mathrm{d}\eta = 0$.

Theorem 2.1 (Darboux theorem). *Given a contact manifold* (M,η) *with* $\dim M = 2n+1$, *around every point* $x \in M$ *there exist local coordinates* $\{q^i, p_i, s\}$, *with* $i = 1,\ldots,n$, *called* Darboux coordinates, *such that*

$$\eta = \mathrm{d}s - p_i \mathrm{d}q^i\,.$$

In these coordinates, $R = \partial/\partial s$.

A proof of the Darboux theorem for contact manifolds can be found in [1, 60].

Example 2.2 (Canonical contact manifold). Consider the product manifold $M = \mathrm{T}^*Q \times \mathbb{R}$, where Q is any manifold. The cotangent bundle T^*Q admits an adapted coordinate system $\{q^1,\ldots,q^n,p_1,\ldots,p_n\}$ and $\mathbb{R}$ has a natural coordinate s, which in turn give rise to a coordinate system $\{q^1,\ldots,q^n,p_1,\ldots,p_n,s\}$ on $\mathrm{T}^*Q \times \mathbb{R}$. Then $\eta = \mathrm{d}s - \theta$, where θ is the pull-back of the Liouville one-form $\theta_\circ \in \Omega^1(\mathrm{T}^*Q)$ relative to the canonical projection $\mathrm{T}^*Q \times \mathbb{R} \to \mathrm{T}^*Q$, is a contact form on M. In the chosen coordinates,

$$\eta = \mathrm{d}s - p_i \mathrm{d}q^i\,, \qquad R = \frac{\partial}{\partial s}\,.$$

The coordinates $\{q^i, p_i, s\}$ are Darboux coordinates on M. It is remarkable that $\theta_\circ$, and thus η, are independent of the coordinates $\{q^1,\ldots,q^n\}$. △

Example 2.2 is a particular case of *contactification* of an exact symplectic manifold. Consider an *exact symplectic manifold* (N,ω), namely a symplectic manifold whose symplectic form, ω, is exact, i.e. $\omega = -\mathrm{d}\theta$ for a differential one-form $\theta \in \Omega^1(N)$. Then, the product manifold $M = N \times \mathbb{R}$ is a contact manifold with the contact form $\eta = \mathrm{d}s - \theta$, where the variable s stands for the canonical coordinate in $\mathbb{R}$ understood as a variable in M in the natural manner.

Let (M,η) be a contact manifold. There exists a vector bundle isomorphism $\flat : \mathrm{T}M \to \mathrm{T}^*M$ given by

$$\flat(v) = \iota_v(\mathrm{d}\eta)_x + (\iota_v\eta_x)\eta_x\,, \qquad \forall v \in \mathrm{T}_xM, \quad \forall x \in M\,.$$

This isomorphism can be extended to a $\mathscr{C}^\infty(M)$-module isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ in the natural way. It is usual to denote both isomorphisms, of vector bundles and of $\mathscr{C}^\infty(M)$-modules, by $\flat$ as this does not lead to any misunderstanding, and the inverse of $\flat$ is denoted by $\sharp = \flat^{-1}$. Taking into account this isomorphism, $R = \sharp(\eta)$.

A *contact Hamiltonian system* [15, 29, 41] is a triple (M,η,h), where (M,η) is a contact manifold and $h \in \mathscr{C}^\infty(M)$. If $Rh = 0$, then h is called a *good Hamiltonian function* [34]. Given a contact Hamiltonian system (M,η,h), there exists a unique vector field $X_h \in \mathfrak{X}(M)$, called the *contact Hamiltonian vector field* of h, satisfying any of the following equivalent conditions:

(1) $\iota_{X_h}\mathrm{d}\eta = \mathrm{d}h - (\mathscr{L}_R h)\eta$ and $\iota_{X_h}\eta = -h$,
(2) $\mathscr{L}_{X_h}\eta = -(\mathscr{L}_R h)\eta$ and $\iota_{X_h}\eta = -h$,
(3) $\flat(X_h) = \mathrm{d}h - (\mathscr{L}_R h + h)\eta$.

A vector field $X \in \mathfrak{X}(M)$ is said to be *Hamiltonian* relative to the contact form η if it is the Hamiltonian vector field of a function $h \in \mathscr{C}^\infty(M)$. Let $\mathfrak{X}_{\mathrm{ham}}(M)$ stand for the space of Hamiltonian vector fields relative to (M,η). Unlike the case of symplectic mechanics, a Hamiltonian function h may not be preserved along the integral curves of the contact Hamiltonian vector field X_h (see [41, 61] for details). More precisely,

$$\mathscr{L}_{X_h} h = -(\mathscr{L}_R h) h\,.$$

A function $f \in \mathscr{C}^\infty(M)$ such that $\mathscr{L}_{X_h} f = -(\mathscr{L}_R f) f$ is called a *dissipated quantity* [41]. In Darboux coordinates, the contact Hamiltonian vector field X_h reads

$$X_h = \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial h}{\partial q^i} + p_i\frac{\partial h}{\partial s}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial h}{\partial p_i} - h\right)\frac{\partial}{\partial s}\,. \tag{2.1}$$

Its integral curves, let us say $\gamma(t) = (q^i(t), p_i(t), s(t))$, satisfy the system of differential equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial h}{\partial p_i}\,, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\left(\frac{\partial h}{\partial q^i} + p_i\frac{\partial h}{\partial s}\right)\,, \quad \frac{\mathrm{d}s}{\mathrm{d}t} = p_j\frac{\partial h}{\partial p_j} - h\,, \qquad i = 1,\ldots,n\,.$$

Example 2.3. Consider the contact Hamiltonian system $(\mathrm{T}^*\mathbb{R}^n \times \mathbb{R}, \eta, h)$, where $\mathbb{R}^n$ has global linear coordinates $\{q^1,\ldots,q^n\}$, while $\eta = \mathrm{d}s - p_i\mathrm{d}q^i$, and

$$h = \frac{p^2}{2m} + V(q) + \gamma s\,,$$

where m is the mass of a particle, $p = \sqrt{p_1^2 + \cdots + p_n^2}$, $\gamma \in \mathbb{R}$, and $V(q)$ is a potential. The Hamiltonian function h describes a mechanical system consisting of a particle under the influence of a potential $V(q)$ and with a friction force proportional to the momenta. The integral curves of the contact Hamiltonian vector field, X_h, satisfy the system of equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{p_i}{m}\,, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial V}{\partial q^i}(q) - \gamma p_i\,, \quad \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{p^2}{2m} - V(q) - \gamma s\,, \quad i = 1,\ldots,n\,.$$

Combining the first two equations, one gets

$$m\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} + \gamma m\frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{\partial V}{\partial q^i}(q) = 0\,, \qquad i = 1,\ldots,n\,.$$

Finally, let us recall that a contact manifold (M,η) gives rise to a Lie bracket [29]

$$\{f,g\} = X_f g + gRf = -\mathrm{d}\eta(X_f, X_g) - fRg + gRf\,, \qquad \forall f,g \in \mathscr{C}^\infty(M)\,. \tag{2.2}$$

It can be proved that the map $f \in \mathscr{C}^\infty(M) \mapsto X_f \in \mathfrak{X}_{\mathrm{ham}}(M)$ is a Lie algebra isomorphism. Using (2.1), one can prove that

$$\{f, gh\} = h\{f,g\} + g\{f,h\} + ghRf\,, \qquad \forall f,g,h \in \mathscr{C}^\infty(M)\,.$$

Hence, (2.2) is a Poisson bracket if and only if $R = 0$. Thus, contact manifolds are Jacobi manifolds but not Poisson ones. Nevertheless, if $\mathscr{C}_g^\infty(M)$ stands for the space of good Hamiltonian functions, the restriction of $\{\cdot,\cdot\}$ to $\mathscr{C}_g^\infty(M)$ becomes a Poisson bracket. In particular, since R is the Hamiltonian vector field of the constant function -1, it follows from the Jacobi identity for $\{\cdot,\cdot\}$ that the Lie bracket of two good Hamiltonian functions is a good one.

The formalism presented in this section has a Lagrangian counterpart [41, 68]. In addition, a geometric formulation for time-dependent contact systems developing the so-called cocontact geometry has been introduced in [25, 70].

2.2. Contact manifolds and other geometric structures

Let us study several geometric structures used to describe particular aspects of contact manifolds. In particular, we will also show why, although contact manifolds can be described via some other structures, such approaches are not appropriate for our purposes in this work.

Definition 2.4. A *Jacobi manifold* is a triple (M,Λ,E), where Λ is a bivector field on M, i.e. a skew-symmetric 2-contravariant tensor field, and E is a vector field on M, such that

$$[\Lambda,\Lambda] = 2E\wedge\Lambda\,,\qquad \mathscr{L}_E\Lambda = [E,\Lambda] = 0\,,$$

where $[\cdot,\cdot]$ denotes the Schouten–Nijenhuis bracket in its original sign convention[3] [62, 71, 79].

Remark 2.5. *Poisson manifolds* are equivalent to Jacobi manifolds with $E=0$. In turn, Poisson manifolds retrieve, as particular cases, symplectic and cosymplectic manifolds [16, 79].

Every bivector field Λ on M induces a vector bundle morphism $\Lambda^\sharp:\mathrm{T}^*M\to\mathrm{T}M$ given by $\Lambda^\sharp(\vartheta_x)=\Lambda_x(\vartheta_x,\cdot)$ for every $\vartheta_x\in\mathrm{T}^*_xM$ and $x\in M$.

A *Hamiltonian vector field* relative to (M,Λ,E) is a vector field X on M of the form

$$X = \Lambda^\sharp(\mathrm{d}f) + fE\,,$$

for a function $f\in\mathscr{C}^\infty(M)$, which is called a *Hamiltonian function* of X. It can be proved that if $E_x\notin\mathrm{Im}\,\Lambda^\sharp_x$ at every point $x\in M$, then each Hamiltonian vector field has a unique Hamiltonian function. Additionally, X is called a *good Hamiltonian vector field* if it admits a Hamiltonian function f satisfying $Ef=0$.

The *characteristic distribution* of (M,Λ,E) is the distribution [79] on M of the form

$$\mathcal{C}_x = \mathrm{Im}\Lambda^\sharp_x + \langle E_x\rangle,\qquad \forall x\in M\,.$$

Note that $\mathcal{C}$ need not have constant rank but it is integrable. The restriction of Λ to an even-dimensional maximal integral submanifold of $\mathcal{C}$ gives rise to a *locally conformally symplectic form*, while the restriction to an odd-dimensional maximal integral submanifold of $\mathcal{C}$ gives rise to a contact manifold [79]. Recall that a contact manifold (M,η) with Reeb vector field R gives rise to a Jacobi manifold $(M,\Lambda,-R)$, where Λ is the bivector field such that $\Lambda^\sharp(\alpha)=\sharp\alpha-(\iota_R\alpha)R$ for every $\alpha\in\mathrm{T}^*M$ for the isomorphism $\sharp=\flat^{-1}:\mathrm{T}^*M\to\mathrm{T}M$ (see [29, 41]). Moreover, every Jacobi manifold (M,Λ,E) gives rise to a Jacobi bracket given by

$$\{f,g\} = \Lambda(\mathrm{d}f,\mathrm{d}g) + fEg - gEf\,,\qquad \forall f,g\in\mathscr{C}^\infty(M)\,.$$

This bracket is not a Poisson bracket in general. Moreover, $\{\cdot,\cdot\}$ becomes a Poisson bracket when restricted to the space of good Hamiltonian functions, $\mathscr{C}^\infty_g(M)$, of the Jacobi manifold (M,Λ,E).

In particular, the Jacobi bracket satisfies

$$\{f,g\} = X_fg - gEf\,,$$

[3] There exists a modern, and sometimes more appropriate, definition of the Schouten–Nijenhuis bracket that differs from ours on a global proportional sign depending on the degree of Λ (see example 2.20 in [50] and references therein).

and it matches the definition of the Lie bracket for contact manifolds when (M,η) is such that

$$\Lambda(\mathrm{d}f,\mathrm{d}g) = -\mathrm{d}\eta(\Lambda^\sharp(\mathrm{d}f),\Lambda^\sharp(\mathrm{d}g))\,, \qquad E = -R\,.$$

The space $\mathfrak{X}_{\mathrm{ham}}(M)$ of Hamiltonian vector fields in a Jacobi manifold is a Lie algebra with respect to the Lie bracket of vector fields. More precisely, if $X_f, X_g \in \mathfrak{X}(M)$ are the Hamiltonian vector fields related to two arbitrary functions $f,g \in \mathscr{C}^\infty(M)$ respectively, one has

$$[X_f, X_g] = X_{\{f,g\}}\,.$$

2.3. Reduction of contact manifolds

Let us describe the contact Marsden–Weinstein reduction theory [81]. We will highlight some unexpected facts about momentum mappings for contact manifolds, which make them special and it will have important consequences hereafter.

Definition 2.6. Let $\Phi : G\times M \to M$ be a Lie group action preserving the contact form of (M,η), i.e. $\Phi_g^*\eta = \eta$ for every $g\in G$. We call Φ a *contact Lie group action*. A *contact momentum map* associated with Φ is a map $J\colon M\to\mathfrak{g}^*$ defined by

$$\langle J(x),\xi\rangle = \iota_{\tilde{\xi}_x}\eta_x\,, \qquad \forall x\in M\,,$$

where $\tilde{\xi}\in\mathfrak{X}(M)$ is the fundamental vector field[4] corresponding to $\xi\in\mathfrak{g}$.

Note that a contact Lie group action has a unique momentum map. In contrast to momentum maps in symplectic manifolds [2], the contact momentum map is always Ad-equivariant, where Ad refers to the adjoint action $\mathrm{Ad} : G\times\mathfrak{g}\to\mathfrak{g}$ of a Lie group G on its Lie algebra $\mathfrak{g}$. In other words, $J\circ\Phi_g = \mathrm{Ad}^T_{g^{-1}}J$ for every $g\in G$ [45]. The momentum map J gives rise to a comomentum map $\lambda\colon \xi\in\mathfrak{g}\mapsto J_\xi\in\mathscr{C}^\infty(M)$ defined by $J_\xi(x) = \langle J(x),\xi\rangle$ for every $x\in M$.

Proposition 2.7 (See [81, proposition. 3.1] for details). *Let $\Phi : G\times M\to M$ be a proper contact Lie group action relative to (M,η). Consider its associated contact momentum map $J\colon M\to\mathfrak{g}^*$. Then,*

(1) The level sets of the momentum map J are invariant under the action of the flow of the Reeb vector field of (M,η).
(2) For every $x\in M$, $v\in \mathrm{T}_xM$, and $\xi\in\mathfrak{g}$, one has

$$\mathrm{d}J_\xi = -\iota_{\tilde{\xi}}\mathrm{d}\eta\,.$$

(3) If $J(x)=0$, we have that $\mathrm{T}(G\cdot x)$ is an isotropic subspace of the symplectic vector space $(\ker\eta_x, \mathrm{d}_x\eta|_{\ker\eta_x})$.
(4) $(\mathrm{Im}\,\mathrm{T}_xJ)^\circ = \{\xi\in\mathfrak{g}\mid\tilde{\xi}_x\in\ker\mathrm{d}_x\eta\}$.

Note that the fundamental vector fields of a contact Lie group action have Hamiltonian functions that are first integrals of the Reeb vector field. This fact is relevant to prove the following proposition.

[4] We define the fundamental vector field of $\Phi : G\times M\to M$ associated with $\xi\in\mathfrak{g}$ as

$$\xi_M(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Phi(\exp(t\xi),x)\,, \quad \forall x\in M\,.$$

.

Proposition 2.8. *Let $J : M \to \mathfrak{g}^*$ be a contact momentum map relative to (M,η) for a contact Lie group action $\Phi : G \times M \to M$. Then, the mapping $\xi \in \mathfrak{g} \mapsto J_\xi \in \mathscr{C}^\infty_g(M)$ is a Lie algebra morphism. Moreover,*

$$J : x \in M \longmapsto J(x) \in \mathfrak{g}^*$$

induces a Poisson algebra morphism $J^* : f \in \mathscr{C}^\infty(\mathfrak{g}^*) \mapsto f \circ J \in \mathscr{C}^\infty_g(M)$ relative to the Kirillov–Kostant–Souriau bracket on $\mathfrak{g}^*$.

Proof. Taking into account that $RJ_\xi = 0$ for every $\xi \in \mathfrak{g}$, we have, for an arbitrary $\nu \in \mathfrak{g}$, that

$$\iota_{\tilde{\xi}} \mathrm{d}J_\nu = -\iota_{\tilde{\xi}}(\iota_{\tilde{\nu}}\mathrm{d}\eta - (RJ_\nu)\eta) = \mathrm{d}\eta(\tilde{\xi},\tilde{\nu}) = -\{J_\xi, J_\nu\} - J_\xi RJ_\nu + J_\nu RJ_\xi = -\{J_\xi, J_\nu\}\,.$$

On the other hand, since Φ is Ad-equivariant, one has

$$\iota_{\tilde{\xi}} \mathrm{d}J_\nu = \tilde{\xi} J_\nu = -\langle J, [\xi,\nu]\rangle = -J_{[\xi,\nu]}\,, \qquad \forall \xi,\nu \in \mathfrak{g}\,,$$

which shows that $\xi \in \mathfrak{g} \mapsto J_\xi \in \mathscr{C}^\infty(M)$ is a Lie algebra morphism.

Since the Kirillov–Kostant–Souriau bracket is a Poisson bracket and the space of good Hamiltonian functions relative to (M,η) is a Poisson algebra relative to the bracket (2.2), for all functions $f, g \in \mathscr{C}^\infty(\mathfrak{g}^*)$ and a basis $\{e_1,\ldots,e_r\}$ of $\mathfrak{g} \simeq \mathfrak{g}^{**}$, it follows that

$$\begin{aligned}\{f,g\}_{\mathfrak{g}^*} \circ J &= \left(\frac{\partial f}{\partial e_i} \circ J\right)\left(\frac{\partial g}{\partial e_j} \circ J\right)(\{e_i,e_j\}_{\mathfrak{g}^*} \circ J) = c_{ijk}\left(\frac{\partial f}{\partial e_i} \circ J\right)\left(\frac{\partial g}{\partial e_j} \circ J\right)(e_k \circ J) \\ &= \left(\frac{\partial f}{\partial e_i} \circ J\right)\left(\frac{\partial g}{\partial e_j} \circ J\right)\{e_i \circ J, e_j \circ J\} = \{f \circ J, g \circ J\},\end{aligned} \tag{2.3}$$

for $[e_i, e_j] = c_{ijk} e_k$ and $i, j = 1,\ldots,r$.

□

Assume that $h \in \mathscr{C}^\infty(M)$ is such that $\bar{\xi} h = 0$ for every $\xi \in \mathfrak{g}$, as frequently assumed in the symplectic Marsden–Weinstein reduction of Hamilton systems [61]. Note that

$$X_h J_\xi = \iota_{X_h} \mathrm{d}J_\xi = -\iota_{X_h}\iota_{\tilde{\xi}}\mathrm{d}\eta = \iota_{\tilde{\xi}}\iota_{X_h}\mathrm{d}\eta = \iota_{\tilde{\xi}}(\mathrm{d}h - (Rh)\eta) = -(Rh)J_\xi\,, \qquad \forall \xi \in \mathfrak{g}\,, \tag{2.4}$$

which means that the level sets of J, unlike in the symplectic case, are not conserved under the evolution of X_h. Nevertheless, X_h is tangent to $J^{-1}(\mathbb{R}^+\mu)$, provided it is a submanifold, since

$$\langle \mathscr{L}_{X_h} J, \xi\rangle = \langle -(Rh)J, \xi\rangle\,, \qquad \forall \xi \in \mathfrak{g}\,.$$

This shows that it is $J^{-1}(\mathbb{R}^+\mu)$ what plays a role in the contact Marsden–Weinstein reduction. More specifically, one can state the following definition and Marsden–Weinstein reduction theorem [81].

Definition 2.9. Let $\Phi : G \times M \to M$ be a proper contact Lie group action with respect to (M,η). Consider its associated contact momentum map $J\colon M \to \mathfrak{g}^*$ and $\mu \in \mathfrak{g}^*$. The *kernel group* of μ is the unique connected Lie subgroup of $K_\mu \subset G_\mu$ with Lie algebra $\mathfrak{k}_\mu = \ker \mu|_{\mathfrak{g}_\mu}$, where $\mathfrak{g}_\mu$ is the Lie algebra of the isotropy group G_μ of the point $\mu \in \mathfrak{g}^*$ relative to the coadjoint action of G on $\mathfrak{g}^*$. The *contact quotient*, or *contact reduction* of M by G at μ is

$$M_\mu = J^{-1}(\mathbb{R}^+\mu)/K_\mu\,.$$

The following theorem is a slightly different formulation of theorem 1 in [81], where M_μ was shown to be an orbifold [78] and η gave rise to a one-form η_μ on M_μ whose kernel is a contact distribution. If we assume that M_μ is a manifold, then η_μ becomes a contact form. In other words, we have the following theorem.

Theorem 2.10. *Let G be a Lie group acting by contactomorphisms with respect to (M,η), and let $J: M \to \mathfrak{g}^*$ be its associated contact momentum map. Let K_μ, with $\mu \in \mathfrak{g}^*$, be the connected Lie subgroup of G_μ with Lie algebra $\mathfrak{k}_\mu = \ker \mu|_{\mathfrak{g}_\mu}$. If*

(i) K_μ *acts properly on* $J^{-1}(\mathbb{R}^+\mu)$*,*
(ii) J *is transverse (see [2] for a definition) to* $\mathbb{R}^+\mu$*,*
(iii) $\ker\mu + \mathfrak{g}_\mu = \mathfrak{g}$*,*

then the quotient $M_\mu = J^{-1}(\mathbb{R}^+\mu)/K_\mu$, if a manifold, is naturally a contact manifold, i.e.

$$\ker\eta \cap \mathrm{T}\left(J^{-1}(\mathbb{R}^+\mu)\right)$$

gives rise to a contact manifold on the quotient M_μ relative to the unique contact one-form, η_μ, on M_μ such that $\pi_\mu^*\eta_\mu = \eta|_{J^{-1}(\mathbb{R}^+\mu)}$ for the canonical projection $\pi_\mu : J^{-1}(\mathbb{R}^+\mu) \to M_\mu$.

It is convenient to state the following contact Marsden–Weinstein reduction theorem for contact Hamiltonian systems. Although it follows immediately from previous comments and theories [3, 47, 49, 81], it seems to be absent in the literature.

Corollary 2.11. *Let the assumptions of theorem 2.10 hold. If (M,η,h) is a contact Hamiltonian system such that $\Phi_g^*h = h$ for every $g \in G$, then X_h is tangent to $J^{-1}(\mathbb{R}^+\mu)$. Moreover, $X_h|_{J^{-1}(\mathbb{R}^+\mu)}$ is projectable onto M_μ, where it becomes a Hamiltonian vector field relative to η_μ with Hamiltonian function $f \in \mathscr{C}^\infty(M_\mu)$ determined univocally by $\pi_\mu^* f = \iota_\mu^* h$, where $\iota_\mu : J^{-1}(\mathbb{R}^+\mu) \to M$ is the natural immersion into M.*

Proof. It follows from (2.4) that X_h is tangent to $J^{-1}(\mathbb{R}^+\mu)$. Since h, η, and $\mathrm{d}\eta$ are invariant relative to the Lie group action $\Phi: G \times M \to M$, i.e. $\Phi_g^*h = h$, $\Phi_g^*\eta = \eta$, and $\Phi_g^*\mathrm{d}\eta = \mathrm{d}\eta$ for every $g \in G$, then the pull-back relative to Φ_g^* of the equations

$$\iota_{X_h}\mathrm{d}\eta = \mathrm{d}h - (Rh)\eta\,, \qquad \iota_{X_h}\eta = -h\,, \tag{2.5}$$

imply that $\Phi_{g*}X_h = X_h$ for every $g \in G$. Hence, X_h is invariant relative to the action of the elements of $K_\mu \subset G$, and the restriction of X_h to $J^{-1}(\mathbb{R}^+\mu)$ can be projected onto M_μ. The expressions (2.5) are projectable and $\pi_{\mu*}X_h$ becomes a Hamiltonian vector field relative to the contact structure η_μ on M_μ with Hamiltonian function $f \in \mathscr{C}^\infty(M_\mu)$ such that $\pi_\mu^* f = \iota_\mu^* h$. □

3. Contact Lie systems

Let V be a Lie algebra with Lie bracket $[\cdot,\cdot]: V \times V \to V$. Given subsets $\mathcal{A},\mathcal{B} \subset V$, we write $[\mathcal{A},\mathcal{B}]$ for the real vector space generated by the Lie brackets between the elements of $\mathcal{A}$ and $\mathcal{B}$. Then, $\mathrm{Lie}(\mathcal{A}, V, [\cdot,\cdot])$, or simply $\mathrm{Lie}(\mathcal{A})$, stands for the smallest Lie subalgebra of V (in the sense of inclusion) containing $\mathcal{A}$.

A *time-dependent vector field* on M is a map $X: \mathbb{R} \times M \to \mathrm{T}M$ such that, for every $t \in \mathbb{R}$, the map $X_t = X(t,\cdot): M \to \mathrm{T}M$ is a vector field. In the applications of our methods, t will mainly refer to the physical time. Due to this, we will call X a *time-dependent vector field*. A time-dependent vector field X on M amounts to a one-parametric family of vector fields X_t on M with $t \in \mathbb{R}$. An *integral curve* of X is an integral curve, $\gamma: t \in \mathbb{R} \mapsto (t, x(t)) \in \mathbb{R} \times M$, of the *autonomisation* of X, namely $\partial/\partial t + X$, which is understood in the natural way as an element in $\mathfrak{X}(\mathbb{R} \times M)$. Every time-dependent vector field, X, on M gives rise to its referred to as *associated system* given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t,x)\,, \qquad \forall t \in \mathbb{R}\,, \quad \forall x \in M\,. \tag{3.1}$$

The curves $\gamma : t \in \mathbb{R} \mapsto (t, x(t)) \in \mathbb{R} \times M$, where $x(t)$ is a solution of the above system of differential equations, are the *integral curves* of X. Conversely, every time-dependent system of first-order ordinary differential equations in normal form in M, that is (3.1), describes the integral curves of a unique time-dependent vector field X on M. Hence, this allows us to identify X with its associated system, namely (3.1), and to use X to refer to both. Such a notation will not lead to any misunderstanding, as it will be clear from context what we mean by X in each case. The *smallest Lie algebra* of a time-dependent vector field X is the Lie algebra $V^X = \mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}})$. Every Lie algebra of vector fields V on M gives rise to an associated distribution on M of the form

$$\mathcal{D}^V_x = \{X_x : X \in V\}\,, \qquad \forall x \in M\,.$$

In particular, a time-dependent vector field X on M gives rise to an associated distribution, $\mathcal{D}^X$, given by $\mathcal{D}^X = \mathcal{D}^{V^X}$. It is worth noting that $\mathcal{D}^V$ does not need to have constant rank at every point of M, namely the subspaces $\mathcal{D}^V_x$ may have different dimension at different points $x \in M$.

A *Lie system* is a time-dependent vector field X on M whose smallest Lie algebra V^X is finite-dimensional [34]. If X takes values in a finite-dimensional Lie algebra of vector fields V, i.e. $\{X_t\}_{t\in\mathbb{R}} \subset V$, we call V a *VG Lie algebra* of X and it is said that X *admits a VG Lie algebra* V. Note that X can be considered as a curve $t \mapsto X_t$ in V. A time-dependent vector field X admits a VG Lie algebra if, and only if, V^X is finite-dimensional. An *automorphic Lie system* is a Lie system, X^G, on a Lie group G admitting a VG Lie algebra given by the space of right-invariant vector fields, $\mathfrak{X}_R(G)$, on G. A *locally automorphic Lie system* is a triple (M, X, V) such that V is a VG Lie algebra of X whose associated distribution, $\mathcal{D}^V$, is equal to $\mathrm{T}M$ and $\dim V = \dim M$ [52].

Example 3.1 (Riccati equations). Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_1(t) + a_2(t)x + a_3(t)x^2\,, \qquad \forall x \in \mathbb{R}, \quad \forall t \in \mathbb{R}, \tag{3.2}$$

where $a_1(t), a_2(t), a_3(t)$ are arbitrary time-dependent functions. System (3.2) is the system associated with the time-dependent vector field on $\mathbb{R}$ given by

$$X(t,x) = \sum_{\alpha=1}^{3} a_\alpha(t) X_\alpha(x)\,, \qquad \forall x \in \mathbb{R}\,, \quad \forall t \in \mathbb{R}\,,$$

where

$$X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = x\frac{\partial}{\partial x}\,, \quad X_3 = x^2\frac{\partial}{\partial x}$$

are vector fields on $\mathbb{R}$. Since

$$[X_1, X_2] = X_1\,, \quad [X_1, X_3] = 2X_2\,, \quad [X_2, X_3] = X_3\,,$$

it follows that X_1, X_2, X_3 span a Lie algebra isomorphic to $\mathfrak{sl}_2$. Thus, X defines a Lie system on $\mathbb{R}$ with VG Lie algebra $\langle X_1, X_2, X_3\rangle \simeq \mathfrak{sl}_2$. △

The main property of Lie systems is the so-called superposition rule [17, 82]. A *superposition rule* for a system X on M is a map $\Phi : M^k \times M \to M$ such that the general solution, $x(t)$, of X can be written in the form $x(t) = \Phi\left(x_{(1)}(t), \ldots, x_{(k)}(t); \rho\right)$, where $x_{(1)}(t), \ldots, x_{(k)}(t)$ is a generic family of particular solutions of X and ρ is any point in M related to the initial conditions of X. The Lie theorem [17, 20, 82] states that a system X admits a superposition rule if and only if it is a Lie system.

A *Lie–Hamilton system* is a triple (M, Λ, X), where X is a Lie system on M admitting a VG Lie algebra of Hamiltonian vector fields relative to a Poisson bivector Λ on M. If $\Lambda^\sharp$ is invertible, it gives rise to a symplectic form ω such that $\omega(v,\cdot) = (\Lambda^\sharp)^{-1}(v)$ for every $v \in \mathrm{T}M$, and we will sometimes denote (M,Λ,X) by (M,ω,X). Lie–Hamilton systems became relevant as symplectic and Poisson techniques were applied to determine superposition rules, Lie symmetries, constants of motion, and other properties in a simple way [34].

Finally, a *Jacobi–Lie system* is a quadruple (M,Λ,R,X), where X is a Lie system on M admitting a VG Lie algebra of Hamiltonian vector fields relative to a Jacobi manifold (M,Λ,E). We call *Jacobi–Lie Hamiltonian system* a quadruple (M,Λ,E,h), where (M,Λ,E) is a Jacobi manifold and $h:(t,x)\in\mathbb{R}\times M\mapsto h_t(x)\in\mathbb{R}$ is a time-dependent function such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}},\{\cdot,\cdot\})$ is a finite-dimensional Lie algebra relative to the Lie bracket $\{\cdot,\cdot\}$ associated with the Jacobi manifold (M,Λ,E). Given a system X on M, we say that X admits a *Jacobi–Lie Hamiltonian system* (M,Λ,E,h) if X_t is a Hamiltonian vector field with Hamiltonian function h_t (with respect to (M,Λ,E)) for each $t\in\mathbb{R}$ [4, 5, 34, 56]. We hereafter write $\mathrm{Cas}(M,\Lambda,E)$ the space of Hamiltonian functions related to a zero vector field with respect to a Jacobi manifold (M,Λ,E).

Definition 3.2. A *contact Lie system* is a triple (M,η,X), where η is a contact form on M and X is a Lie system on M whose smallest Lie algebra, V^X, is a finite-dimensional real Lie algebra of contact Hamiltonian vector fields relative to η. A contact Lie system is *of Liouville type* if the Hamiltonian functions of the vector fields in V^X are first integrals of the Reeb vector field of (M,η).

The term 'Liouville' refers to the fact that contact Lie systems of Liouville type satisfy an analogue of the Liouville theorem in symplectic geometry, as it will be proved in proposition 3.5. Note that a contact Lie system of Liouville type amounts to a contact Lie system (M,η,X) that is invariant relative to the flow of the Reeb vector field, R, of η, namely $\mathscr{L}_R X_t = 0$ for every $t\in\mathbb{R}$.

A Lie system X can be considered as a curve in V^X. In contact manifolds, every Hamiltonian vector field gives rise to a unique Hamiltonian function. Therefore, V^X gives rise to a linear space of functions $\mathfrak{W}$ and X defines a curve in $\mathfrak{W}$. Due to the isomorphism of Lie algebras between the space of Hamiltonian vector fields of (M,η) and $\mathscr{C}^\infty(M)$, it turns out that $\mathfrak{W}$ is a Lie algebra. This suggests us the following definition.

Definition 3.3. A *contact Lie–Hamiltonian system* is a triple (M,η,h) where $h:\mathbb{R}\times M\to\mathbb{R}$ gives rise to a time-parametrised family of functions $h_t: x\in M\mapsto h(t,x)\in\mathbb{R}$, with $t\in\mathbb{R}$, which is contained in a finite-dimensional Lie algebra of functions relative to the Lie bracket in $\mathscr{C}^\infty(M)$ induced by (M,η). We call h a *contact Lie–Hamiltonian* (relative to η).

Every contact Lie system gives rise a unique contact Lie–Hamiltonian system and conversely. If M and η are known from the context, h can be called a contact Lie–Hamiltonian system. Note that the fact that the functions $\{h_t\}_{t\in\mathbb{R}}$ span a finite-dimensional Lie algebra or not depends on η and, formally, a contact Lie–Hamiltonian system must be defined as a triple (M,η,h).

Example 3.4 (A simple control system). Consider the system of differential equations in $\mathbb{R}^3$ given by

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = b_1(t)\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b_2(t)\,, \\ \frac{\mathrm{d}z}{\mathrm{d}t} = b_2(t)x\,, \end{cases} \qquad \forall (x,y,z)\in\mathbb{R}^3, \tag{3.3}$$

where $b_1(t), b_2(t)$ are two arbitrary functions depending only on time. The relevance of this system is due to its occurrence in control problems [67].

System (3.3) describes the integral curves of the time-dependent vector field on $\mathbb{R}^3$ given by

$$X = b_1(t)X_1 + b_2(t)X_2\,, \tag{3.4}$$

where

$$X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\,.$$

The vector fields X_1, X_2, and $X_3 = \partial/\partial z$, span a three-dimensional VG Lie algebra $V = \langle X_1, X_2, X_3\rangle \simeq \mathfrak{h}_3$ of X, where $\mathfrak{h}_3$ is the so-called three-dimensional Heisenberg Lie algebra. Indeed, the commutations relations for X_1, X_2, X_3 read

$$[X_1, X_2] = X_3\,, \quad [X_1, X_3] = 0\,, \quad [X_2, X_3] = 0\,.$$

The vector fields X_1, X_2, X_3 are contact Hamiltonian vector fields with respect to the contact form on $\mathbb{R}^3$ given by

$$\eta_c = \mathrm{d}z - y\,\mathrm{d}x\,,$$

with Hamiltonian functions

$$h_1 = y\,, \quad h_2 = -x\,, \quad h_3 = -1\,,$$

respectively. It follows that all the elements of V^X are Hamiltonian vector fields relative to $(\mathbb{R}^3, \eta_c)$. Hence, the time-dependent Hamiltonian for (3.3) relative to $(\mathbb{R}^3, \eta_c)$ is given by $(\mathbb{R}^3, \eta_c, h)$ for

$$h_t = b_1(t)y - b_2(t)x\,, \qquad \forall t \in \mathbb{R}.$$

Thus, $(\mathbb{R}^3, \eta_c, X)$ is a contact Lie system. Since h_1, h_2, h_3 are first integrals of $X_3 = \partial/\partial z$, which is the Reeb vector field of η_c, then $(\mathbb{R}^3, \eta_c, X)$ is of Liouville type. In fact, $[X_3, X_t] = 0$ for every $t \in \mathbb{R}$.

Note that η_c gives rise to a volume form $\Omega_{\eta_c} = \eta_c \wedge \mathrm{d}\eta_c$ on $\mathbb{R}^3$. Moreover the evolution of (3.3) leaves Ω_{η_c} invariant. △

Let us study the behaviour of the volume form, $\Omega_\eta = \eta \wedge (\mathrm{d}\eta)^n$, induced by a $(2n+1)$-dimensional contact manifold (M, η) relative to the dynamics of a contact Lie system (M, η, X).

Proposition 3.5. *Let (M, η, X) be a contact Lie system on a $(2n+1)$-dimensional contact manifold (M, η) and let $\Omega_\eta = \eta \wedge (\mathrm{d}\eta)^n$. Then,*

$$\mathscr{L}_{X_t}\Omega_\eta = 0\,, \qquad \forall t \in \mathbb{R}$$

if and only if (M, η, X) is of Liouville type.

Proof. Assume that (M, η, X) is of Liouville type. The vector fields of the smallest Lie algebra, V^X, are of the form X_f for a certain $f \in \mathscr{C}^\infty(M)$ such that $Rf = 0$. Then,

$$\begin{aligned}\mathscr{L}_{X_f}\Omega_\eta &= \mathscr{L}_{X_f}(\eta \wedge (\mathrm{d}\eta)^n)\\ &= (\mathscr{L}_{X_f}\eta) \wedge (\mathrm{d}\eta)^n + n\eta \wedge (\mathrm{d}\mathscr{L}_{X_f})\eta \wedge (\mathrm{d}\eta)^{n-1} = -(n+1)(Rf)\Omega_\eta\,,\end{aligned} \tag{3.5}$$

since $\mathscr{L}_{X_f}\eta = -(Rf)\eta$. As $Rf = 0$, and $X = \sum_{\alpha=1}^r b_\alpha(t)X_\alpha$ for some basis $X_1, \ldots, X_r$ of V^X and functions $b_1(t), \ldots, b_r(t)$, then $\mathscr{L}_{X_t}\Omega_\eta = 0$.

Conversely, if $\mathscr{L}_{X_t}\Omega_\eta = 0$ for every $t \in \mathbb{R}$, then the functions $h_t = -\iota_{X_t}\eta$, with $t \in \mathbb{R}$, satisfy that

$$0 = \mathscr{L}_{X_t}\Omega_\eta = -(n+1)(Rh_t)\Omega_\eta, \qquad \forall t \in \mathbb{R},$$

which implies that $Rh_t = 0$ for every $t \in \mathbb{R}$. Moreover X admits a unique time-dependent Hamiltonian function $h = \sum_{\alpha=1}^{r} b_\alpha(t) h_\alpha$, where $h_1, \ldots, h_r$ are some linearly independent Hamiltonian functions whose Hamiltonian vector fields belong to the linear space $\langle X_t \rangle_{t\in\mathbb{R}} \subset V^X$, where the span is over the reals, and the $b_1(t), \ldots, b_r(t)$ are linearly independent too. Note that $X_{h_1}, \ldots, X_{h_r}$ do not need to span the full V^X, while $\{h_1, \ldots, h_r\}$ is a basis of the linear span, over the reals, of the form $\langle h_t \rangle_{t\in\mathbb{R}} = \mathcal{W}$. Let us assume, without loss of generality, that $h_1, \ldots, h_s$ span the space of good Hamiltonian functions in $\mathcal{W}$. The condition $Rh_t = 0$ for every $t \in \mathbb{R}$ implies that $\sum_{\alpha=s+1}^{r} b_\alpha(t) Rh_\alpha = 0$. Then, $Rh_\alpha = 0$ for $\alpha = s+1, \ldots, r$. This implies that $r = s$ and $\mathcal{W}$ is spanned by first integrals of R. Moreover, $\{h_i, h_j\}$, with $i, j = 1, \ldots, r$, and their successive Lie brackets will also be first integrals of R. Hence, the Lie algebra of Hamiltonian functions of the vector fields of V^X consists only of first integrals of the Reeb vector field R. Hence, (M, η, X) is a contact Lie system of Liouville type. □

Note that the space of Hamiltonian vector fields on M relative to (M, η) and admitting a Hamiltonian function being a first integral of R is a Lie subalgebra of $\mathfrak{X}_{\mathrm{ham}}(M)$.

Theorem 3.6 (Gromov's non-squeezing theorem). *Let (M, ω) be a symplectic manifold and let $\{q^1, \ldots, q^n, p_1, \ldots, p_n\}$ be Darboux coordinates for ω on an open subset $U \subset M$, i.e. $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$. Given the set of points*

$$B(r) = \left\{ (q,p) \in U : \sum_{i=1}^{n} \left[(q^i - q_0^i)^2 + (p_i - p_i^0)^2 \right] \leqslant r^2 \right\},$$

where $(q_0^1, \ldots, q_0^n, p_1^0, \ldots, p_n^0) \in U$ and $r \in \mathbb{R}_+$, if the image of $B(\mathrm{r})$ under a symplectomorphism $\phi : M \to M$ is such that $\phi(B(r)) \subset C_\rho$, where

$$C_\rho = \left\{ (q,p) \in U : (q^1 - q_0^1)^2 + (p_1 - p_1^0)^2 \leqslant \rho^2 \right\},$$

for $\rho \in \mathbb{R}^+$, then $r \leqslant \rho$.

Our interest in the Gromov's non-squeezing theorem is due to the fact that it applies to the Hamiltonian system relative to a symplectic form appearing as the projection of a contact Lie system of Liouville type (M, η, X) onto the space of integral submanifolds of R in M, let us say M/R, if the latter admits a manifold structure [2]. In other words, one has the following.

Proposition 3.7 (Non-squeezing contact theorem). *Let (M, η, X) be a contact Lie system of Liouville type. Given a family of Darboux coordinates $\{q^1, \ldots, q^n, p_1, \ldots, p_n, z\}$ for η on an open subset $U \subset M$, i.e. $\eta = \mathrm{d}z - p_i \mathrm{d}q^i$, and the set of points*

$$B(r) = \left\{ (q,p) \in U : \sum_{i=1}^{n} \left[(q^i - q_0^i)^2 + (p_i - p_i^0)^2 \right] \leqslant r^2 \right\},$$

where $(q_0^1, \ldots, q_0^n, p_1^0, \ldots, p_n^0, z) \in U$ and $r \in \mathbb{R}^+$, if the image of $B(\mathrm{r})$ under a symplectomorphism $\phi : M \to M$ is such that $\phi(B(r)) \subset C_\rho$, where

$$C_\rho = \left\{ (q,p) \in U : (q^1 - q_0^1)^2 + (p_1 - p_1^0)^2 \leqslant \rho^2 \right\},$$

for $\rho \in \mathbb{R}^+$, then $r \leqslant \rho$.

Proof. Since X is a Lie system of Liouville type, it follows that $[R, X_t] = 0$ for every $t \in \mathbb{R}$. Hence, X can be projected onto a time-dependent vector field Y_t on M/R. Moreover, $\iota_{X_t}\mathrm{d}\eta = \mathrm{d}h_t$ for every $t \in \mathbb{R}$. Since X_t, $\mathrm{d}\eta$ and the $\{h_t\}_{t\in\mathbb{R}}$ are invariant relative to R, corollary 2.11 implies that each vector field Y_t, with $t \in \mathbb{R}$, on M_μ is Hamiltonian relative to the symplectic manifold $(M/R, \omega)$, where ω is the only two-differential form on M_μ such that $\pi_R^*\omega = \mathrm{d}\eta$ for the canonical projection $\pi_R : M \to M/R$. The coordinates $\{q^1, \ldots, q^n, p_1, \ldots, p_n\}$ of a Darboux coordinate system $\{q^1,\dots,q^n,p_1,\dots,p_n,z\}$ on M can be considered as the pull-back to M via π_R of a Darboux coordinate system $\{q^1,\dots,q^n,p_1,\dots,p_n\}$ on $(M/R,\omega)$, which are denoted in the same manner for simplicity. Then, one can apply the non-squeezing Gromov theorem to Y on $(M/R,\omega)$, which gives rise to the statement of our contact non-squeezing theorem for contact Lie systems of Liouville type. □

3.1. Contact Lie systems and other classes of Lie systems

Recall that Lie–Hamilton systems are Lie systems admitting a VG Lie algebra of Hamiltonian vector fields relative to a Poisson bivector. They were the first studied type of Lie systems admitting a VG Lie algebra of Hamiltonian vector fields relative to a geometric structure [18, 19]. Despite that, they were insufficient for studying many types of Lie systems [34]. Let us study why contact Lie systems are interesting on their own and their relations to other types of Lie systems. Let us start by the next proposition, which is a no-go result for the existence of a Poisson bivector turning the vector fields of a VG Lie algebra of a Lie system into Hamiltonian vector fields. It is indeed a version of proposition 5.1 in [23].

Proposition 3.8. *If (M,Λ,X) is a Lie–Hamilton system and $\mathcal{D}^X = \mathrm{T}M$, then M is even-dimensional.*

Proof. Since (M,Λ,X) is a Lie–Hamilton system, the vector fields of V^X are Hamiltonian relative to Λ and $\mathcal{D}^X = \mathrm{T}M$ is spanned by Hamiltonian vector fields. Hence, the characteristic distribution of Λ, which is spanned by all Hamiltonian vector fields and has even rank at every point [79], must be $\mathrm{T}M$. Hence, $\mathrm{T}M$ has even rank and M is even dimensional. □

Proposition 3.8 shows that Lie–Hamilton systems are not appropriate to describe Lie systems admitting certain smallest Lie algebras. Note that, for instance, example 3.4 describes a Lie system whose smallest Lie algebra satisfies the conditions of proposition 3.8 when the vectors $(b_1(t),b_2(t))$, with $t\in\mathbb{R}$, span $\mathbb{R}^2$ and, therefore, $V^X = \langle X_1,X_2,X_3\rangle$ while $\mathcal{D}^X = \mathrm{T}\mathbb{R}^3$. This illustrates the need for describing Lie systems admitting VG Lie algebras of Hamiltonian vector fields relative to other geometric structures, like contact manifolds.

The following proposition shows how contact Lie systems of Liouville type induce some Lie–Hamilton systems on other spaces.

Proposition 3.9. *If (M,η,X) is a contact Lie system of Liouville type, the space of integral curves of the Reeb vector field R, let us say M/R, is a manifold, and $\pi_R : M\to M/R$ is the canonical projection, then $(M/R,\omega,\pi_*X)$, where ω is the only differential form on M/R such that $\pi_R^*\omega = \mathrm{d}\eta$, is a Lie–Hamilton system relative to the symplectic form ω on M/R.*

Proof. Since (M,η,X) is of Liouville type, the Lie derivative of the Reeb vector field R with the Hamiltonian vector fields of V^X is zero. Therefore, all the elements of V^X are projectable onto M/R. Moreover, $\mathscr{L}_R\mathrm{d}\eta = 0$ and $\iota_R\mathrm{d}\eta = 0$. Hence, $\mathrm{d}\eta$ can be projected onto M/R. In other words, there exists a unique two-form, ω, on M/R such that $\pi_R^*\omega = \mathrm{d}\eta$. Note that ω is closed. Moreover, if $\iota_{Y_{[x]}}\omega_{[x]} = 0$ for a tangent vector $Y_{[x]}\in\mathrm{T}_{[x]}(M/R)$, then there exists a tangent vector $\widetilde{Y}_x\in\mathrm{T}_xM$ projecting onto $Y_{[x]}$ via $\mathrm{T}_x\pi_R$. Then, $\mathrm{T}_x\pi_R^T\iota_{Y_{[x]}}\omega_{[x]} = \iota_{\widetilde{Y}_x}(\mathrm{d}\eta)_x = 0$. Hence, $\widetilde{Y}_x$ takes values in the kernel of $(\mathrm{d}\eta)_x$ and it is proportional to R_x. Hence, $\pi_{R*x}Y_x = 0$ and ω is non-degenerate. Since ω is closed, it becomes a symplectic form and the vector fields of $\pi_{R*}V^X$ span a finite-dimensional Lie algebra of Hamiltonian vector fields relative to ω. Therefore, the time-dependent vector field $\pi_{R*}X$, namely the t-parametric family of vector fields $(\pi_{R*}X)_t = \pi_{R*}X_t$ for every $t\in\mathbb{R}$, becomes a Lie–Hamilton system relative to ω. □

Since the vector fields of a VG Lie algebra of a contact Lie system are Hamiltonian vector fields relative to its associated Jacobi manifold, one may ask whether contact Lie systems are interesting on its own. There are several reasons for their study. For instance, contact structures have particular properties that are not shared by general Jacobi manifolds and they are specific. For example, every Hamiltonian function determines a unique Hamiltonian vector field and conversely, which make some results more specific, e.g. every contact Lie system admits a contact Lie–Hamiltonian system. Moreover, as we focus on contact Lie systems on contact manifolds admitting a contact form, several results related to the contact form are available.

Proposition 3.10. *Every contact Lie system (M,η,X_h) gives rise to a Lie–Hamilton system $(\mathbb{R}\times M, e^{-s}(\mathrm{d}\eta+\eta\wedge\mathrm{d}s), -(Rh)\partial/\partial s+X_h)$, where* s *is the natural variable in $\mathbb{R}$ understood as a variable in $\mathbb{R}\times M$ in the natural way.*

Proposition 3.10 may be inappropriate to study contact Hamiltonian systems on M via Hamiltonian systems on symplectic manifolds. This is due to the fact that the dynamics of a contact Hamiltonian vector field on M may significantly differ from the Hamiltonian system on $\mathbb{R}\times M$ used to study it. For example, a contact Hamiltonian vector field X on M may have stable points, while $-(Rh)\partial/\partial s+X_h$, which is its associated Hamiltonian vector field on $\mathbb{R}\times M$, has not. This has relevance in certain theories, like the energy–momentum method [61]. Moreover, every contact Lie system can be understood as the projection of a Lie–Hamilton system on a homogeneous symplectic manifold (see [47]). Anyhow, the latter approach is not appropriate for our purposes for a number of reasons, e.g. considering Lie systems on manifolds of larger dimension may make the study of contact Lie systems harder to solve. Examples of this problem will be given in section 4.

Finally, let us recall that a multisymplectic Lie system is triple (M,Θ,X), where X is a Lie system on M admitting a VG Lie algebra of Hamiltonian vector fields relative to the multisymplectic form Θ on M (see [52, 54] for details). The following proposition, whose proof is immediate, relates contact Lie systems of Liouville type to multisymplectic Lie systems.

Corollary 3.11. *If (M,η,X) is a contact Lie system of Liouville type, then (M,Ω_η,X) is a multisymplectic Lie system.*

Multisymplectic techniques significantly differ from contact ones and, although the above result has several applications [54], the contact approach may be more interesting, for instance, to use reduction techniques for Lie systems, since there exists no general multisymplectic reduction so far.

4. Existence of invariant contact forms for Lie systems

Let us analyse the existence of contact forms turning the elements of a VG Lie algebra into good Hamiltonian vector fields. Our results will help us to determine Lie systems that can be considered as contact Lie systems of Liouville type. In particular, the classification of automorphic Lie systems on three-dimensional Lie groups with a non-abelian Lie algebra and admitting a left-invariant contact form will be given.

Lemma 4.1. *Let (M,X,V^X) be a locally automorphic Lie system. If η is a differential form on M such that $\mathscr{L}_Y\eta=0$ for every $Y\in V^X$, then the value of η at a point of M determines the value of η on the whole M.*

Proof. Let $x \in M$ be a fixed arbitrary point and let $m = \dim M$. Since the vector fields in V^X span the distribution $\mathrm{T}M$, it follows from the Orbit theorem [76, theorem 4.1] that x can be connected to any other point $y \in M$ by a local diffeomorphism of the form

$$\phi_{xy} = \exp(t_1 X_{i_1}) \circ \exp(t_2 X_{i_2}) \circ \cdots \circ \exp(t_k X_{i_k}), \tag{4.1}$$

where $k \in \mathbb{N}$ is a natural number or zero, $\exp(tY)$ is the t-parametric group of local diffeomorphisms related to any $Y \in \mathfrak{X}(M)$, the vector fields $X_1, \ldots, X_m$ form a basis (over the reals) of V^X, and $t_1, \ldots, t_k \in \mathbb{R}$ while $i_1, \ldots, i_k \in \{1, \ldots, m\}$. Since $\mathscr{L}_Y \eta = 0$ for every $Y \in V^X$ and due to (4.1), it follows that $\phi_{xy}^* \eta_y = \eta_x$ and the value of η_y is determined by η_x. □

Proposition 4.2. *Let (M, X, V^X) be a locally automorphic Lie system on an odd-dimensional manifold. Then, there exists a bijection between the space $\mathcal{C}$ of contact forms turning the elements of V^X into good Hamiltonian vector fields and the one-chains, ϑ, of the Chevalley–Eilenberg cohomology of $\mathfrak{g}$ isomorphic to V^X such that $\vartheta \wedge (\delta\vartheta)^k$ is a non-zero $(2k+1)$-covector with* $\dim M = 2k+1$.

Proof. By lemma 4.1 and our assumptions, a contact form on M is determined by its value at one point $x \in M$. Every locally automorphic Lie system (M, V^X, X) is locally diffeomorphic to an automorphic Lie system (see [52] for details). In our case, there exists a local diffeomorphism $\varphi : G \to M$ such that X is φ-related to a Lie system on a Lie group G with Lie algebra $\mathfrak{g}$ so that

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R(g), \qquad \forall g \in G, \tag{4.2}$$

for a basis of right-invariant vector fields $\{X_1^R, \ldots, X_r^R\}$ on G and some time-dependent functions $b_1(t), \ldots, b_r(t)$. Since V^X is the smallest Lie algebra containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$, while $\mathcal{D}^{V^X} = \mathrm{T}M$, and (4.2) is locally diffeomorphic to X, it follows that the smallest Lie algebra of (4.2) is $\langle X_1^R, \ldots, X_r^R \rangle$, which spans $\mathrm{T}G$. Since the elements of V^X are good Hamiltonian vector fields, the contact form η for X satisfies that $\mathscr{L}_Y \eta = 0$ for every $Y \in V^X$. Then, the local diffeomorphism φ maps the invariant contact form for X to a left-invariant contact form η^L for (4.2).

If η^L is a left-invariant contact form on G, then $\eta^L \wedge (\mathrm{d}\eta^L)^k$ is a volume form on G for $2k+1 = \dim G = \dim V^X = \dim M$. Moreover,

$$\mathrm{d}\eta^L(X_i^L, X_j^L) = -\eta^L([X_i^L, X_j^L]), \qquad i,j = 1, \ldots, r.$$

Define $\delta : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ to be minus the transpose of $[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$. On the other hand, $\eta^L \wedge (\mathrm{d}\eta^L)^k$ being a volume form amounts to the fact that its value at the neutral element e is different from zero. But $\eta_e^L \wedge (\mathrm{d}\eta^L)_e^k = \eta_e^L \wedge (\delta\eta_e^L)^k$. Setting $\eta_e^L = \vartheta$, the results follows. □

The conditions in proposition 4.2 can be verified for every automorphic Lie system on a three-dimensional Lie group with a smallest Lie algebra given by the right-invariant vector fields on the Lie group, as their Lie algebras are completely classified. The abelian case is immediate: every left-invariant one-form has zero differential and no left-invariant contact form exists. Moreover, every real non-abelian three-dimensional Lie algebra is isomorphic to $(E, [\cdot,\cdot])$, where E is a three-dimensional vector space and the Lie bracket is given on a canonical basis $\{e_1, e_2, e_3\}$ of E by one of the cases in table 1 (see e.g. [36, 38]).

Despite the interest of associating contact manifolds with symplectic ones of larger dimension to study global geometric properties of the former (as done in [8, 47]), the idea is not, in many cases, appropriate to study contact Hamiltonian systems. For instance, we already commented after proposition 3.10 that such methods relate Hamiltonian vector fields on a contact manifold with equilibrium points to new ones in a symplectic manifold without them, which radically changes their stability properties (see [47, 74]). In our case now, it is not useful to relate contact Lie systems on three-dimensional Lie groups to Hamiltonian Lie systems in four-dimensional manifolds because the classification problem of the latter is much harder. This is due two facts: to the larger dimension of the related manifold and the need for relating the obtained classification with the initial, searched, one.

Let us now classify left-invariant contact forms for automorphic Lie systems on three-dimensional Lie groups Lie algebras $\mathfrak{sl}_2$, $\mathfrak{r}_{3,\lambda}$, and $\mathfrak{r}'_{3,\lambda\neq 0}$. More specifically, we will study the conditions required for $\vartheta=\lambda_1 e^1+\lambda_2 e^2+\lambda_3 e^3\in \mathrm{T}^*_eG\simeq\mathfrak{g}^*$, where $\{e^1,e^2,e^3\}$ is the dual basis to the basis $\{e_1,e_2,e_3\}$ of T_eG and $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$, to be the value of a left-invariant contact form at the neutral element of a non-abelian three-dimensional Lie groups related to $\mathfrak{sl}_2$, $\mathfrak{r}_{3,\lambda}$.

- Case $\mathfrak{sl}_2$: The corresponding Lie bracket is an skew-symmetric bilinear function that can be understood univocally as a mapping $[\cdot,\cdot]:\mathfrak{sl}_2\wedge\mathfrak{sl}_2\to\mathfrak{sl}_2$. Defining the map $\delta:\mathfrak{sl}_2^*\to\mathfrak{sl}_2^*\wedge\mathfrak{sl}_2^*$ as $\delta=-[\cdot,\cdot]^T$, we have

$$\delta(e^1)=-e^1([\cdot,\cdot])=\frac{1}{2}e^3\wedge e^2\,,\quad \delta(e^2)=-e^2([\cdot,\cdot])=-\frac{1}{2}e^1\wedge e^2\,,\tag{4.3}$$

$$\delta(e^3)=-e^3([\cdot,\cdot])=\frac{1}{2}e^1\wedge e^3\,,\tag{4.4}$$

and thus,

$$\delta=\frac{1}{2}e_1\otimes e^3\wedge e^2-\frac{1}{2}e_2\otimes e^1\wedge e^2+\frac{1}{2}e_3\otimes e^1\wedge e^3\,.$$

In this case, $k=1$ and

$$\begin{aligned}0&\neq\delta(\lambda_1e^1+\lambda_2e^2+\lambda_3e^3)\wedge(\lambda_1e^1+\lambda_2e^2+\lambda_3e^3)\\&=\frac{1}{2}\left(\lambda_1e^3\wedge e^2-\lambda_2e^1\wedge e^2+\lambda_3e^1\wedge e^3\right)\wedge(\lambda_1e^1+\lambda_2e^2+\lambda_3e^3)\\&=-\frac{1}{2}\left(\lambda_1^2+2\lambda_2\lambda_3\right)e^1\wedge e^2\wedge e^3\,.\end{aligned}$$

Then, the differential one-form $\eta^L=\sum_{\alpha=1}^3\lambda_\alpha\eta^L_\alpha$ on $SL(2,\mathbb{R})$, where $\eta^L_\alpha(e)=e_\alpha$ for $\alpha=1,2,3$, is a contact form if and only if $\lambda_1^2+2\lambda_2\lambda_3\neq 0$.

- Case $\mathfrak{r}_{3,\lambda}$, with $\lambda\in(-1,1)$. As previously, define the map $\delta:\mathfrak{r}^*_{3,\lambda}\to\mathfrak{r}^*_{3,\lambda}\wedge\mathfrak{r}^*_{3,\lambda}$ as $\delta=-[\cdot,\cdot]^T$. Then,

$$\delta(e^1)=-e^1([\cdot,\cdot])=\frac{1}{2}e^1\wedge e^3\,,\quad \delta(e^2)=-e^2([\cdot,\cdot])=-\frac{\lambda}{2}e^3\wedge e^2\,,\tag{4.5}$$

$$\delta(e^3)=-e^3([\cdot,\cdot])=0\,,\tag{4.6}$$

and thus,

$$\delta=\frac{1}{2}e_1\otimes e^1\wedge e^3-\frac{1}{2}\lambda e_2\otimes e^3\wedge e^2\,.$$

Therefore,

$$\begin{aligned}0 &\neq \delta(\lambda_1 e^1 + \lambda_2\lambda e^2 + \lambda_3 e^3) \wedge (\lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 e^3) \\ &= \left(\frac{\lambda_1}{2} e^1 \wedge e^3 - \frac{\lambda_2\lambda}{2} e^3 \wedge e^2\right) \wedge (\lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 e^3) \\ &= \frac{1}{2}\lambda_1\lambda_2(1-\lambda) e^1 \wedge e^2 \wedge e^3 .\end{aligned}$$

Then, the left-invariant contact forms on a Lie group with Lie algebra isomorphic to $\mathfrak{r}_{3,\lambda}$ are characterised by the condition $\lambda_1\lambda_2 \neq 0$.

- Case $\mathfrak{r}'_{3,\lambda\neq 0}$. Defining the map $\delta : \mathfrak{r}'^{*}_{3,\lambda\neq 0} \to \mathfrak{r}'^{*}_{3,\lambda\neq 0} \wedge \mathfrak{r}'^{*}_{3,\lambda\neq 0}$ as $\delta = -[\cdot,\cdot]^T$, we have

$$\delta(e^1) = \frac{\lambda}{2} e^1 \wedge e^3 - \frac{1}{2} e^3 \wedge e^2 , \quad \delta(e^2) = -\frac{1}{2} e^1 \wedge e^3 - \frac{\lambda}{2} e^3 \wedge e^2 , \quad \delta(e^3) = 0 ,$$

and thus,

$$\delta = \frac{\lambda}{2} e_1 \otimes e^1 \wedge e^3 - \frac{1}{2} e_1 \otimes e^3 \wedge e^2 - \frac{1}{2} e_2 \otimes e^1 \wedge e^3 - \frac{\lambda}{2} e_2 \otimes e^3 \wedge e^2 .$$

In this case,

$$\begin{aligned}0 &\neq \delta(\lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 e^3) \wedge (\lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 e^3) \\ &= \left(\frac{\lambda\lambda_1}{2} e^1 \wedge e^3 - \frac{\lambda_1}{2} e^3 \wedge e^2 - \frac{\lambda_2}{2} e^1 \wedge e^3 - \frac{\lambda\lambda_2}{2} e^3 \wedge e^2\right) \wedge (\lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 e^3) \\ &= \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right) e^1 \wedge e^2 \wedge e^3 .\end{aligned}$$

Then, the differential one-form $\eta^L = \sum_{\alpha=1}^{3} \lambda_\alpha \eta^L_\alpha$ on each Lie group with Lie algebra $\mathfrak{r}'_{3,\lambda\neq 0}$, where $\eta^L_\alpha(e) = e^\alpha$, with $\alpha = 1,2,3$, is a contact form if and only if $\lambda_1^2 + \lambda_2^2 > 0$.

The other cases can be computed similarly, as summarised in the following theorem.

Theorem 4.3. *Let G be a Lie group with a three-dimensional non-abelian Lie algebra $\mathfrak{g}$ whose dual has a basis $\{e^1, e^2, e^3\}$. Then, the left-invariant one-form $\eta^L = \sum_{\alpha=1}^{3} \lambda_\alpha \eta^L_\alpha$ on G, where $(\eta^L_\alpha)_e = e^\alpha$ for $\alpha = 1,2,3$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, is a contact form if and only if the condition for the value of η^L_e in table 1 for the Lie algebra $\mathfrak{g}$ of G is satisfied.*

The following proposition takes a deeper look at the properties of left-invariant contact forms on Lie groups and show some of their properties. In particular, it proves that the space of left-invariant contact forms on a Lie group must be invariant under the natural action of $\mathrm{Aut}(G)$, namely the space of Lie group automorphisms of G, on $\mathfrak{g}^*$. Recall that $\mathrm{Aut}(G)$ acts on G, which gives rise to a Lie group action $(f, v) \in \mathrm{Aut}(G) \times \mathfrak{g} \mapsto \mathrm{T}_e f(v) \in \mathfrak{g}$ and its dual one on $\mathfrak{g}^*$.

Proposition 4.4. *Let $Aut(G)$ be the Lie group of Lie group automorphisms of G and let $\varphi : Aut(G) \times \mathfrak{g} \to \mathfrak{g}$ be its associated action on $\mathfrak{g}$. Then, the space $\mathcal{C}$ of left-invariant contact forms on G is invariant relative to the induced action of $Aut(G)$ on $\mathfrak{g}^*$.*

Proof. Let us prove that every $f \in \mathrm{Aut}(G)$ maps left-invariant one-forms on G into left-invariant one-forms on G. First,

$$f L_g(h) = f(gh) = f(g) f(h) = L_{f(g)} f(h), \qquad \forall g, h \in G, \qquad \forall f \in \mathrm{Aut}(G).$$

Table 1. Classification of left-invariant contact forms on three-dimensional Lie groups with non-abelian Lie algebras. Note that $\lambda \in (-1,1)$. The value of the left-invariant contact form at the neutral element e is of the form $\eta^L_e = \sum_{i=1}^3 \lambda_i e^i$ for the dual basis $\{e^1, e^2, e^3\}$ to the basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$.

Lie algebra	$[e_1, e_2]$	$[e_1, e_3]$	$[e_3, e_2]$	Contact condition
$\mathfrak{sl}_2$	e_2	$-e_3$	$-e_1$	$\lambda_1^2 + 2\lambda_2\lambda_3 > 0$
$\mathfrak{su}_2$	e_3	$-e_2$	$-e_1$	$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 > 0$
$\mathfrak{h}_3$	e_3	0	0	$\lambda_3 \neq 0$
$\mathfrak{r}'_{3,0}$	$-e_3$	e_2	0	$\lambda_2^2 + \lambda_3^2 > 0$
$\mathfrak{r}_{3,-1}$	e_2	$-e_3$	0	$\lambda_2\lambda_3 \neq 0$
$\mathfrak{r}_{3,1}$	e_2	e_3	0	$\nexists$
$\mathfrak{r}_3$	0	$-e_1$	$e_1 + e_2$	$\lambda_1 \neq 0$
$\mathfrak{r}_{3,\lambda}$	0	$-e_1$	λe_2	$\lambda_1\lambda_2 \neq 0$
$\mathfrak{r}'_{3,\lambda\neq 0}$	0	$e_2 - \lambda e_1$	$\lambda e_2 + e_1$	$\lambda_1^2 + \lambda_2^2 > 0$

Then, given a left-invariant one-form η^L on G, one has $f^*\eta^L = f^* L_g^* \eta^L = L^*_{f(g)} f^* \eta^L$ and since $f(g)$ gives every element of G for an appropriate element g, one has that $f^*\eta^L$ is a new left-invariant one-form. Hence, if η^L is a left-invariant contact form on G and $\dim G = 2k+1$, one has that

$$0 \neq f^*[(\mathrm{d}\eta^L)^k \wedge \eta^L] = [\mathrm{d}f^*\eta^L]^k \wedge f^*\eta^L, \qquad \forall f \in \mathrm{Aut}(G).$$

And $f^*\eta^L$ is a new contact form. Moreover, the value of $(\eta^L)_e$ at the neutral element e of G is such that $[f^*\eta^L]_e = (\mathrm{T}_e f)^T[\eta^L_e]$. Hence, if an element of $\mu \in \mathfrak{g}^*$ determines the value at e of a left-invariant contact form, all left-invariant one-forms with values at e within the orbit of μ relative to the action of $\mathrm{Aut}(G)$ on $\mathfrak{g}^*$ give rise to contact forms. □

Let $\mathrm{Aut}(\mathfrak{g})$ stand for the group of Lie algebra automorphisms of $\mathfrak{g}$. Since the tangent map at $e \in G$ to every element of $\mathrm{Aut}(G)$ is an element of $\mathrm{Aut}(\mathfrak{g})$ and vice versa, the action induced by $\mathrm{Aut}(G)$ on $\mathfrak{g}^*$ is indeed a Lie group action of $\mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{g}^*$.

5. Examples

Let us apply the techniques devised in previous sections to some particular examples of physical and mathematical interest.

5.1. The Brockett control system

Let us consider a first example of contact Lie system. The Brockett control system [67] in $\mathbb{R}^3$ is given by

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = b_1(t), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b_2(t), \\ \frac{\mathrm{d}z}{\mathrm{d}t} = b_2(t)x - b_1(t)y, \end{cases} \qquad \forall (x,y,z) \in \mathbb{R}^3, \tag{5.1}$$

where $b_1(t)$ and $b_2(t)$ are arbitrary time-dependent functions. System (5.1) is associated with the time-dependent vector field on $\mathbb{R}^3$ given by

$$X = b_1(t)X_1 + b_2(t)X_2,$$

where

$$X_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\,, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\,,$$

along with the vector field $X_3 = 2\frac{\partial}{\partial z}$, span a three-dimensional VG Lie algebra $V = \langle X_1, X_2, X_3\rangle$ on $\mathbb{R}^3$ with commutation relations

$$[X_1, X_2] = X_3\,, \quad [X_1, X_3] = 0\,, \quad [X_2, X_3] = 0\,.$$

As in example 3.4, the vector space $\langle X_1, X_2, X_3\rangle$ is a VG Lie algebra isomorphic to the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ (see figure 1).

The Lie algebra of Lie symmetries of V, i.e. the vector fields on $\mathbb{R}^3$ commuting with all the elements of V, is spanned by the vector fields

$$Y_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\,, \quad Y_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\,, \quad Y_3 = 2\frac{\partial}{\partial z}\,,$$

which have commutation relations

$$[Y_1, Y_2] = -Y_3\,, \quad [Y_1, Y_3] = 0\,, \quad [Y_2, Y_3] = 0\,.$$

Let us denote the Lie algebra of Lie symmetries of V by $\mathrm{Sym}(V)$. The dual base of one-forms to $\{Y_1, Y_2, Y_3\}$ is

$$\eta_1 = \mathrm{d}x\,, \quad \eta_2 = \mathrm{d}y\,, \quad \eta_3 = \frac{1}{2}(\mathrm{d}z - y\mathrm{d}x + x\mathrm{d}y)\,.$$

It is clear that $\mathrm{d}\eta_3 = \mathrm{d}x \wedge \mathrm{d}y$. Since $\eta_3 \wedge \mathrm{d}\eta_3 = \frac{1}{2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0$, we have that η_3 is a contact form in $\mathbb{R}^3$.

A short calculation shows that X_1, X_2, X_3 are contact Hamiltonian vector fields with respect to the contact form given by η_3 with Hamiltonian functions

$$h_1 = y\,, \quad h_2 = -x\,, \quad h_3 = -1\,,$$

respectively. Therefore, $\langle X_1, X_2, X_3\rangle$ are also Hamiltonian vector fields relative to $(\mathbb{R}^3, \eta_3, X)$. Thus, the triple $(\mathbb{R}^3, \eta_3, X)$ is a contact Lie system with a VG Lie algebra $\langle X_1, X_2, X_3\rangle \simeq \mathfrak{h}_3$. Moreover, the Reeb vector field is given by $Y_3 = X_3$. Then, $(\mathbb{R}^3, \eta_3, X)$ is a contact Lie system Liouville. Moreover, the time-dependent Hamiltonian

$$h_t = b_1(t)h_1 + b_2(t)h_2, \qquad \forall t \in \mathbb{R},$$

gives rise to a contact Lie–Hamiltonian system $(\mathbb{R}^3, \eta_3, h)$.

The projection of the original Hamiltonian contact system (5.1) onto $\mathbb{R}^2/X_3$, the space of orbits of X_3, via the natural projection $\pi : (x, y, z) \in \mathbb{R}^3 \mapsto (x, y) \in \mathbb{R}^2 \simeq \mathbb{R}^3/X_3$, reads

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b_1(t)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = b_2(t)\,. \tag{5.2}$$

As foreseen by proposition 3.9, system 5.2 is Hamiltonian relative to the symplectic form $\Omega = \mathrm{d}x \wedge \mathrm{d}y$ that is determined by the condition $\mathrm{d}\eta = \pi^*\Omega$. The Liouville theorem for Ω on $\mathbb{R}^2$ tells us that the evolution of (5.2) on $\mathbb{R}^2$ leaves invariant the area of any surface, but since $\{x, y\}$ are Darboux coordinates for Ω, the non-squeezing theorem also says that given a ball in $\mathbb{R}^2$ centred at a point of radius r, then if the image of such a ball under the dynamics of (5.2) is inside a ball in $\mathbb{R}^2$ of radius R with centre matching the centre of the original ball, then $R \geqslant r$. In fact, the evolution of (5.2) is given by

$$x' = x + \int_0^t b_1(t')\mathrm{d}t'\,, \qquad y' = y + \int_0^t b_2(t')\mathrm{d}t'\,.$$

Then, the image of a ball with centre at a point (x,y) at the time $t_0=0$ evolved relative to the evolution given by (5.2) until t is a new ball with centre at (x',y') and the same radius.

By the Liouville theorem for contact Lie systems of Liouville type, one has that the volume of a space of solutions in $\mathbb{R}^3$ does not vary on time. Hence, (5.1) is then a Hamiltonian system relative to a multisymplectic form Ω_η, and therefore the methods developed in [54] can be applied to the study of its properties.

5.2. *The Schwarz equation*

Consider a Schwarz equation [10, 64] of the form

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3}=\frac{3}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1}\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)^2+2b_1(t)\frac{\mathrm{d}x}{\mathrm{d}t}\,, \tag{5.3}$$

where $b_1(t)$ is any non-constant time-dependent function. Equation (5.3) is of relevance since it appears when dealing with Ermakov systems [58] and the Schwarzian derivative [23].

It is well known that equation (5.3) is a *higher-order Lie system* [22], i.e. the associated first-order system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=v\,,\quad \frac{\mathrm{d}v}{\mathrm{d}t}=a\,,\quad \frac{\mathrm{d}a}{\mathrm{d}t}=\frac{3}{2}\frac{a^2}{v}+2b_1(t)v\,, \tag{5.4}$$

is a Lie system. Indeed, the latter system is associated with the time-dependent vector field $X=X_3+b_1(t)X_1$ defined on $\mathcal{O}=\{(x,v,a)\in\mathbb{R}^3\mid v\neq 0\}$, where

$$X_1=2v\frac{\partial}{\partial a}\,,\quad X_2=v\frac{\partial}{\partial v}+2a\frac{\partial}{\partial a}\,,\quad X_3=v\frac{\partial}{\partial x}+a\frac{\partial}{\partial v}+\frac{3}{2}\frac{a^2}{v}\frac{\partial}{\partial a}\,.$$

These vector fields satisfy the commutation relations

$$[X_1,X_2]=X_1\,,\quad [X_1,X_3]=2X_2\,,\quad [X_2,X_3]=X_3\,,$$

and thus span a three-dimensional VG Lie algebra $V=\langle X_1,X_2,X_3\rangle\simeq\mathfrak{sl}_2$.

The Schwarz equation, when written as a first-order system (5.4), i.e. the hereafter called *Schwarz system*, admits a Lie algebra of Lie symmetries, denoted by $\mathrm{Sym}(V)$, spanned by the vector fields (see [33] for details)

$$Y_1=\frac{\partial}{\partial x}\,,\quad Y_2=x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}+a\frac{\partial}{\partial a}\,,\quad Y_3=x^2\frac{\partial}{\partial x}+2vx\frac{\partial}{\partial v}+2(ax+v^2)\frac{\partial}{\partial a}\,.$$

These Lie symmetries satisfy the commutation relations

$$[Y_1,Y_2]=Y_1\,,\quad [Y_1,Y_3]=2Y_2\,,\quad [Y_2,Y_3]=Y_3\,,$$

and thus $V\simeq\mathrm{Sym}(V)$. The basis $\{Y_1,Y_2,Y_3\}$ admits a dual basis of one-forms $\{\eta_1,\eta_2,\eta_3\}$ given by

$$\eta_1=\mathrm{d}x-\frac{x(ax+2v^2)}{2v^3}\mathrm{d}v+\frac{x^2}{2v^2}\mathrm{d}a\,,\quad \eta_2=\frac{ax+v^2}{v^3}\mathrm{d}v-\frac{x}{v^2}\mathrm{d}a\,, \tag{5.5}$$

$$\eta_3=-\frac{a}{2v^3}\mathrm{d}v+\frac{1}{2v^2}\mathrm{d}a\,. \tag{5.6}$$

Since

$$\eta_2\wedge\mathrm{d}\eta_2=\frac{1}{v^3}\mathrm{d}x\wedge\mathrm{d}v\wedge\mathrm{d}a\,,$$

we have that $(\mathcal{O},\eta_2)$ is a contact manifold. The vector fields X_1,X_2,X_3 are contact Hamiltonian vector fields with Hamiltonian functions

$$h_1=\frac{2x}{v}\,,\quad h_2=\frac{ax-v^2}{v^2}\,,\quad h_3=\frac{a(ax-2v^2)}{2v^3}\,,$$

respectively. Hence, V consists of Hamiltonian vector fields relative to $(\mathcal{O},\eta_2)$. Thus, $(\mathcal{O},\eta_2,X)$ becomes a contact Lie system and its Reeb vector field is Y_2.

The coordinates $\{x,v,a\}$ are not Darboux coordinates for η_2. Consider a new coordinate system on $\mathcal{O}$ given by

$$q=\frac{a}{v}\,,\qquad p=\frac{x}{v}\,,\qquad z=\ln v\,.$$

Using these coordinates, we obtain $\eta_2=\mathrm{d}z-p\,\mathrm{d}q$. Hence, they become Darboux coordinates for η_2. Now, the Reeb vector field, Y_2, becomes $\partial/\partial z$, and

$$X_1=2\frac{\partial}{\partial q}\,,\quad X_2=q\frac{\partial}{\partial q}-p\frac{\partial}{\partial p}+\frac{\partial}{\partial z}\,,\quad X_3=\frac{q^2}{2}\frac{\partial}{\partial q}+(1-pq)\frac{\partial}{\partial p}+q\frac{\partial}{\partial z}\,.$$

In Darboux coordinates $\{q,p,z\}$, the Lie symmetries Y_1,Y_2,Y_3 of V read

$$Y_1=\frac{1}{e^z}\frac{\partial}{\partial p}\,,\qquad Y_2=\frac{\partial}{\partial z}\,,\qquad Y_3=e^z\left(2\frac{\partial}{\partial q}-p^2\frac{\partial}{\partial p}+2p\frac{\partial}{\partial z}\right)\,.$$

The vector fields X_1,X_2,X_3 have Hamiltonian functions

$$h_1=2p\,,\qquad h_2=pq-1\,,\qquad h_3=\frac{1}{2}q^2p-q\,,$$

respectively. Moreover,

$$X=X_3+b_1(t)X_1=\left(\frac{q^2}{2}+2b_1(t)\right)\frac{\partial}{\partial q}+(1-pq)\frac{\partial}{\partial p}+q\frac{\partial}{\partial z}\,,$$

defines the system of ordinary differential equations

$$\begin{cases}\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{q^2}{2}+2b_1(t)\,,\\ \frac{\mathrm{d}p}{\mathrm{d}t}=1-pq\,,\\ \frac{\mathrm{d}z}{\mathrm{d}t}=q\,.\end{cases}\tag{5.7}$$

The dynamics portrait of system (5.7) is depicted in figure 1. It is a well-known result in contact dynamics [29, 41] that the evolution of the Hamiltonian function h along a solution of X_h is given by

$$\mathscr{L}_{X_h}h=-(\mathscr{L}_R h)h\,,$$

where R denotes the Reeb vector field. Since our Reeb vector field is $Y_2=\partial/\partial z$ and the Hamiltonian functions h_1,h_2,h_3 do not depend on the coordinate z, we have that our system preserves the energy along the solutions. Then, it is of Liouville type.

Note that system (5.7) can be projected onto $\mathcal{O}/Y_2\simeq\mathbb{R}^2$, which is a consequence of proposition 3.9. The projected system reads

$$\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{q^2}{2}+2b_1(t)\,,\qquad \frac{\mathrm{d}p}{\mathrm{d}t}=1-pq\,,\tag{5.8}$$

which is Hamiltonian relative to the symplectic form $\Omega=\mathrm{d}q\wedge\mathrm{d}p$. Indeed, its Hamiltonian function reads

$$k(t,q,p)=\frac{1}{2}q^2p+2b_1(t)p\,.$$

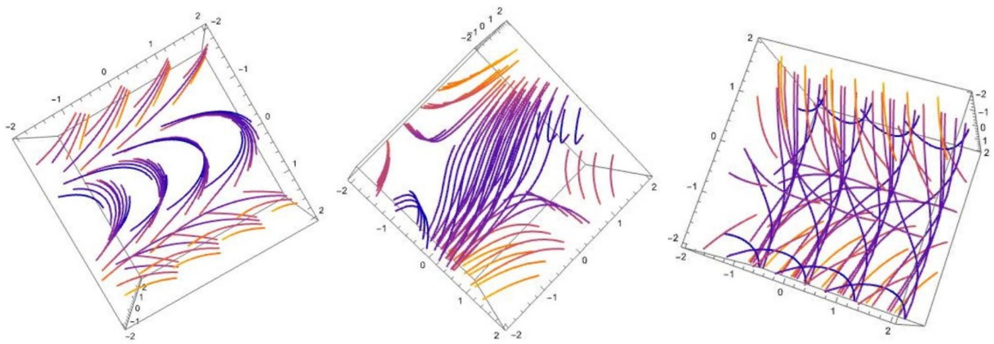

Figure 1. Dynamics portrait of system (5.7) from three different perspectives.

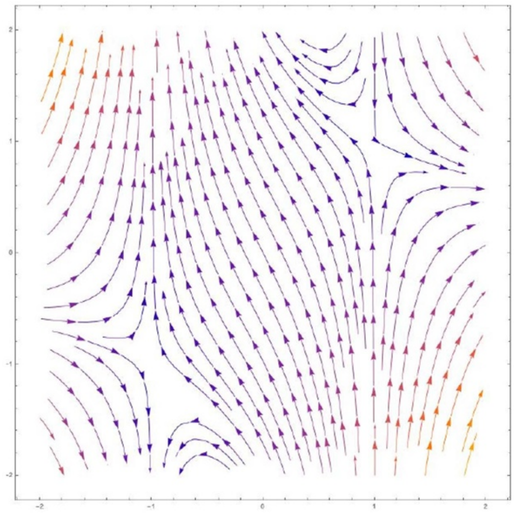

Figure 2. Phase portrait of the reduced Schwarz system (5.8). One can see its two saddle points at $(-1,-1)$ and $(1,1)$.

System (5.8) has no equilibrium points for $b_1(t) \geqslant 0$. Meanwhile, system (5.8) and two equilibrium points at

$$q = \pm 2\sqrt{-b_1(t)}, \quad p = \frac{\pm 1}{2\sqrt{-b_1(t)}}$$

for $b_1(t) < 0$. Setting $b_1(t) = -1/4$, system (5.8) has the form

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{q^2}{2} - \frac{1}{2}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = 1 - pq, \tag{5.9}$$

and has equilibrium points $(1,1)$ and $(-1,-1)$. Both equilibria are saddle points. The phase portrait for system (5.9) is depicted in figure 2.

As commented in the previous section, the volume of the evolution of a ball under the dynamics of (5.9) is constant, as can be seen in figure 3, but if the initial ball has radius r and origin at $(0,0)$, then the evolution of the ball cannot be bounded by a ball of radius smaller than r with centre at the origin.

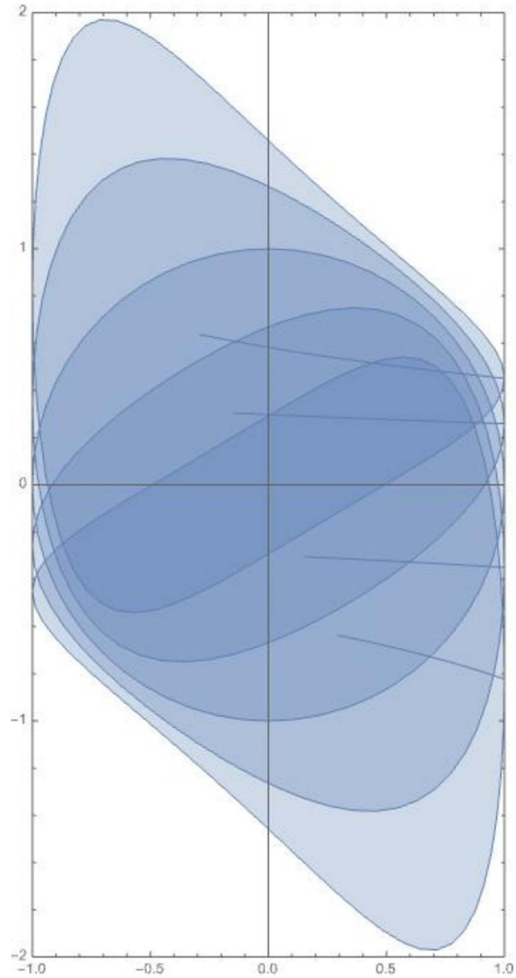

Figure 3. Evolution of a ball under the reduced Schwarz system (5.8). One can see that although the ball is deformed, its area is preserved.

5.3. A quantum contact Lie system

Let us illustrate how contact Marsden–Weinstein reduction can be used to study and to reduce contact Lie systems. Consider the linear space over the real numbers, $\mathfrak{W} = \langle i\widehat{H}_1, \ldots, i\widehat{H}_5\rangle$, spanned by the basis of skew-Hermitian operators on $\mathbb{R}^2$ given by

$$i\widehat{H}_1 = i\hat{x}\,, \quad i\widehat{H}_2 = i\hat{p}_x = \frac{\partial}{\partial x}\,, \quad i\widehat{H}_3 = i\hat{y}\,, \quad i\widehat{H}_4 = i\hat{p}_y = \frac{\partial}{\partial y}\,, \quad i\widehat{H}_5 = i\mathrm{Id}\,,$$

where the only non-vanishing commutation relations between the elements of the basis read

$$[i\widehat{H}_1, i\widehat{H}_2] = -i\widehat{H}_5\,, \quad [i\widehat{H}_3, i\widehat{H}_4] = -i\widehat{H}_5\,.$$

The Lie algebra $\mathfrak{W}$ appears in quantum mechanical problems. Let us consider the Lie algebra morphism $\rho : \mathfrak{W} \mapsto \mathfrak{X}(\mathbb{R}^5)$ satisfying that

$$\rho(i\widehat{H}_1) = X_1 = \frac{\partial}{\partial x_1}, \quad \rho(i\widehat{H}_2) = X_2 = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_5}, \quad \rho(i\widehat{H}_3) = X_3 = \frac{\partial}{\partial x_3},$$
$$\rho(i\widehat{H}_4) = X_4 = \frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_5}, \quad \rho(i\widehat{H}_5) = X_5 = \frac{\partial}{\partial x_5}.$$

Consider the Lie system on $\mathbb{R}^5$ associated with the time-dependent vector field

$$X^Q(t,x) = \sum_{\alpha=1}^{5} b_\alpha(t) X_\alpha(x), \qquad \forall t \in \mathbb{R}, \quad x \in \mathbb{R}^5,$$

with arbitrary time-dependent functions $b_1(t), \ldots, b_5(t)$, which has a VG Lie algebra $V^Q = \langle X_1, \ldots, X_5 \rangle$. The Lie algebra of Lie symmetries of V^Q is spanned by the vector fields

$$Y_1 = \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_5}, \qquad Y_2 = \frac{\partial}{\partial x_2}, \qquad Y_3 = \frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_5}, \quad Y_4 = \frac{\partial}{\partial x_4}, \qquad Y_5 = \frac{\partial}{\partial x_5}.$$

Since $Y_1 \wedge \ldots \wedge Y_5 \neq 0$ at every point of $\mathbb{R}^5$, there exists a basis of one-forms on $\mathbb{R}^5$ dual to $\{Y_1, \ldots, Y_5\}$ given by

$$\eta_1 = \mathrm{d}x_1, \quad \eta_2 = \mathrm{d}x_2, \quad \eta_3 = \mathrm{d}x_3, \quad \eta_4 = \mathrm{d}x_4, \quad \eta_5 = \mathrm{d}x_5 + x_2\mathrm{d}x_1 + x_4\mathrm{d}x_3,$$

i.e. $\iota_{Y_j}\eta_i = \delta_{ij}$, for $i,j = 1, \ldots, 5$, where δ_{ij} is the Kronecker's delta function. Then, $\eta_5 \wedge (\mathrm{d}\eta_5)^2 = 2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5$ is a volume form on $\mathbb{R}^5$ and thus η_5 becomes a contact form on $\mathbb{R}^5$. Moreover, X_1, X_2, X_3, X_4, X_5 are contact Hamiltonian vector fields with Hamiltonian functions

$$h_1 = -x_2, \quad h_2 = x_1, \quad h_3 = -x_4, \quad h_4 = x_3, \quad h_5 = -1,$$

respectively. Thus, X^Q admits a VG Lie algebra V^Q of Hamiltonian vector fields relative to η_5, and $(\mathbb{R}^5, \eta_5, X^Q)$ becomes a contact Lie system. The Reeb vector field of η_5 is given by $X_5 = Y_5$. Since the Hamiltonian functions $h_1, \ldots, h_5$ are first integrals of the Reeb vector field, $(\mathbb{R}^5, \eta_5, X)$ is a contact Lie system of Liouville type. It is relevant that many important techniques for studying contact Lie systems will be only available for contact Lie systems of Liouville type.

Let us consider the Lie algebra of symmetries of V^Q spanned by

$$V^S = \langle Y_1, Y_2, Y_5 \rangle.$$

This Lie algebra is isomorphic to the Heisenberg three-dimensional Lie algebra $\mathfrak{h}_3$. Moreover, the vector fields of V^S are also Hamiltonian relative the contact form η_5. The momentum map $J : \mathbb{R}^5 \to \mathfrak{h}_3^*$ associated with V^S is such that $\iota_{X_i}\eta_5 = J^i$ for $i = 1, 2, 5$, where

$$J^1 = x_2, \qquad J^2 = -x_1, \qquad J^5 = -1,$$

Note that J is not a submersion, but its tangent map has constant rank. By the Constant Rank Theorem, $J^{-1}(\mu)$ is a submanifold for every $\mu \in \mathfrak{h}_3^*$ and the tangent space at one of its points $x \in \mathbb{R}^5$ is given by the kernel of T_xJ, whatever $\mu \in \mathfrak{h}_3^*$ is. By theorem 2.10, the submanifold $J^{-1}(\mathbb{R}_+\mu)$ is invariant relative to the evolution of the contact Lie system.

Let us give the integral curves of the vector fields X_1, X_2, X_5:

$$\begin{aligned}
&X_1 \to x_1' = x_1 + \lambda_1, && x_2' = x_2, && x_3' = x_3, && x_4' = x_4, && x_5' = x_5,\\
&X_2 \to x_1' = x_1, && x_2' = x_2 + \lambda_2, && x_3' = x_3, && x_4' = x_4, && x_5' = x_5 - \lambda_2 x_1,\\
&X_5 \to x_1' = x_1, && x_2' = x_2, && x_3' = x_3, && x_4' = x_4, && x_5' = x_5 + \lambda_3,
\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Thus, $\mathscr{L}_{X_5} J = 0$ and $\lambda(\mu_1, \mu_2, -1) = (\lambda\mu_1, \lambda\mu_2, -\lambda) \notin \operatorname{Im} J$ unless $\lambda = 1$. Then, $J^{-1}(\mathbb{R}_+\mu) = J^{-1}(\mu) = \{x_1, x_2\} \times \mathbb{R}^3$. The isotropy subgroup of μ is the Lie subgroup $G_\mu = G_5$, whose action on $\mathbb{R}^5$ has fundamental vector fields spanned by X_5. Note that $\ker \mu|_{\mathfrak{g}_\mu} = 0$ and its related connected subgroup, K_μ, is the neutral element. Moreover,

$$J^{-1}(\mathbb{R}_+\mu)/K_\mu = \{x_1, x_2\} \times \mathbb{R}^3 .$$

Therefore, $J^{-1}(\mathbb{R}_+\mu)/K_\mu$ admits coordinates $\{x_3, x_4, x_5\}$. It is simple to prove that conditions for the contact Marsden–Weinstein reduction hold. Note that the projection of the initial contact Lie system onto $J^{-1}(\mathbb{R}_+\mu)/K_\mu$ reads

$$\bar{X}^Q(t,x) = \sum_{\alpha=2}^{5} b_\alpha(t)\widehat{X}_\alpha(x), \qquad \forall t \in \mathbb{R}, \qquad \forall x \in \mathbb{R}^3,$$

while the projection of the initial VG Lie algebra is spanned by the vector fields

$$\widehat{X}_2 = -x_1\frac{\partial}{\partial x_5}, \qquad \widehat{X}_3 = \frac{\partial}{\partial x_3}, \qquad \widehat{X}_4 = \frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_5}, \qquad \widehat{X}_5 = \frac{\partial}{\partial x_5} .$$

These are Hamiltonian vector fields relative to the contact form $\mathrm{d}x_5 + x_4\mathrm{d}x_3$ with Hamiltonian functions

$$\bar{h}_2 = x_1, \qquad \bar{h}_3 = -x_4, \qquad \bar{h}_4 = x_3, \qquad \bar{h}_5 = -1 .$$

Since $\widehat{X}_5$ is the Reeb vector fields on $\mathbb{R}^3$ relative to $\mathrm{d}x_5 + x_4\mathrm{d}x_3$, the reduced contact Lie system is also of Liouville type. In fact, it could be projected onto $\mathbb{R}^3/\widehat{X}_5 \simeq \mathbb{R}^2$, giving rise to a Lie–Hamilton system on $\mathbb{R}^2$ of the form

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = b_3(t), \qquad \frac{\mathrm{d}x_4}{\mathrm{d}t} = b_4(t)$$

relative to $\omega = \mathrm{d}x_4 \wedge \mathrm{d}x_3$.

5.4. A non-Liouville example

Consider the manifold $M = \mathbb{R}^3$ equipped with linear coordinates $\{q, p, z\}$. The manifold M has a natural contact form given by the one-form $\eta = \mathrm{d}z - p\,\mathrm{d}q$. Its associated Reeb vector field is $R = \partial/\partial z$. Consider the vector fields on M given by

$$X_1 = \frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial q}, \qquad X_3 = z\frac{\partial}{\partial q} - p^2\frac{\partial}{\partial p} .$$

These vector fields are Hamiltonian relative to $(\mathbb{R}^3, \eta)$ with Hamiltonian functions

$$h_1 = -1, \qquad h_2 = p, \qquad h_3 = pz,$$

and span a three-dimensional VG Lie algebra with commutation relations

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = X_2, \qquad [X_2, X_3] = 0,$$

isomorphic to $\mathfrak{h}_3$. This allows us to define a contact Lie system on $\mathbb{R}^3$ relative to η given by $(\mathbb{R}^3, \eta, X)$ with

$$X = \sum_{\alpha=1}^{3} b_\alpha(t)X_\alpha . \tag{5.10}$$

where $b_1(t), b_2(t), b_3(t)$ are time-dependent functions such that $V^X = \langle X_1, X_2, X_3\rangle$. Since the Hamiltonian function of X_3 is not a first integral of the Reeb vector field R, then X is a non-Liouville contact Lie system. Note also that X is associated with the time-dependent Hamiltonian function

$$h = \sum_{\alpha=1}^{3} b_\alpha(t) h_\alpha \,,$$

namely each X_t is the Hamiltonian vector field related to h_t for every $t \in \mathbb{R}$. As a consequence, the volume form related to the contact form, namely

$$\Omega = \mathrm{d}\eta \wedge \eta = \mathrm{d}q \wedge \mathrm{d}p \wedge \mathrm{d}z \,,$$

is not invariant relative to the vector fields of the VG Lie algebra and Ω is not preserved by the evolution of X. More specifically, if $F_{t_0} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is the flow starting from the point t_0 of X, namely $F_{t_0}(t_0, x_0) = x_0$ and $F_{t_0}(t, x_0) = x(t)$, where $x(t)$ is the particular solution to (5.10) with $x(t_0) = x_0$, for every $x_0 \in \mathbb{R}^3$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \int_{F_{t_0}(t,A)} \Omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \int_A F^*_{t_0,t}\Omega = \int_A \mathscr{L}_X \Omega \,,$$

for every subset $A \subset \mathbb{R}^3$. But $\mathscr{L}_X \Omega = 2(Rh)\Omega$. Hence,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \int_{F_{t_0}(t,A)} \Omega = 2\int_A (Rh)\Omega = 2\int_A \left(\sum_{\alpha=1}^{3} b_\alpha(t) R h_\alpha\right) \Omega \,.$$

Note that if $V = V^X$, then

$$\sum_{\alpha=1}^{3} b_\alpha(t) R h_\alpha = b_3(t) p \neq 0$$

for a generic value of $p \in \mathbb{R}$ and $t \in \mathbb{R}$.

6. Coalgebra method and superposition rules of Jacobi–Lie systems

Let us provide a method to derive superposition rules for contact Lie systems via Poisson coalgebras. Our method is a modification of the coalgebra method for deriving superposition rules for Dirac–Lie systems devised in [23]. It is worth noting that the coalgebra method does not work for contact Lie systems *per se* since, as proved next, the diagonal prolongations of a contact Lie system will not always be a contact Lie system.

Let us start by defining the diagonal prolongation of the sections of a vector bundle, as this is a key for developing the coalgebra method for Lie systems admitting VG Lie algebras of Hamiltonian vector fields relative to different geometric structures.

The *diagonal prolongation to M^k of a vector bundle $\tau : F \to M$* on M is defined to be the Whitney sum of k-times the vector bundle τ with itself, namely the vector bundle $\tau^{[k]} : F^k = F \times \overset{(k)}{\cdots} \times F \mapsto M^k = M \times \overset{(k)}{\cdots} \times M$, understood as a vector bundle over $\mathrm{M^k}$ in the natural way, i.e.

$$\tau^{[k]}(f_{(1)}, \ldots, f_{(k)}) = (\tau(f_{(1)}), \ldots, \tau(f_{(k)})) \,, \qquad \forall f_{(1)}, \ldots, f_{(k)} \in F \,,$$

and

$$F^k{}_{(x_{(1)}, \ldots, x_{(k)})} = F_{x_{(1)}} \oplus \cdots \oplus F_{x_{(k)}} \,, \qquad \forall (x_{(1)}, \ldots, x_{(k)}) \in M^k \,.$$

Every section $e : M \to F$ of the vector bundle τ has a natural *diagonal prolongation* to a section $e^{[k]}$ of the vector bundle $\tau^{[k]}$ given by

$$e^{[k]}(x_{(1)},\ldots,x_{(k)}) = e(x_{(1)}) + \cdots + e(x_{(k)}), \qquad \forall (x_{(1)},\ldots,x_{(k)}) \in M^k.$$

The *diagonal prolongation of a function* $f \in \mathscr{C}^\infty(M)$ to M^{k} is the function defined as

$$f^{[k]}(x_{(1)},\ldots,x_{(k)}) = f(x_{(1)}) + \ldots + f(x_{(k)}).$$

Consider also the sections $e^{(j)}$ of $\tau^{[k]}$, where $j \in \{1,\ldots,n\}$ and e is a section of τ, given by

$$e^{(j)}(x_{(1)},\ldots,x_{(k)}) = 0 + \cdots + e(x_{(j)}) + \cdots + 0, \qquad \forall (x_{(1)},\ldots,x_{(k)}) \in M^k. \tag{6.1}$$

If $\{e_1,\ldots,e_r\}$ is a basis of local sections of the vector bundle τ, then $e_i^{(j)}$, with $j = 1,\ldots,k$ and $i = 1,\ldots,r$, is a basis of local sections of $\tau^{[k]}$.

Due to the obvious canonical isomorphisms

$$(\mathrm{T}M)^{[k]} \simeq \mathrm{T}M^k \quad \text{and} \quad (\mathrm{T}^*M)^{[k]} \simeq \mathrm{T}^*M^k,$$

the diagonal prolongation $X^{[k]}$ of a vector field $X \in \mathfrak{X}(M)$ can be understood as a vector field $\widetilde{X}^{[k]}$ on M^{k}, and the diagonal prolongation, $\alpha^{[k]}$, of a one-form α on M can be understood as a one-form $\widetilde{\alpha}^{[k]}$ on M^{k}. If k is assumed to be fixed, we will simply write $\widetilde{X}$ and $\widetilde{\alpha}$ for their diagonal prolongations.

The proofs of proposition 6.1, its corollaries 6.2 and 6.3, and proposition 6.4 below are straightforward as they rely, almost entirely, on the definition of diagonal prolongations. Anyhow, as Jacobi manifolds with a non-vanishing Reeb vector field give rise to a Dirac manifold, they can also be considered as slight modifications of the results given for Dirac structures in [23].

Proposition 6.1. *Let (M,Λ,E) be a Jacobi manifold with bracket $\{\cdot,\cdot\}_{\Lambda,E}$. Let X and f be a vector field and a function on M, respectively. Then,*

(a) $(M^k,\Lambda^{[k]},E^{[k]})$ is a Jacobi manifold for every $k \in \mathbb{N}$.
(b) If f is a Hamiltonian function for a Hamiltonian vector field X relative to (M,Λ,E), its diagonal prolongation $f^{[k]}$ to M^k is a Hamiltonian function of the diagonal prolongation, $X^{[k]}$, to M^k with respect to $(M^k,\Lambda^{[k]},E^{[k]})$.
(c) If $f \in Cas(M,\Lambda,E)$, then $f^{[k]} \in Cas(M^k,\Lambda^{[k]},E^{[k]})$.
(d) The map $\lambda : (\mathscr{C}^\infty(M),\{\cdot,\cdot\}_{\Lambda,E}) \ni f \mapsto f^{[k]} \in (\mathscr{C}^\infty(M^k),\{\cdot,\cdot\}_{\Lambda^{[k]},E^{[k]}})$ is an injective Lie algebra morphism.

Corollary 6.2. *Let $h_1,\ldots,h_r : M \to \mathbb{R}$ be a family of functions on a Jacobi manifold (M,Λ,E) spanning a finite-dimensional real Lie algebra of functions with respect to the Lie bracket $\{\cdot,\cdot\}_{\Lambda,E}$. Then, their diagonal prolongations to M^k, $\tilde{h}_1,\ldots,\tilde{h}_r$, close an isomorphic Lie algebra of functions with respect to the Lie bracket $\{\cdot,\cdot\}_{\Lambda,E}$ induced by the Jacobi manifold $(M^k,\Lambda^{[k]},E^{[k]})$.*

Corollary 6.3. *Let (M,Λ,E,X) be a Jacobi–Lie system that admits a Jacobi–Lie Hamiltonian system (M,Λ,E,h). Then, $(M^k,\Lambda^{[k]},E^{[k]},X^{[k]})$ is a Jacobi–Lie system admitting a Jacobi–Lie Hamiltonian system $(M^k,\Lambda^{[k]},E^{[k]},h^{[k]})$, where $h_t^{[k]} = \tilde{h}_t^{[k]}$ is the diagonal prolongation of h_t to M^k.*

Proposition 6.4. *If X be a system possessing a time-independent constant of the motion f and Y is a time-independent Lie symmetry of X, then:*

1. The diagonal prolongation $f^{[k]}$ is a time-independent constant of the motion for $X^{[k]}$.

2. *The vector field $Y^{[k]}$ is a time-independent Lie symmetry of $X^{[k]}$.*
3. *If h is a time-independent constant of the motion for $X^{[k]}$, then $Y^{[k]}h$ is another time-independent constant of the motion for $X^{[k]}$.*

Proposition 6.5. *The diagonal prolongation to M^k of a Jacobi–Lie system (M,Λ,E,X) is a Jacobi–Lie system $(M^k,\Lambda^{[k]},E^{[k]},X^{[k]})$.*

Note that the diagonal prolongation of a contact Lie system do not need to be a contact Lie system as it may not be defined on an odd-dimensional manifold.

For the sake of completeness, let us prove the following result.

Proposition 6.6. *Let (M,Λ,E,X) be a Jacobi–Lie system possessing a Jacobi–Lie Hamiltonian system (M,Λ,E,h). A function $f\in\mathscr{C}^\infty(M)$ is a constant of the motion for X if and only if it commutes with all the elements of $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}},\{\cdot,\cdot\})$.*

Proof. The function f is a constant of the motion for X if

$$0=X_tf=\{h_t,f\}\,,\qquad \forall t\in\mathbb{R}\,. \tag{6.2}$$

Hence,

$$\{f,\{h_t,h_{t'}\}\}=\{\{f,h_t\},h_{t'}\}\}+\{h_t,\{f,h_{t'}\}\}\,,\qquad \forall t,t'\in\mathbb{R}\,.$$

Inductively, f is shown to commute with all the elements of $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$.

Conversely, if f commutes with all $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ relative to $\{\cdot,\cdot\}$, in particular, (6.2) holds and f is a constant of the motion of X.

□

The bracket for Jacobi manifolds is not, in general, a Poisson bracket. It becomes only a Poisson bracket for good Hamiltonian functions. Nevertheless, when a Lie group action gives rise to a momentum map, the components of the momentum map are first integrals of R. As a consequence, the following proposition, which can be considered as an adaptation of [23, proposition 8.4], is satisfied. Recall that if (M,Λ,E) is a Jacobi manifold, then $\mathscr{C}^\infty(M^k)$ becomes a Lie algebra relative to the Lie bracket $\{\cdot,\cdot\}_k$ related to $\Lambda^{[k]}$ and $\mathscr{C}^\infty(\mathfrak{W}^*)$ is a Poisson algebra relative to the Kirillov–Kostant–Souriau bracket.

Proposition 6.7. *Let (M,Λ,R,X) be a Jacobi–Lie system with an associated Jacobi–Lie Hamiltonian system (M,Λ,R,h) such that $\{h_t\}_{t\in\mathbb{R}}$ is contained in a finite-dimensional Lie algebra of good functions $(\mathfrak{W},\{\cdot,\cdot\})$. Let $\{v_1,\dots v_r\}$ be a basis of linear coordinates on $\mathfrak{W}^*$. Given the good momentum map $J:M\to\mathfrak{W}^*$, the pull-back J^*C of any Casimir function* C *on $\mathfrak{W}^*$ is a constant of the motion for* X*. Moreover, if $h_i=J^*v_i$ for $i=1,\dots,r$, and $C=C(v_1,\dots,v_r)$, then*

$$C\left(\sum_{a=1}^k h_1(x_{(a)}),\dots,\sum_{a=1}^k h_r(x_{(a)})\right), \tag{6.3}$$

is a constant of the motion of $X^{[k]}$.

The coalgebra method takes its name from the fact that it analyses the use of Poisson coalgebras and a so-called *coproduct* to obtain superposition rules. In fact, the coproduct is responsible for the form of (6.3).

6.1. A coalgebra method example for contact Lie systems of Liouville type

Finally, let us provide an example of the coalgebra method for contact Lie systems. Let us consider the Lie group $\mathrm{SL}(2,\mathbb{R})$ of 2×2 matrices with determinant one and real entries, i.e.

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, \alpha\delta - \beta\gamma = 1 \right\},$$

and the automorphic Lie system

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \sum_{\alpha=1}^{3} b_\alpha(t) X_\alpha^R(g), \qquad \forall t\in\mathbb{R}, \quad \forall g\in \mathrm{SL}(2,\mathbb{R}), \tag{6.4}$$

where $b_1(t), b_2(t), b_3(t)$ are arbitrary time-dependent functions. Observe that α,β,γ become a coordinate system of $\mathrm{SL}(2,\mathbb{R})$ close to the identity matrix. The Lie algebra of right-invariant vector fields on $\mathrm{SL}(2,\mathbb{R})$ is spanned by the vector fields

$$X_1^R = \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} - \gamma\frac{\partial}{\partial\gamma}, \qquad X_2^R = \gamma\frac{\partial}{\partial\alpha} + \frac{1+\beta\gamma}{\alpha}\frac{\partial}{\partial\beta}, \qquad X_3^R = \alpha\frac{\partial}{\partial\gamma},$$

and their commutation relations are

$$[X_1^R, X_2^R] = -2X_2^R, \qquad [X_2^R, X_3^R] = -X_1^R, \qquad [X_1^R, X_3^R] = 2X_3^R.$$

Meanwhile, the left-invariant vector fields are spanned by

$$X_1^L = \alpha\frac{\partial}{\partial\alpha} - \beta\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\gamma}, \qquad X_2^L = \alpha\frac{\partial}{\partial\beta}, \qquad X_3^L = \beta\frac{\partial}{\partial\alpha} + \frac{1+\beta\gamma}{\alpha}\frac{\partial}{\partial\gamma}.$$

Moreover,

$$[X_1^L, X_2^L] = 2X_2^L, \qquad [X_2^L, X_3^L] = X_1^L, \qquad [X_1^L, X_3^L] = -2X_3^L. \tag{6.5}$$

Consider the set of the left-invariant differential forms on $\mathrm{SL}(2,\mathbb{R})$ given by

$$\eta_1^L = \frac{1+\beta\gamma}{\alpha}\mathrm{d}\alpha - \beta\mathrm{d}\gamma, \qquad \eta_2^L = \frac{\beta(1+\beta\gamma)}{\alpha^2}\mathrm{d}\alpha + \frac{1}{\alpha}\mathrm{d}\beta - \frac{\beta^2}{\alpha}\mathrm{d}\gamma, \tag{6.6}$$

$$\eta_3^L = -\gamma\mathrm{d}\alpha + \alpha\mathrm{d}\gamma, \tag{6.7}$$

which become a basis of the space of left-invariant differential forms on $\mathrm{SL}(2,\mathbb{R})$. It is relevant that

$$\mathrm{d}\eta_1^L = \eta_2^L\wedge\eta_3^L \Rightarrow \mathrm{d}\eta_1^L\wedge\eta_1^L \neq 0.$$

Hence, η_1 becomes a left-invariant contact form on $\mathrm{SL}(2,\mathbb{R})$ with a Reeb vector field X_1^L. Therefore, the vector fields X_1^R, X_2^R, X_3^R admit the Hamiltonian functions

$$h_1 = -\eta_1^L(X_1^R) = -1-2\beta\gamma, \qquad h_2 = -\eta_1^L(X_2^R) = -\frac{\gamma}{\alpha}(1+\beta\gamma), \tag{6.8}$$

$$h_3 = -\eta_1^L(X_3^R) = \alpha\beta. \tag{6.9}$$

These Hamiltonian functions satisfy the commutation relations

$$\{h_1,h_2\} = -2h_2, \qquad \{h_1,h_3\} = 2h_3, \qquad \{h_2,h_3\} = -h_1.$$

Hence, all Hamiltonian functions for the right-invariant vector fields relative to the contact form η_1^L are first integrals of the Reeb vector field of η_1^L, namely X_1^L. This can be used to obtain the superposition rule for Lie systems on $\mathrm{SL}(2,\mathbb{R})$. Let us explain this. Let $\{e^1,e^2,e^3\}$ be a basis of $\mathfrak{sl}_2^*$ dual to $\{X_1^L(e), X_2^L(e), X_3^L(e)\}$. Given the action of $\mathrm{SL}(2,\mathbb{R})$ on itself on the left, whose fundamental vector fields are given by the linear space of right-invariant vector fields on $\mathrm{SL}(2,\mathbb{R})$, one can define an associated good momentum map

$$J : A \in \mathrm{SL}(2,\mathbb{R}) \longmapsto -(1+2\beta\gamma)e^1 - \frac{\gamma}{\alpha}(1+\beta\gamma)e^2 + \alpha\beta e^3 \in \mathfrak{sl}_2^*\,.$$

This allows us to obtain a superposition rule using the coalgebra method and proposition 6.7. The theory of Lie systems states that, to determine a superposition rule for a Lie system, one has to determine the smallest $k \in \mathbb{N}$ so that the vector fields $[X_1^R]^{[k]}, [X_2^R]^{[k]}, [X_3^R]^{[k]}$ are linearly independent at a generic point (see [34]). Since X_1^R, X_2^R, X_3^R are linearly independent at every point of $\mathrm{SL}(2,\mathbb{R})$, it follows that $k=1$. Hence, a superposition rule for (6.4) can be obtained by deriving three common first integrals for $[X_1^R]^{[k+1]}, [X_2^R]^{[k+1]}, [X_3^R]^{[k+1]}$, let us say I_1, I_2, I_3, satisfying

$$\frac{\partial(I_1,I_2,I_3)}{\partial(\alpha,\beta,\gamma)} \neq 0\,.$$

Standard Lie systems method, derive such functions by solving systems of PDEs [17]. Instead, we can do it by means of contact manifolds and our coalgebra method. A good Hamiltonian function that Poisson commutes with h_1, h_2, h_3 is given by

$$C_1 = 4h_2(\alpha,\beta,\gamma)h_3(\alpha,\beta,\gamma) + h_1(\alpha,\beta,\gamma)^2 \in \mathscr{C}^\infty(\mathrm{SL}(2,\mathbb{R}))\,,$$

where α,β,γ are assumed to be functions on $\mathrm{SL}(2,\mathbb{R})$. The above is indeed the pull-back to $\mathrm{SL}(2,\mathbb{R})$ of the famous Casimir function for $\mathfrak{sl}_2^*$ given by $4v_2v_3 + v_1^2$, where $\{v_1,v_2,v_3\}$ is the basis of $\mathfrak{sl}_2$ with commutation relations as in (6.5) understood as coordinates on $\mathfrak{sl}_2^*$. Similarly,

$$\begin{aligned}
&h_1^{[2]} = -(1+2\beta\gamma) - (1+2\beta'\gamma')\,, \qquad h_2^{[2]} = -\frac{\gamma}{\alpha}(1+\beta\gamma) - \frac{\gamma'}{\alpha'}(1+\beta'\gamma'),\\
&h_3^{[2]} = \alpha\beta + \alpha'\beta'
\end{aligned}$$

become the Hamiltonian functions of

$$\begin{aligned}
&[X_1^R]^{[2]} = \alpha\frac{\partial}{\partial\alpha} - \beta\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\gamma} + \alpha'\frac{\partial}{\partial\alpha'} - \beta'\frac{\partial}{\partial\beta'} + \gamma'\frac{\partial}{\partial\gamma'},\\
&[X_2^R]^{[2]} = \alpha\frac{\partial}{\partial\beta} + \alpha'\frac{\partial}{\partial\beta'},\\
&[X_3^R]^{[2]} = \beta\frac{\partial}{\partial\alpha} + \frac{1+\beta\gamma}{\alpha}\frac{\partial}{\partial\gamma} + \beta'\frac{\partial}{\partial\alpha'} + \frac{1+\beta'\gamma'}{\alpha'}\frac{\partial}{\partial\gamma'}.
\end{aligned}$$

Hence, a common first integral for $[X_1^R]^{[2]}, [X_2^R]^{[2]}, [X_3^R]^{[2]}$ is given by

$$I_1 = 4h_2^{[2]}h_3^{[2]} - (h_1^{[2]})^2 = -\frac{4(\beta\gamma\alpha' + \alpha' - \alpha\beta\gamma')(\gamma\alpha'\beta' - \alpha(\beta'\gamma'+1))}{\alpha\alpha'}\,.$$

Note that this is indeed an application of (6.3) to our problem.

To obtain the remaining two first integrals for $[X_1^R]^{[2]}, [X_2^R]^{[2]}, [X_3^R]^{[2]}$, we derive

$$\begin{aligned}
I_2 &= [X_2^L]^{[2]}I_1 = -\frac{4(\gamma\alpha' - \alpha\gamma')\big((1+\beta\gamma)\alpha'^2 - \alpha(\alpha - \gamma\alpha'\beta' + \beta\alpha'\gamma' + \alpha\beta'\gamma')\big)}{\alpha\alpha'}\,,\\
I_3 &= [X_3^L]^{[2]}I_1\\
&= -\frac{4(\alpha\beta(\beta'\gamma'+1) - (\beta\gamma+1)\alpha'\beta')\left(\alpha(\beta'\gamma'\alpha + \alpha - \gamma\alpha'\beta' + \beta\alpha'\gamma') - (\beta\gamma+1)\alpha'^2\right)}{\alpha^2\alpha'^2}\,.
\end{aligned}$$

Since the determinant of

$$\frac{\partial(I_1,I_2,I_3)}{\partial(\alpha,\beta,\gamma)} = \begin{pmatrix} \frac{\partial I_1}{\partial\alpha} & \frac{\partial I_1}{\partial\beta} & \frac{\partial I_1}{\partial\gamma} \\ \frac{\partial I_2}{\partial\alpha} & \frac{\partial I_2}{\partial\beta} & \frac{\partial I_2}{\partial\gamma} \\ \frac{\partial I_3}{\partial\alpha} & \frac{\partial I_3}{\partial\beta} & \frac{\partial I_3}{\partial\gamma} \end{pmatrix}$$

is different from zero at a generic point in $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, the system of algebraic equations

$$I_1=\lambda_1\,,\qquad I_2=\lambda_2\,,\qquad I_3=\lambda_3\,, \tag{6.10}$$

allows us to obtain α,β,γ in terms of α',β',γ' and $\lambda_1,\lambda_2,\lambda_3$, which gives rise to a superposition rule. Its expression may be complicated, but can be derived using any program of mathematical manipulation.

Anyway, there is a simpler method to obtain the superposition rule for automorphic Lie systems in general and (6.4) in particular. Since (6.4) is an automorphic Lie system with a VG Lie algebra of right-invariant vector fields, it is known that a superposition rule is given by the multiplication on the right

$$\Phi : (g,h)\in G\times G \longmapsto gh\in G\,.$$

Since the vector fields $[X_1^L]^{[2]},[X_2^L]^{[2]},[X_3^L]^{[2]}$ span a distribution of rank three on $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, which is three-codimensional as the dimension of $\mathrm{SL}(2,\mathbb{R})$, it was proved in [20] that the superposition rule must be unique. Hence, this superposition rule must be the one obtained by solving the algebraic system (6.10).

The example shown in this section just depicts how the coalgebra method works for a particular example that has a known unique superposition rule. The coalgebra method for contact Lie systems of Liouville type will be, in general, simpler to use than previous known procedures to obtain superposition rules, which are based on solving systems of partial or ordinary differential equations.

7. Conclusions and further research

This paper, introduces the notion of contact Lie system: systems of first-order differential equations describing the integral curves of a time-dependent vector field taking values in a finite-dimensional Lie algebra of Hamiltonian vector fields relative to a contact manifold. In particular, we have studied families of contact Lie systems of Liouville type, i.e. those that are invariant with respect to the flow of the Reeb vector field. We have also developed Liouville theorems, a contact reduction, and a Gromov non-squeezing theorem for certain classes of contact Lie systems. We have also studied locally automorphic contact Lie systems of Liouville type on three-dimensional manifolds. In particular, the classification of left-invariant contact forms on three-dimensional Lie groups has been derived, which is useful to classify automorphic Lie systems of Liouville type on three-dimensional Lie groups with VG Lie algebras given by spaces of right-invariant vector fields. To illustrate our results, we have worked out several examples, such as the Brockett control system, the Schwarz equation, an automorphic Lie system on $\mathrm{SL}(2,\mathbb{R})$, a quantum contact Lie system, and a non-Liouville contact Lie system.

The reduction procedures developed by Willet [81] and Albert [3] open the door to develop an energy-momentum method [61] for contact Lie systems, both Liouville and non-Liouville. This will allow us to study the relative equilibria points of these systems. We also believe that a new type of contact reduction can be achieved by interpreting contact forms in a new, distributional, manner. This is currently being developed and will hopefully be published in a future work.

Note that our techniques in this work are focused on Hamiltonian vector fields relative to a contact manifold. Nevertheless, contact geometry is concerned with evolution and gradient vector fields too [75]. It will be interesting to extend our theory to such a case as well as analysing the possible extensions of Marsden–Weinstein reductions, energy–momentum methods, and other theories and techniques to evolution and gradient vector fields.

Recently, the contact formulation for non-conservative mechanical systems has been generalised via the so-called k-contact [40, 42, 55], k-cocontact [69], and multicontact [27, 80] formulations. It would be interesting to study the Lie systems whose VG Lie algebra consists of Hamiltonian vector fields relative to these structures. We also plan to study the case of Lie systems admitting a VG Lie algebra of Hamiltonian vector fields relative to a locally conformally symplectic manifold. It would also be interesting to classify contact Lie systems possessing a transitive primitive VG Lie algebra [72, 73].

Data availability statement

No new data were created or analysed in this study.

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Conflict of interest

Data sharing is not applicable to this article, as no data sets were generated or analysed during the current study. The authors have no conflicts of interest to declare. All coauthors have seen and agree with the contents of the manuscript and there is no financial interest to report.

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References

[1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* (*AMS Chelsea Publishing*) vol 364 (New York: Benjamin/Cummings Pub. Co.)

[2] Abraham R, Marsden J E and Ratiu T 1988 *Manifolds, Tensor Analysis and Applications* (*Applied Mathematical Sciences* vol 75) (New York: Springer)

[3] Albert C 1989 Le théorème de réduction de Marsden–Weinstein en géométrie cosymplectique et de contact *J. Geom. Phys.* **6** 627–49

[4] Amirzadeh-Fard H, Haghighatdoost G, Kheradmandynia P and Rezaei-Aghdam A 2020 Jacobi structures on real two- and three-dimensional Lie groups and their Jacobi–Lie systems *Theor. Math. Phys.* **205** 1393–410

[5] Amirzadeh-Fard H, Haghighatdoost G and Rezaei-Aghdam A 2022 Jacobi–Lie Hamiltonian systems on real low-dimensional Jacobi–Lie groups and their Lie symmetries *J. Math. Phys. Anal. Geom.* **18** 33–56

[6] Ballesteros A, Blasco A, Herranz F, de Lucas J and Sardón C 2015 Lie–Hamilton systems on the plane: properties, classification and applications *J. Differ. Equ.* **258** 2873–907

[7] Ballesteros A, Cariñena J, Herranz F, de Lucas J and Sardón C 2013 From constants of motion to superposition rules for Lie–Hamilton systems J. Phys. A: Math. Theor. **46** 285203

[8] Banyaga A and Houenou D F 2016 *A Brief Introduction to Symplectic and Contact Manifolds* vol 15 (Singapore: World Scientific Publishing Co. Pte. Ltd.)

[9] Beckers J, Gagnon L, Hussin V and Winternitz P 1990 Superposition formulas for nonlinear superequations *J. Math. Phys.* **31** 2528–34

[10] Berkovich L M 2007 Method of factorization of ordinary differential operators and some of its applications *Appl. Anal. Discret. Math.* **1** 122–49

[11] Blacker C 2021 Reduction of multisymplectic manifolds *Lett. Math. Phys.* **111** 64

[12] Blasco A, Herranz F, de Lucas J and Sardón C 2015 Lie–Hamilton systems on the plane: applications and superposition rules J. Phys. A: Math. Theor. **48** 345202

[13] Bravetti A 2017 Contact Hamiltonian dynamics: the concept and its use *Entropy* **10** 535

[14] Bravetti A 2018 Contact geometry and thermodynamics *Int. J. Geom. Methods Mod. Phys.* **16** 1940003

[15] Bravetti A, Cruz H and Tapias D 2017 Contact Hamiltonian mechanics *Ann. Phys.* **376** 17–39

[16] Cappelletti-Montano B, De Nicola A and Yudin I 2013 A survey on cosymplectic geometry *Rev. Math. Phys.* **25** 1343002

[17] Cariñena J and de Lucas J 2011 Lie systems: theory, generalisations and applications *Dissertationes Math.* **479** 1–162

[18] Cariñena J, de Lucas J and Sardón C 2013 Lie–Hamilton systems: theory and applications *Int. J. Geom. Methods Mod. Phys.* **10** 1350047

[19] Cariñena J, Grabowski J and Marmo G 2000 *Lie–Scheffers Systems: A Geometric Approach* (Naples: Bibliopolis)

[20] Cariñena J, Grabowski J and Marmo G 2007 Superposition rules, Lie theorem and partial differential equations *Rep. Math. Phys.* **60** 237–58

[21] Cariñena J F, Clemente-Gallardo J, Jover-Galtier J A and de Lucas J 2019 Application of Lie systems to quantum mechanics: superposition rules *Proc. 60 Years Alberto Ibort Fest Classical and Quantum Physics: Geometry, Dynamics and Control* (Cham: Springer International Publishing) pp 85–119

[22] Cariñena J F, Grabowski J and de Lucas J 2012 Superposition rules for higher-order differential equations and their applications *J. Phys. A: Math. Theor.* **45** 185202

[23] Cariñena J F, Grabowski J, de Lucas J and Sardón C 2014 Dirac–Lie systems and Schwarzian equations *J. Differ. Equ.* **257** 2303–40

[24] Ciaglia F M, Cruz H and Marmo G 2018 Contact manifolds and dissipation, classical and quantum *Ann. Phys.* **398** 159–79

[25] de León M, Gaset J, Gràcia X, Muñoz-Lecanda M and Rivas X 2022 Time-dependent contact mechanics *Mon.hefte Math.* **201** 1149–83

[26] de León M, Gaset J, Lainz-Valcázar M, Rivas X and Román-Roy N 2020 Unified Lagrangian-Hamiltonian formalism for contact systems *Fortschr. Phys.* **68** 2000045

[27] de León M, Gaset J, Muñoz-Lecanda M C, Rivas X and Román-Roy N 2022 Multicontact formalism for non-conservative field theories *J. Phys. A: Math. Theor.* **56** 025201

[28] de León M, Jiménez V M and Lainz-Valcázar M 2021 Contact Hamiltonian and Lagrangian systems with nonholonomic constraints *J. Geom. Mech.* **13** 25–53
[29] de León M and Lainz-Valcázar M 2019 Contact Hamiltonian systems *J. Math. Phys.* **60** 102902
[30] de León M and Lainz-Valcázar M 2019 Singular Lagrangians and precontact Hamiltonian systems *Int. J. Geom. Methods Mod. Phys.* **16** 1950158
[31] de León M and Lainz-Valcázar M 2020 Infinitesimal symmetries in contact Hamiltonian systems *J. Geom. Phys.* **153** 103651
[32] de León M and Sardón C 2017 Cosymplectic and contact structures to resolve time-dependent and dissipative Hamiltonian systems *J. Phys. A: Math. Theor.* A **50** 255205
[33] de Lucas J and Sardón C 2013 On Lie systems and Kummer-Schwarz equations *J. Math. Phys.* **54** 033505
[34] de Lucas J and Sardón C 2020 *A Guide to Lie Systems With Compatible Geometric Structures* (Singapore: World Scientific)
[35] de Lucas J and Vilariño S 2015 *k*-symplectic Lie systems: theory and applications *J. Differ. Equ.* **258** 2221–55
[36] de Lucas J and Wysocki D 2020 A Grassmann and graded approach to coboundary Lie bialgebras, their classification and Yang–Baxter equations *J. Lie Theory* **30** 1161–94
[37] Echeverría-Enríquez A, Muñoz-Lecanda M C and Román-Roy N 2018 Remarks on multisymplectic reduction *Rep. Math. Phys.* **81** 415–24
[38] Farinati M A and Jancsa A P 2015 Three dimensional real Lie bialgebras *Rev. Un. Mat. Argentina* **56** 27–62
[39] Flores-Espinoza R 2011 Periodic first integrals for Hamiltonian systems of Lie type *Int. J. Geom. Methods Mod. Phys.* **8** 1169–77
[40] Gaset J, Gràcia X, Muñoz-Lecanda M C, Rivas X and Román-Roy N 2020 A contact geometry framework for field theories with dissipation *Ann. Phys.* **414** 168092
[41] Gaset J, Gràcia X, Muñoz-Lecanda M C, Rivas X and Román-Roy N 2020 New contributions to the Hamiltonian and Lagrangian contact formalisms for dissipative mechanical systems and their symmetries *Int. J. Geom. Methods Mod. Phys.* **17** 2050090
[42] Gaset J, Gràcia X, Muñoz-Lecanda M C, Rivas X and Román-Roy N 2021 A *k*-contact Lagrangian formulation for nonconservative field theories *Rep. Math. Phys.* **87** 347–68
[43] Gaset J, López-Gordón A and Rivas X 2023 Symmetries, conservation and dissipation in time-dependent contact systems *Fortschr. Phys.* 2300048
[44] Gaset J and Mas A 2023 A variational derivation of the field equations of an action-dependent Einstein–Hilbert Lagrangian *J. Geom. Mech.* **15** 357–74
[45] Geiges H 2008 *An Introduction to Contact Topology* (*Cambridge Studies in Advanced Mathematics*) vol 109 (Cambridge: Cambridge University Press)
[46] Goto S 2016 Contact geometric descriptions of vector fields on dually flat spaces and their applications in electric circuit models and nonequilibrium statistical mechanics *J. Math. Phys.* **57** 102702
[47] Grabowska K and Grabowski J 2022 A geometric approach to contact Hamiltonians and contact Hamilton–Jacobi theory *J. Phys. A: Math. Theor.* **55** 435204
[48] Grabowska K and Grabowski J 2022 Contact geometric mechanics: the Tulczyjew triples (arXiv:2209.03154)
[49] Grabowska K and Grabowski J 2023 Reductions: precontact versus presymplectic *Ann. Mat. Pura Appl.* (https://doi.org/10.1007/s10231-023-01341-y)
[50] Grabowski J 2013 Brackets *Int. J. Geom. Methods Mod. Phys.* **10** 1360001
[51] Grabowski J and de Lucas J 2013 Mixed superposition rules and the Riccati hierarchy *J. Differ. Equ.* **254** 179–98
[52] Gràcia X, de Lucas J, Muñoz Lecanda M and Vilariño S 2019 Multisymplectic structures and invariant tensors for Lie systems *J. Phys. A: Math. Theor.* **52** 215201
[53] Grundland A M and de Lucas J 2019 On the geometry of the Clairin theory of conditional symmetries for higher-order systems of PDEs with applications *Differ. Geom. Appl.* **67** 101557
[54] Gràcia X, de Lucas J, Rivas X, Román-Roy N and Vilariño S 2022 Reduction and reconstruction of multisymplectic Lie systems *J. Phys. A: Math. Theor.* A **55** 295204
[55] Gràcia X, Rivas X and Román-Roy N 2022 Skinner–Rusk formalism for *k*-contact systems *J. Geom. Phys.* **172** 104429

[56] Herranz F, de Lucas J and Sardón C 2015 Jacobi–Lie systems: fundamentals and low-dimensional classification *Dynamical Systems, Differential Equations and Applications, 2015 (10th AIMS Conf.. Suppl.)* pp 605–14
[57] Kholodenko A L 2013 *Applications of Contact Geometry and Topology in Physics* (Singapore: World Scientific)
[58] Leach P G L and Andriopoulus K 2008 Ermakov equation: a commentary *Appl. Anal. Dis. Math.* **2** 146–57
[59] Lewandowski M and de Lucas J 2017 Geometric features of Vessiot–Guldberg Lie algebras of conformal and Killing vector fields on $\mathbb{R}^2$ *Banach Center Publ.* **113** 243–62
[60] Libermann P and Marle C-M 1987 *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordretch: Springer Netherlands)
[61] Marsden J E and Simo J C 1990 The Energy-Momentum Method *La "Mécanique analytique" de Lagrange et son héritage* 1 (*Acta Academiae Scientiarum Taurinensis* vol 124) (Torino: Accademia delle scienze di Torino) pp 245–68
[62] Nijenhuis A 1955 Jacobi–type identities for bilinear differential concomitants of certain tensor fields. I, II *Indag. Math.* A **58** 390–403
[63] Odzijewicz A and Grundland A M 2000 The superposition principle for the Lie type first-order PDEs *Rep. Math. Phys.* **45** 293–306
[64] Ovsienko V and Tabachnikov S 2009 What is the Schwarzian derivative *Not. AMS* **56** 34–36
[65] Paiva J A P, Lazo M J and Zanchin V T 2022 Generalized nonconservative gravitational field equations from Herglotz action principle *Phys. Rev.* D **105** 124023
[66] Ramirez H, Maschke B and Sbarbaro D 2017 Partial stabilization of input-output contact systems on a Legendre submanifold *IEEE Trans. Autom. Control* **62** 1431–7
[67] Ramos A 2011 Sistemas de Lie y sus aplicaciones en física y teoría de control *PhD Thesis* Universidad de Zaragoza
[68] Rivas X 2021 Geometrical aspects of contact mechanical systems and field theories *PhD Thesis* Universitat Politècnica de Catalunya (UPC)
[69] Rivas X 2023 Nonautonomous k-contact field theories *J. Math. Phys.* **64** 033507
[70] Rivas X and Torres D 2022 Lagrangian–Hamiltonian formalism for cocontact systems *J. Geom. Mech.* **15** 1–26
[71] Schouten J A 1953 On the differential operators of first order in tensor calculus *Math. Centrum* 1953–012
[72] Shnider S and Winternitz P 1984 Classification of systems of nonlinear ordinary differential equations with superposition principles *J. Math. Phys.* **25** 3155–65
[73] Shnider S and Winternitz P 1984 Nonlinear equations with superposition principles and the theory of transitive primitive Lie algebras *Lett. Math. Phys.* **8** 69–78
[74] Simo J C, Lewis D and Marsden J E 1991 Stability of relative equilibria. I. The reduced energy-momentum method *Arch. Ration. Mech. Anal.* **115** 15–59
[75] Simoes A A, de León M, Lainz-Valcázar M and Martín de Diego D 2020 Contact geometry for simple thermodynamical systems with friction *Proc. R. Soc.* A **476** 20200244
[76] Sussmann H J 1973 Orbits of families of vector fields and integrability of distributions *Trans. Am. Math. Soc.* **180** 171–88
[77] Sussmann H J 1999 *Geometry and Optimal Control* (*Mathematical Control Theory*) (New York: Springer)
[78] Thurston W P 2022 *Geometry and Topology of Three-Manifolds* vol IV (Providence, RI: American Mathematical Society)
[79] Vaisman I 1994 *Lectures on the Geometry of Poisson Manifolds* (*Progress in Mathematics*) vol 118 (Basel: Birkhäuser Verlag)
[80] Vitagliano L 2015 L_∞-algebras from multicontact geometry *Differ. Geom. Appl.* **59** 147–65
[81] Willett C 2002 Contact reduction *Trans. Am. Math. Soc.* **354** 4245–60
[82] Winternitz P 1983 *Lie Groups and Solutions of Nonlinear Differential Equations* (*Nonlinear Phenomena, Lecture Notes in Physics* vol 189) (Oaxtepec: Springer)