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Nonautonomous k-contact field theories

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ABSTRACT

This paper provides a new geometric framework to describe non-conservative field theories with explicit dependence on the space–time coordinates by combining the k-cosymplectic and k-contact formulations. This geometric framework, the k-cocontact geometry, permits the development of Hamiltonian and Lagrangian formalisms for these field theories. We also compare this new formulation in the autonomous case with the previous k-contact formalism. To illustrate the theory, we study the nonlinear damped wave equation with external time-dependent forcing.

I. INTRODUCTION

During the second half of the 20th century, geometric methods have been widely applied to mechanics and field theory with the aim of providing geometric descriptions of a large variety of systems in applied mathematics, physics, engineering, etc. Some of the most frequent geometric structures involved in geometric mechanics and field theory are symplectic, multisymplectic, or k-symplectic manifolds (see, for instance, Refs. 1–10 and references therein). In general, all these geometric methods are applied to Lagrangian and Hamiltonian conservative systems, that is, without damping.

In recent years, there has been a growing interest in non-conservative systems. In particular, contact geometry[11–13] has been used to study mechanical systems with dissipation.[14–19] This has many applications in thermodynamics,[20,21] quantum mechanics,[22] circuit theory,[23] and control theory[24] among others.[25–30] Recently, contact mechanics have been generalized in order to deal with time-dependent contact systems.[31,32] It is worth pointing out that contact geometry allows us to study more systems than just dissipative ones.[33] In recent years, a generalization of both contact and k-symplectic structures was devised to describe autonomous field theories with damping[34–36] in both the Hamiltonian and Lagrangian formulations.

The main goal of this paper is to extend the k-contact formulation to non-autonomous field theories by combining it with k-cosymplectic geometry.[37,38] This leads to the definition of a k-*cocontact structure* as a couple of families of k differential one-forms: the first family accounting for the space–time coordinates, and the other one encoding the dampings or dissipations inspired by the contact formulation. It is worth noting that the number of independent variables ina the system coincides with the number of "dissipation coordinates." This new geometry enables us to introduce the notion of a k-cocontact Hamiltonian system as a k-cocontact manifold together with a Hamiltonian function. With these elements, we can state the k-cocontact Hamilton equations, which indeed add dissipation terms to the well-known Hamiltonian field equations.[6]

In addition, we also generalize the Lagrangian formulation of field theories to consider non-autonomous non-conservative ones. In this new formalism, the phase bundle is $M = \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$, where the direct sum has to be understood as a fibered sum of vector bundles, with adapted coordinates $(t^\alpha, q^i, v^i_\alpha, z^\alpha)$. Then, given a Lagrangian function $L : M \to \mathbb{R}$, we define a family η^α_L of one-forms which, when L is regular, constitute along with the forms $\mathrm{d}t^\alpha$ a k-cocontact structure on M. Then, the k-cocontact Lagrangian field equations are the k-cocontact Hamiltonian field equations for the Lagrangian energy. When written in coordinates, they are the Euler–Lagrange equations with some additional damping terms. It is worth pointing out that the field equations obtained via the k-contact and k-cocontact formalisms coincide with the ones obtained by generalizing the Herglotz variational principle[39] to the case of field theories.[40]

We also compare the k-cocontact formalism introduced in this work in the autonomous case with the previous k-contact formalism and see that they are partially equivalent, in the same way that autonomous k-cosymplectic systems are closely related to k-symplectic systems.[6]

Finally, we apply this formalism to the nonlinear damped wave equation with a time-dependent external force, both in the Hamiltonian and Lagrangian formulations.

The structure of the paper is as follows: in Sec. II, we provide a review of the k-contact formalism for non-conservative autonomous field theories. In particular, we provide the main results on k-contact geometry and a brief description of the Hamiltonian and Lagrangian formalisms. Section III is devoted to the introduction of the notion of k-cocontact structure and the study of its geometry. More precisely, we prove the existence of two families of Reeb vector fields and the existence of two types of special sets of coordinates: adapted coordinates and, by adding an extra hypothesis, Darboux coordinates.

In Sec. IV, we develop a Hamiltonian formalism for non-autonomous field theories with damping, generalizing the De Donder–Weyl formulation for field theories. We provide field equations both for k-vector fields and integral sections, and we prove the existence (but not the uniqueness) of solutions to these equations. We begin Sec. V by describing the geometry of the phase bundle of k-cocontact Lagrangian field theories. We also present the field equations, generalizing the Euler–Lagrange equations, and give the conditions for a Lagrangian function to be regular, that is, to yield a k-cocontact structure. Finally, we study a particularly interesting type of Lagrangian functions: the Lagrangians with holonomic damping term.

Section VI is devoted to comparing the k-contact formalism introduced in Ref. 34 with the k-cocontact setting presented in this work in the autonomous case. In order to illustrate the geometric formalism introduced in Secs. II–VI, in Sec. VII, we study the example of a nonlinear damped wave equation, describing all the geometric objects involved, both in the Lagrangian and Hamiltonian formulations.

Unless otherwise stated, all maps are assumed to be $\mathscr{C}^\infty$, and all manifolds are smooth, connected, and second countable. The sum of crossed and repeated indices is understood. The direct sum of two vector bundles over the same base space is to be understood as the Whitney sum of vector bundles.

II. REVIEW ON k-CONTACT SYSTEMS

In this section, we review the k-contact formalism for non-conservative field theories. In the first place, we introduce the geometric framework: k-contact structures. Then, the Hamiltonian[34] and the Lagrangian[35] formalisms are presented.

A. k-contact manifolds

Consider an m-dimensional manifold M. A *generalized distribution* on M is a subset of $D \subset \mathrm{T}M$ such that $D_x \subset \mathrm{T}_x M$ is a vector subspace for every $x \in M$. A distribution D is said to be *smooth* if it can be locally spanned by a family of vector fields, and *regular* if it is smooth and of locally constant rank. A *codistribution* on M is a subset of $C \subset \mathrm{T}^*M$ such that $C_x \subset \mathrm{T}^*_x M$ is a vector subspace for every $x \in M$.

Given a distribution D, the anihilator D° of D is a codistribution. If D is not regular, D° may not be smooth. Using the usual identification $E^{**} = E$ of finite-dimensional linear algebra, it is clear that $(D^\circ)^\circ = D$.

A differential one-form $\eta \in \Omega^1(M)$ generates a smooth codistribution, denoted by $\langle \eta \rangle \subset \mathrm{T}^*M$. This codistribution has rank 1 at every point where η does not vanish. Its annihilator is a distribution $\langle \eta \rangle^\circ \subset \mathrm{T}M$ that can be described as the kernel of the linear vector bundle morphism $\widehat{\eta} : \mathrm{T}M \to M \times \mathbb{R}$ defined by η, where $\mathrm{T}M$ and $M \times \mathbb{R}$ are understood as vector bundles over M. This codistribution has corank 1 at every point where η does not vanish.

In the same way, every two-form $\omega \in \Omega^2(M)$ induces a linear morphism $\widehat{\omega} : \mathrm{T}M \to \mathrm{T}^*M$, defined as $\widehat{\omega}(v) = i(v)\omega$. The kernel of this morphism $\widehat{\omega}$ is a distribution $\ker \widehat{\omega} \subset \mathrm{T}M$.

Given a family of k differential one-forms $\eta^1, \dots, \eta^k \in \Omega^1(M)$, we will denote

- $\mathcal{C}^{\mathrm{C}} = \langle \eta^1, \dots, \eta^k \rangle \subset \mathrm{T}^*M$.
- $\mathcal{D}^{\mathrm{C}} = \left(\mathcal{C}^{\mathrm{C}}\right)^\circ = \ker \widehat{\eta^1} \cap \dots \cap \ker \widehat{\eta^k} \subset \mathrm{T}M$.
- $\mathcal{D}^{\mathrm{R}} = \ker \widehat{\mathrm{d}\eta^1} \cap \dots \cap \ker \widehat{\mathrm{d}\eta^k} \subset \mathrm{T}M$.
- $\mathcal{C}^{\mathrm{R}} = \left(\mathcal{D}^{\mathrm{R}}\right)^\circ \subset \mathrm{T}^*M$.

With the preceding notations, a *k-contact structure* on a manifold M is a family of k differential one-forms $\eta^1, \dots, \eta^k \in \Omega^1(M)$ such that $\mathcal{D}^{\mathrm{C}} \subset \mathrm{T}M$ is a regular distribution of corank k, $\mathcal{D}^{\mathrm{R}} \subset \mathrm{T}M$ is a regular distribution of rank k and $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}} = \{0\}$. We call $\mathcal{C}^{\mathrm{C}}$ the *contact codistribution*, $\mathcal{D}^{\mathrm{C}}$ the *contact distribution*, $\mathcal{D}^{\mathrm{R}}$ the *Reeb distribution*, and $\mathcal{C}^{\mathrm{R}}$ the *Reeb codistribution*. A manifold M endowed with a k-contact structure $\eta^1, \dots, \eta^k \in \Omega^1(M)$ is a *k-contact manifold*.

Remark 2.1. In the particular case $k = 1$, a 1-contact structure is given by a one-form η. In this case, we recover the notion of a contact manifold.[17,18]

Given a k-contact manifold (M, η^α), the Reeb distribution $\mathcal{D}^{\mathrm{R}}$ is involutive and, therefore, integrable, and there exists a unique family of k vector fields $R_\alpha \in \mathfrak{X}(M)$, called *Reeb vector fields* of M, such that $i(R_\alpha)\eta^\beta = \delta_\alpha^\beta$ and $i(R_\alpha)\mathrm{d}\eta^\beta = 0$. The Reeb vector fields commute and span the Reeb distribution $\mathcal{D}^{\mathrm{R}} = \langle R_1, \dots, R_k \rangle$.

Example 2.2 (Canonical k-contact structure). Consider $k \geq 1$ and let Q be a smooth manifold. The manifold product $M = \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ has a canonical k-contact structure given by the one-forms $\eta^1, \ldots, \eta^k \in \Omega^1(M)$ defined as

$$\eta^\alpha = \mathrm{d}z^\alpha - \theta^\alpha,$$

where $(z^1, \ldots, z^k)$ are the canonical coordinates of $\mathbb{R}^k$ and θ^α is the pull-back of the Liouville one-form θ of the cotangent bundle T^*Q with respect to the projection $\mathrm{pr}^\alpha : M \to \mathrm{T}^*Q$ to the α-th component. Take coordinates (q^i) on Q. Then, M has natural coordinates $(q^i, p_i^\alpha, z^\alpha)$. Using these coordinates, we have

$$\eta^\alpha = \mathrm{d}z^\alpha - p_i^\alpha \mathrm{d}q^i, \qquad \mathcal{D}^{\mathrm{R}} = \left\langle \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^k} \right\rangle, \qquad R_\alpha = \frac{\partial}{\partial z^\alpha}.$$

Example 2.3 (Contactification of a k-symplectic manifold). Consider a k-symplectic manifold (P, ω^α) such that $\omega^\alpha = -d\theta^\alpha$ and the product manifold $M = P \times \mathbb{R}^k$. Let (z^α) be the Cartesian coordinates of $\mathbb{R}^k$ and denote also by θ^α the pull-back of θ^α to the product manifold M. Consider the one-forms $\eta^\alpha = dz^\alpha - \theta^\alpha \in \Omega^1(M)$.

Then, (M, η^α) is a k-contact manifold because $\mathcal{C}^{\mathrm{C}} = \langle \eta^1, \ldots, \eta^k \rangle$ has rank k, $d\eta^\alpha = -d\theta^\alpha$, and $\mathcal{D}^{\mathrm{R}} = \bigcap_\alpha \ker \widehat{\mathrm{d}\theta^\alpha} = \langle \partial/\partial z^1, \ldots, \partial/\partial z^k \rangle$ has rank k since (P, ω^α) is k-symplectic, and the last condition is immediate.

Notice that the so-called canonical k-contact structure described in Example 2.2 is just the contactification of the k-symplectic manifold $P = \oplus^k \mathrm{T}^* Q$.

Theorem 2.4 (k-contact Darboux Theorem). *Consider a k-contact manifold (M, η^α) of dimension $\dim M = n + kn + k$ such that there exists an integrable subdistribution $\mathcal{V}$ of $\mathcal{D}^{\mathrm{C}}$ with $\operatorname{rank} \mathcal{V} = nk$. Then, around every point of M there exists a local chart $(U, q^i, p_i^\alpha, z^\alpha)$, $1 \leq \alpha \leq k$, where $1 \leq i \leq n$, such that*

$$\eta^\alpha|U = \mathrm{d}z^\alpha - p_i^\alpha \mathrm{d}q^i, \quad \mathcal{D}^{\mathrm{R}}|U = \left\langle R_\alpha = \frac{\partial}{\partial z^\alpha} \right\rangle, \quad \mathcal{V}|U = \left\langle \frac{\partial}{\partial p_i^\alpha} \right\rangle.$$

These coordinates are called the Darboux coordinates of the k-contact manifold (M, η^α).

B. Hamiltonian formalism for k-contact systems

The geometric setting introduced in Sec. II A allows us to introduce the notion of a k-contact Hamiltonian system.[34]

Definition 2.5. A *k-contact Hamiltonian system* is a family (M, η^α, h), where (M, η^α) is a k-contact manifold and $h \in \mathscr{C}^\infty(M)$ is called a *Hamiltonian function*. Consider a map $\psi : D \subset \mathbb{R}^k \to M$. The *$k$-contact Hamilton–De Donder–Weyl equations for the map ψ* are

$$\begin{cases} i(\psi'_\alpha)\mathrm{d}\eta^\alpha = \left(\mathrm{d}h - (\mathscr{L}_{R_\alpha} h)\eta^\alpha\right) \circ \psi, \\ i(\psi'_\alpha)\eta^\alpha = -h \circ \psi, \end{cases} \tag{1}$$

where $\psi' : \mathbb{R}^k \to \bigoplus^k \mathrm{T}M$ is the canonical lift of ψ to $\bigoplus^k \mathrm{T}M$.

In Darboux coordinates, if the map ψ has the local expression $\psi(t) = (q^i(t), p_i^\alpha(t), z^\alpha(t))$, Eq. (1) read

$$\begin{cases} \dfrac{\partial q^i}{\partial t^\alpha} = \dfrac{\partial h}{\partial p_i^\alpha} \circ \psi, \\ \dfrac{\partial p_i^\alpha}{\partial t^\alpha} = -\left(\dfrac{\partial h}{\partial q^i} + p_i^\alpha \dfrac{\partial h}{\partial z^\alpha} \right) \circ \psi, \\ \dfrac{\partial z^\alpha}{\partial t^\alpha} = \left(p_i^\alpha \dfrac{\partial h}{\partial p_i^\alpha} - h \right) \circ \psi. \end{cases} \tag{2}$$

Definition 2.6. Consider a k-contact Hamiltonian system (M, η^α, h) and a k-vector field $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(M)$. The *$k$-contact Hamilton–De Donder–Weyl equations for the k-vector field* $\mathbf{X}$ are

$$\begin{cases} i(X_\alpha)\mathrm{d}\eta^\alpha = \mathrm{d}h - (\mathscr{L}_{R_\alpha} h)\eta^\alpha, \\ i(X_\alpha)\eta^\alpha = -h. \end{cases} \tag{3}$$

A k-vector field solution to these equations is a *k-contact Hamiltonian k-vector field*.

Proposition 2.7. The k-contact Hamilton–De Donder–Weyl equations (3) admit solutions. They are not unique if $k>1$.

Consider a k-vector field $\mathbf{X}=(X_1,\dots,X_k)\in\mathfrak{X}^k(M)$ with local expression in Darboux coordinates,

$$X_\alpha=(X_\alpha)^i\frac{\partial}{\partial q^i}+(X_\alpha)_i^\beta\frac{\partial}{\partial p_i^\beta}+(X_\alpha)^\beta\frac{\partial}{\partial z^\beta}.$$

Then, Eq. (3) yields the conditions

$$\begin{cases}(X_\alpha)^i=\dfrac{\partial h}{\partial p_i^\alpha},\\ (X_\alpha)_i^\alpha=-\left(\dfrac{\partial h}{\partial q^i}+p_i^\alpha\dfrac{\partial h}{\partial z^\alpha}\right),\\ (X_\alpha)^\alpha=p_i^\alpha\dfrac{\partial h}{\partial p_i^\alpha}-h.\end{cases}$$

Proposition 2.8. Consider an integrable k-vector field $\mathbf{X}\in\mathfrak{X}^k(M)$. Then, every integral section $\psi:D\subset\mathbb{R}^k\to M$ of $\mathbf{X}$ satisfies the k-contact Hamilton–De Donder–Weyl equations (1) if, and only if, $\mathbf{X}$ is a solution to (3).

Proposition 2.9. The k-contact Hamilton–De Donder–Weyl equations (3) are equivalent to

$$\begin{cases}\mathscr{L}_{X_\alpha}\eta^\alpha=-(\mathscr{L}_{R_\alpha}h)\eta^\alpha,\\ i(X_\alpha)\eta^\alpha=-h.\end{cases}$$

C. Lagrangian formalism for k-contact systems

The Hamiltonian formalism presented in Sec. II B has a Lagrangian counterpart. Consider the phase bundle $\oplus^k\mathrm{T}Q\times\mathbb{R}^k$ endowed with adapted coordinates $(q^i,v_\alpha^i,z^\alpha)$ with the usual canonical structures: the Liouville vector field $\Delta=v_\alpha^i\frac{\partial}{\partial v_\alpha^i}$ and the canonical k-tangent structure $J^\alpha=\frac{\partial}{\partial v_\alpha^i}\otimes\mathrm{d}q^i$ (see Ref. 35 for details). A k-vector field $\mathbf{X}=(X_\alpha)\in\mathfrak{X}^k(\oplus^k\mathrm{T}Q\times\mathbb{R}^k)$ is a *second-order partial differential equation* (SOPDE) if $J^\alpha(X_\alpha)=\Delta$.

Given a Lagrangian function $L:\oplus^k\mathrm{T}Q\times\mathbb{R}\to\mathbb{R}$, the *Lagrangian energy* associated with the Lagrangian L is the function $E_L=\Delta(L)-L$, and the *contact one-forms* $\eta_L^\alpha\in\Omega^1(\oplus^k\mathrm{T}Q\times\mathbb{R}^k)$ associated with L are given by $\eta_L^\alpha=\mathrm{d}z^\alpha-\theta_L^\alpha$, where $\theta_L^\alpha={}^tJ^\alpha\circ\mathrm{d}L$.

The Lagrangian L is *regular*, namely, $\frac{\partial^2L}{\partial v_\alpha^i\partial v_\beta^j}$ is non-degenerate, if and only if the contact one-forms η_L^α define a k-contact structure on $\oplus^k\mathrm{T}Q\times\mathbb{R}^k$. Therefore, we can consider the k-contact Hamiltonian system $(\oplus^k\mathrm{T}Q\times\mathbb{R}^k,\eta_L^\alpha,E_L)$, whose corresponding field equations read

$$\begin{cases}\dfrac{\partial}{\partial t^\alpha}\left(\dfrac{\partial L}{\partial v_\alpha^i}\circ\psi\right)=\left(\dfrac{\partial L}{\partial q^i}+\dfrac{\partial L}{\partial z^\alpha}\dfrac{\partial L}{\partial v_\alpha^i}\right)\circ\psi,\\ \dfrac{\partial(z^\alpha\circ\psi)}{\partial t^\alpha}=L\circ\psi,\end{cases}$$

and are called *k-contact Euler–Lagrange equations* (for more details on the k-contact Lagrangian formulation, see Refs. 35 and 41).

III. k-COCONTACT GEOMETRY

Let $\tau^1,\dots,\tau^k\in\Omega^1(M)$ be a family of closed one-forms on M, and let $\eta^1,\dots,\eta^k\in\Omega^1(M)$ be a family of one-forms on M. We will use the following notations:

- $\mathcal{C}^{\mathrm{C}}=\langle\eta^1,\dots,\eta^k\rangle\subset\mathrm{T}^*M$.
- $\mathcal{D}^{\mathrm{C}}=\left(\mathcal{C}^{\mathrm{C}}\right)^\circ=\ker\widehat{\eta^1}\cap\cdots\cap\ker\widehat{\eta^k}\subset\mathrm{T}M$.
- $\mathcal{D}^{\mathrm{R}}=\ker\widehat{\mathrm{d}\eta^1}\cap\cdots\cap\ker\widehat{\mathrm{d}\eta^k}\subset\mathrm{T}M$.
- $\mathcal{C}^{\mathrm{R}}=\left(\mathcal{D}^{\mathrm{R}}\right)^\circ\subset\mathrm{T}^*M$.
- $\mathcal{C}^{\mathrm{S}}=\langle\tau^1,\dots,\tau^k\rangle\subset\mathrm{T}^*M$.
- $\mathcal{D}^{\mathrm{S}}=\left(\mathcal{C}^{\mathrm{S}}\right)^\circ=\ker\widehat{\tau^1}\cap\cdots\cap\ker\widehat{\tau^k}\subset\mathrm{T}M$.

With these notations, we can define the following notion of k-cocontact structure:

Definition 3.1. A *k-cocontact structure* on a manifold M is a family of k closed differential one-forms $\tau^1, \dots, \tau^k \in \Omega^1(M)$ and a family of k differential one-forms $\eta^1, \dots, \eta^k \in \Omega^1(M)$ such that, with the preceding notations,

(1) $\mathcal{D}^{\mathrm{C}} \subset \mathrm{T}M$ is a regular distribution of corank k.
(2) $\mathcal{D}^{\mathrm{S}} \subset \mathrm{T}M$ is a regular distribution of corank k.
(3) $\mathcal{D}^{\mathrm{R}} \subset \mathrm{T}M$ is a regular distribution of rank $2k$.
(4) $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{S}}$ is a regular distribution of corank $2k$, $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}}$ is a regular distribution of rank k, and $\mathcal{D}^{\mathrm{S}} \cap \mathcal{D}^{\mathrm{R}}$ is a regular distribution of rank k.
(5) $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}} \cap \mathcal{D}^{\mathrm{S}} = \{0\}$.

We call $\mathcal{C}^{\mathrm{C}}$ the *contact codistribution*, $\mathcal{D}^{\mathrm{C}}$ the *contact distribution*, $\mathcal{D}^{\mathrm{R}}$ the *Reeb distribution*, $\mathcal{C}^{\mathrm{R}}$ the *Reeb codistribution*, $\mathcal{C}^{\mathrm{S}}$ the *space–time codistribution*, and $\mathcal{D}^{\mathrm{S}}$ the *space–time distribution*.

A manifold M endowed with a k-cocontact structure $\tau^1, \dots, \tau^k, \eta^1, \dots, \eta^k \in \Omega^1(M)$ is a *k-cocontact manifold*.

Notice that the condition $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}} \cap \mathcal{D}^{\mathrm{S}} = \{0\}$ implies that

$$\mathrm{T}^*M = \mathcal{C}^{\mathrm{C}} \oplus \mathcal{C}^{\mathrm{R}} \oplus \mathcal{C}^{\mathrm{S}}.$$

Remark 3.2. In the particular case $k = 1$, a 1-cocontact structure is given by two one-forms τ, η, with $\mathrm{d}\tau = 0$. The conditions in Definition 3.1 mean the following: (1) $\eta \neq 0$ everywhere, (2) $\tau \neq 0$ everywhere, (4) $\tau \wedge \eta \neq 0$, (5) $\ker \widehat{\tau} \cap \ker \widehat{\eta} \cap \ker \widehat{\mathrm{d}\eta} = \{0\}$, which implies that $\ker \widehat{\mathrm{d}\eta}$ has rank 0, 1, or 2, and (3) implies that $\ker \widehat{\mathrm{d}\eta}$ has rank 2. Therefore, a 1-cocontact structure coincides with the cocontact structure introduced in Ref. 31 to describe time-dependent contact mechanics.

Lemma 3.3. The Reeb distribution $\mathcal{D}^{\mathrm{R}}$ and the space–time distribution $\mathcal{D}^{\mathrm{S}}$ are involutive and, therefore, integrable.

Proof. Given X, Y two sections of $\mathcal{D}^{\mathrm{R}}$ and applying the relation

$$i_{[X,Y]} = \mathscr{L}_X i_Y - i_Y \mathscr{L}_X = \mathrm{d} i_X i_Y + i_X \mathrm{d} i_Y - i_Y \mathrm{d} i_X - i_Y i_X \mathrm{d},$$

to the closed two-form $\mathrm{d}\eta^\alpha$, the result is zero. In the same way, one can check that $\mathcal{D}^{\mathrm{S}}$ is also involutive. □

As a consequence, the distribution $\mathcal{D}^{\mathrm{R}} \cap \mathcal{D}^{\mathrm{S}}$ is also involutive and, therefore, integrable. Moreover, the distribution $\mathcal{D}^{\mathrm{R}} \cap \mathcal{D}^{\mathrm{C}}$ is also involutive and integrable. The following theorem characterizes a family of vector fields spanning the Reeb distribution $\mathcal{D}^{\mathrm{R}}$:

Theorem 3.4. *Let $(M, \tau^\alpha, \eta^\alpha)$ be a k-cocontact manifold. Then, there exists a unique family $R_1^t, \dots, R_k^t, R_1^z, \dots, R_k^z \in \mathfrak{X}(M)$ such that*

$$\begin{aligned} i(R_\alpha^t)\mathrm{d}\eta^\beta &= 0, & i(R_\alpha^t)\eta^\beta &= 0, & i(R_\alpha^t)\tau^\beta &= \delta_\alpha^\beta, \\ i(R_\alpha^z)\mathrm{d}\eta^\beta &= 0, & i(R_\alpha^z)\eta^\beta &= \delta_\alpha^\beta, & i(R_\alpha^z)\tau^\beta &= 0. \end{aligned}$$

The vector fields R_α^t are called space–time Reeb vector fields. The vector fields R_α^z are called contact Reeb vector fields.

In addition, the Reeb vector fields commute and span the Reeb distribution introduced in Definition 3.1,

$$\mathcal{D}^{\mathrm{R}} = \langle R_1^t, \dots, R_k^t, R_1^z, \dots, R_k^z \rangle,$$

therefore, motivating its name.

Proof. Consider $\mathrm{T}^*M = \mathcal{C}^{\mathrm{C}} \oplus \mathcal{C}^{\mathrm{R}} \oplus \mathcal{C}^{\mathrm{S}}$. The family of one-forms $\{\eta^\beta\}$ is a global frame of the contact codistribution $\mathcal{C}^{\mathrm{C}}$ and the family of one-forms $\{\tau^\beta\}$ is a global frame of the space–time codistribution $\mathcal{C}^{\mathrm{S}}$. We can find a global frame $\bar{\eta}^\mu$ of the Reeb codistribution $\mathcal{C}^{\mathrm{R}}$ so that $(\eta^\beta, \bar{\eta}^\mu, \tau^\beta)$ is a global frame of T^*M. Let $(R_\alpha^z, R_\nu, R_\alpha^t)$ be the corresponding dual frame of $\mathrm{T}M$, where the vector fields R_α^z and R_α^t are uniquely determined by the conditions

$$\begin{aligned} \langle \eta^\beta, R_\alpha^z \rangle &= \delta_\alpha^\beta, & \langle \bar{\eta}^\mu, R_\alpha^z \rangle &= 0, & \langle \tau^\beta, R_\alpha^z \rangle &= 0, \\ \langle \eta^\beta, R_\alpha^t \rangle &= 0, & \langle \bar{\eta}^\mu, R_\alpha^t \rangle &= 0, & \langle \tau^\beta, R_\alpha^t \rangle &= \delta_\alpha^\beta. \end{aligned}$$

Notice that the relations involving the $\bar{\eta}^\mu$ do not depend on the choice of the one-forms $\bar{\eta}^\mu$, this means that the vector fields R_α^z and R_α^t are sections of the Reeb distribution $(\mathcal{C}^{\mathrm{R}})^\circ = \mathcal{D}^{\mathrm{R}}$. This amounts to $i(R_\alpha^z)\mathrm{d}\eta^\beta = 0$ and $i(R_\alpha^t)\mathrm{d}\eta^\beta = 0$ for every $\alpha = 1, \dots, k$. Since the one-forms η^β and τ^β are globally defined, so are the vector fields R_α^z and R_α^t.

To prove that the Reeb vector fields R^z_α, R^t_β commute, notice that

$$i_{[X,Y]}\eta^\gamma = 0,\qquad i_{[X,Y]}\mathrm{d}\eta^\gamma = 0,\qquad i_{[X,Y]}\tau^\gamma = 0,$$

for every $X, Y \in \langle R^z_\alpha, R^t_\beta\rangle$, which is a consequence of their definition. □

The following proposition proves the existence of a special set of coordinates, the so-called adapted coordinates:

Proposition 3.5. Consider a k-cocontact manifold $(M,\tau^\alpha,\eta^\alpha)$. Then, around every point in M, there exist local coordinates $(t^\alpha, x^I, z^\alpha)$ such that

$$R^t_\alpha = \frac{\partial}{\partial t^\alpha},\qquad \tau^\alpha = \mathrm{d}t^\alpha,\qquad R^z_\alpha = \frac{\partial}{\partial z^\alpha},\qquad \eta^\alpha = \mathrm{d}z^\alpha - f^\alpha_I(x^J)\mathrm{d}x^I,$$

where the functions f^α_I only depend on the coordinates x^I. These coordinates are called adapted coordinates.

Proof. Since the Reeb vector fields commute, there exists a set of local coordinates $(t^\alpha, x^I, z^\alpha)$ simultaneously straightening out the Reeb vector fields (see Ref. 42, p. 234 for details),

$$R^t_\alpha = \frac{\partial}{\partial t^\alpha},\qquad R^z_\alpha = \frac{\partial}{\partial z^\alpha}.$$

Let us write the forms τ^β and η^β using these coordinates. The conditions $i(R^t_\alpha)\eta^\beta = 0$ and $i(R^z_\alpha)\eta^\beta = \delta^\beta_\alpha$ imply that $\eta^\beta = \mathrm{d}z^\beta - f^\beta_I(t^\alpha, x^J, z^\alpha)\mathrm{d}x^I$. On the other hand, we have that $\mathrm{d}\eta^\beta = \mathrm{d}x^I\wedge\mathrm{d}f^\beta_I$. In this case, the conditions $i(R^t_\alpha)\mathrm{d}\eta^\beta = 0$ and $i(R^z_\alpha)\mathrm{d}\eta^\beta = 0$ imply that $\partial f^\beta_I/\partial t^\alpha = 0$ and that $\partial f^\beta_I/\partial z^\alpha = 0$, and thus

$$\eta^\beta = \mathrm{d}z^\beta - f^\beta_I(x^J)\mathrm{d}x^I.$$

Repeating this process for the forms τ^β, taking into account that $\mathrm{d}\tau^\beta = 0$, and redefining the coordinates t^α, we obtain the desired result. □

Example 3.6 (Canonical k-cocontact structure). Let Q be a smooth n-dimensional manifold with coordinates (q^i) and let $k \geq 1$. Consider the product manifold $M = \mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ endowed with natural coordinates $(t^\alpha; q^i, p^\alpha_i; z^\alpha)$. We have the canonical projections

$$\begin{tikzcd}
\mathbb{R} & \mathbb{R}^k\times\bigoplus^k \mathrm{T}^*Q\times\mathbb{R}^k \arrow[l, "\pi_1^\alpha"'] \arrow[r, "\pi_3^\alpha"] \arrow[d, "\pi_2"] \arrow[dr, "\pi_2^\alpha"] \arrow[dd, bend right, "\pi_0"'] & \mathbb{R} \\
 & \bigoplus^k \mathrm{T}^*Q \arrow[r, "\pi^\alpha"'] & \mathrm{T}^*Q \\
 & \mathbb{R}^k\times Q\times\mathbb{R}^k &
\end{tikzcd}.$$

Let θ be the Liouville one-form on T^*Q with local expression in natural coordinates $\theta = p_i dq^i$. Then, the family $(\tau^\alpha, \eta^\alpha)$ where $\tau^\alpha = \pi_1^{\alpha\,*}(\mathrm{d}t)$ with t the canonical coordinate of $\mathbb{R}$ and $\eta^\alpha = \mathrm{d}z^\alpha - \pi_2^{\alpha\,*}\theta$, is a k-cocontact structure on M. In natural coordinates,

$$\tau^\alpha = \mathrm{d}t^\alpha,\qquad \eta^\alpha = \mathrm{d}z^\alpha - p^\alpha_i\mathrm{d}q^i.$$

Therefore, the Reeb vector fields are $R^t_\alpha = \partial/\partial t^\alpha$ and $R^z_\alpha = \partial/\partial z^\alpha$.

The following theorem is an upgrade of Proposition 3.5 and states the existence of Darboux-like coordinates in a k-cocontact manifold provided the existence of a certain subdistribution $\mathcal{V}\subset\mathcal{D}^{\mathrm{C}}$:

Theorem 3.7 (Darboux theorem for k-cocontact manifolds). *Let $(M,\tau^\alpha,\eta^\alpha)$ be a k-cocontact manifold with dimension $\dim M = k + n + kn + k$ such that there exists an integrable subdistribution $\mathcal{V}\subset\mathcal{D}^{\mathrm{C}}$ with $\operatorname{rank}\mathcal{V} = nk$. Then, around every point of M there exist local coordinates $(t^\alpha, q^i, p^\alpha_i, z^\alpha)$, where $1\leq\alpha\leq k$ and $1\leq i\leq n$, such that, locally,*

$$\tau^\alpha = \mathrm{d}t^\alpha, \qquad \eta^\alpha = \mathrm{d}z^\alpha - p_i^\alpha \mathrm{d}q^i.$$

Using these coordinates,

$$\mathcal{D}^{\mathrm{R}} = \left\langle R_\alpha^t = \frac{\partial}{\partial t^\alpha},\ R_\alpha^z = \frac{\partial}{\partial z^\alpha}\right\rangle, \qquad \mathcal{V} = \left\langle \frac{\partial}{\partial p_i^\alpha}\right\rangle.$$

These coordinates are called the Darboux coordinates of the k-cocontact manifold $(M, \tau^\alpha, \eta^\alpha)$.

Proof. By Proposition 3.5, there exist local coordinates $(t^\alpha, x^I, z^\alpha)$ such that

$$R_\alpha^t = \frac{\partial}{\partial t^\alpha}, \qquad \tau^\alpha = \mathrm{d}t^\alpha, \qquad R_\alpha^z = \frac{\partial}{\partial z^\alpha}, \qquad \eta^\alpha = \mathrm{d}z^\alpha - f_I^\alpha(x^J)\mathrm{d}x^I.$$

Since the distribution $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}} = \langle R_\alpha^t = \frac{\partial}{\partial t^\alpha}\rangle$ is involutive and, therefore, integrable, we can consider (at least locally) the quotient manifold $\widetilde{M} = M/(\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}})$, with the projection $\rho : M \to \widetilde{M}$ and local coordinates (x^I, z^α).

The one-forms η^α, the vector fields R_α^z and the distribution $\mathcal{V}$ can be projected to $\widetilde{M}$, and the distribution $\widetilde{\mathcal{D}^{\mathrm{C}}}$ induced by $\mathcal{D}^{\mathrm{C}}$ is $\widetilde{\mathcal{D}^{\mathrm{C}}} = \langle R_\alpha^z\rangle$. It is easy to check that the manifold $(\widetilde{M}, \widetilde{\eta}^\alpha)$, where $\widetilde{\eta}^\alpha$ are the projections of η^α to $\widetilde{M}$, is a k-contact manifold. Since the projected distribution $\widetilde{\mathcal{V}}$ has rank nk, by Theorem 2.4, around every point there exists a local chart $(\widetilde{U}; \widetilde{q}^{\,i}, \widetilde{p}_i^\alpha, \widetilde{z}^{\,\alpha})$ in $\widetilde{M}$ such that

$$\widetilde{\eta}^{\,\alpha} = \mathrm{d}\widetilde{z}^{\,\alpha} - \widetilde{p}_i^\alpha \mathrm{d}\widetilde{q}^{\,i}, \quad \widetilde{\mathcal{V}} = \left\langle \frac{\partial}{\partial \widetilde{p}_i^\alpha}\right\rangle.$$

With all this in mind, in $U = \rho^{-1}(\widetilde{U}) \subset M$, we can take coordinates $(t^\alpha, x^I, z^\alpha) = (t^\alpha, q^i, p_i^\alpha, z^\alpha)$, with $q^i = \widetilde{q}^{\,i} \circ \rho$, $p_i^\alpha = \widetilde{p}_i^\alpha \circ \rho$ and $z^\alpha = \widetilde{z}^{\,\alpha} \circ \rho$ fulfilling the conditions of the theorem. □

Taking into account the previous theorem, we can consider the manifold introduced in Example 3.6 as the canonical model for k-cocontact structures.

IV. HAMILTONIAN FORMALISM

This section introduces the notion of the k-cocontact Hamiltonian system and its Hamilton–De Donder–Weyl equations. The existence of solutions to these equations is proved. We provide local expressions of the Hamilton–De Donder–Weyl equations for maps and k-vector fields in both adapted and Darboux coordinates.

Definition 4.1. A *k-cocontact Hamiltonian system* is a tuple $(M, \tau^\alpha, \eta^\alpha, h)$, where $(\tau^\alpha, \eta^\alpha)$ is a k-cocontact structure on the manifold M and $h : M \to \mathbb{R}$ is a *Hamiltonian function*. Given a map $\psi : D \subset \mathbb{R}^k \to M$, the *k-cocontact Hamilton–De Donder–Weyl equations for the map* ψ are

$$\begin{cases} i(\psi'_\alpha)\mathrm{d}\eta^\alpha = \left(\mathrm{d}h - (\mathscr{L}_{R_\alpha^t} h)\tau^\alpha - (\mathscr{L}_{R_\alpha^z} h)\eta^\alpha\right) \circ \psi, \\ i(\psi'_\alpha)\eta^\alpha = -h \circ \psi, \\ i(\psi'_\alpha)\tau^\beta = \delta_\alpha^\beta. \end{cases} \tag{4}$$

Now we are going to look at the expression in coordinates of the Hamilton–De Donder–Weyl equations (4).

Consider first the adapted coordinates $(t^\alpha, x^I, z^\alpha)$. In these coordinates,

$$R_\alpha^t = \frac{\partial}{\partial t^\alpha}, \quad \tau^\alpha = \mathrm{d}t^\alpha, \quad R_\alpha^z = \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha = \mathrm{d}z^\alpha - f_I^\alpha(x^J)\mathrm{d}x^I, \quad \mathrm{d}\eta^\alpha = \frac{1}{2}\omega_{IJ}^\alpha \mathrm{d}x^I \wedge \mathrm{d}x^J,$$

where $\omega_{IJ}^\alpha = \frac{\partial f_J^\alpha}{\partial x^I} - \frac{\partial f_I^\alpha}{\partial x^J}$. Consider a map $\psi : D \subset \mathbb{R}^k \to M$ with local expression $\psi(s) = (t^\alpha(s), x^I(s), z^\alpha(s))$. Then,

$$\psi'_\alpha = \left(t^\beta, x^I, z^\beta; \frac{\partial t^\beta}{\partial s^\alpha}, \frac{\partial x^I}{\partial s^\alpha}, \frac{\partial z^\beta}{\partial s^\alpha}\right).$$

Then, the Hamilton–De Donder–Weyl equations in adapted coordinates read

$$\begin{cases} \dfrac{\partial x^J}{\partial s^\alpha}\omega^\alpha_{JI} = \left(\dfrac{\partial h}{\partial x^I} + \dfrac{\partial h}{\partial z^\alpha} f^\alpha_I\right)\circ\psi, \\ \dfrac{\partial z^\alpha}{\partial s^\alpha} - f^\alpha_I \dfrac{\partial x^I}{\partial s^\alpha} = -h\circ\psi, \\ \dfrac{\partial t^\alpha}{\partial s^\beta} = \delta^\alpha_\beta. \end{cases}$$

On the other hand, if the local expression in Darboux coordinates of a map $\psi : D\subset\mathbb{R}^k \to M$ is $\psi(r) = (t^\alpha(r), q^i(r), p^\alpha_i(r), z^\alpha(r))$, where $r = (r^1,\ldots,r^k)\in\mathbb{R}^k$. Then, the Hamilton–De Donder–Weyl equations in Darboux coordinates read

$$\begin{cases} \dfrac{\partial t^\beta}{\partial r^\alpha} = \delta^\beta_\alpha, \\ \dfrac{\partial q^i}{\partial r^\alpha} = \dfrac{\partial h}{\partial p^\alpha_i}\circ\psi, \\ \dfrac{\partial p^\alpha_i}{\partial r^\alpha} = -\left(\dfrac{\partial h}{\partial q^i} + p^\alpha_i\dfrac{\partial h}{\partial z^\alpha}\right)\circ\psi, \\ \dfrac{\partial z^\alpha}{\partial r^\alpha} = \left(p^\alpha_i\dfrac{\partial h}{\partial p^\alpha_i} - h\right)\circ\psi. \end{cases} \tag{5}$$

Definition 4.2. Consider a k-cocontact Hamiltonian system $(M,\tau^\alpha,\eta^\alpha,h)$. The *$k$-cocontact Hamilton–De Donder–Weyl equations for a k-vector field* $\mathbf{X} = (X_\alpha)\in\mathfrak{X}^k(M)$ are

$$\begin{cases} i(X_\alpha)\mathrm{d}\eta^\alpha = \mathrm{d}h - (\mathscr{L}_{R^t_\alpha}h)\tau^\alpha - (\mathscr{L}_{R^z_\alpha}h)\eta^\alpha, \\ i(X_\alpha)\eta^\alpha = -h, \\ i(X_\alpha)\tau^\beta = \delta^\beta_\alpha. \end{cases} \tag{6}$$

A k-vector field solution to these equations is a *k-cocontact Hamiltonian k-vector field*. We will denote this set of k-vector fields by $\mathfrak{X}^k_{\mathrm{ham}}(M)$.

Proposition 4.3. The k-cocontact Hamilton–De Donder–Weyl equations (6) admit solutions. They are not unique if $k > 1$.

Proof. Consider the bundle maps

$$\rho : \mathrm{T}M \to \bigoplus^k \mathrm{T}^*M, \qquad \sigma : \bigoplus^k \mathrm{T}M \to \mathrm{T}^*M,$$

given by

$$\rho(X) = (i_X\mathrm{d}\eta^1,\ldots,i_X\mathrm{d}\eta^k), \qquad \sigma(X_1,\ldots,X_k) = i_{X_\alpha}\mathrm{d}\eta^\alpha.$$

These morphisms can be extended to $\mathscr{C}^\infty(M)$-modules. Notice that $\ker\rho = \mathcal{D}^{\mathrm{R}}$ is the Reeb distribution. Using the natural identification $(E\oplus F)^* = E^*\oplus F^*$, the transposed morphism of τ is ${}^t\tau = -\rho$, taking into account that ${}^t\mathrm{d}\eta^\alpha = -\mathrm{d}\eta^\alpha$.

The first Hamilton–De Donder–Weyl equation for a k-vector field $\mathbf{X}$ can be written as

$$\tau\circ\mathbf{X} = \mathrm{d}h - R^z_\alpha(h)\eta^\alpha - R^t_\alpha(h)\tau^\alpha.$$

A sufficient condition for this linear equation to have solutions $\mathbf{X}$ is that the right-hand-side must be in the image of τ, that is, annihilated by any section of $\mathcal{D}^{\mathrm{R}} = \ker{}^t\tau$. However, since

$$i_R(\mathrm{d}h - R^z_\alpha(h)\eta^\alpha - R^t_\alpha(h)\tau^\alpha) = 0, \qquad \text{for every } R\in\mathcal{D}^{\mathrm{R}},$$

we can conclude that the first Hamilton–De Donder–Weyl has solutions. Notice that if $\mathbf{X}$ is a solution to the first equation, $\mathbf{X}+\mathbf{R}$, where $\mathbf{R}$ is a k-vector field whose components are in $\mathcal{D}^{\mathrm{R}}$, is also a solution. On the other hand, the second and third equations have common solutions $\mathbf{R}$ whose components belong to the Reeb distribution, for instance, $\mathbf{R} = (-hR^z_1 + R^t_1, R^t_2,\ldots,R^t_k)$.

The non-uniqueness for $k > 1$ is obvious. □

Consider a k-vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ with local expression in adapted coordinates

$$X_\alpha = A_\alpha^\beta \frac{\partial}{\partial t^\beta} + B_\alpha^I \frac{\partial}{\partial x^I} + D_\alpha^\beta \frac{\partial}{\partial z^\beta}.$$

Therefore, Eq. (6) in adapted coordinates read

$$\begin{cases} A_\alpha^\beta = \delta_\alpha^\beta, \\ B_\alpha^J \omega_{JI}^\alpha = \dfrac{\partial h}{\partial x^I} + \dfrac{\partial h}{\partial z^\alpha} f_I^\alpha, \\ D_\alpha^\alpha - f_I^\alpha B_\alpha^I = -h. \end{cases}$$

On the other hand, consider a k-vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ with local expression in Darboux coordinates

$$X_\alpha = A_\alpha^\beta \frac{\partial}{\partial t^\beta} + B_\alpha^i \frac{\partial}{\partial q^i} + C_{\alpha i}^\beta \frac{\partial}{\partial p_i^\beta} + D_\alpha^\beta \frac{\partial}{\partial z^\beta}.$$

Imposing Eq. (6), we get the conditions

$$\begin{cases} A_\alpha^\beta = \delta_\alpha^\beta, \\ B_\alpha^i = \dfrac{\partial h}{\partial p_i^\alpha}, \\ C_{\alpha i}^\alpha = -\left(\dfrac{\partial h}{\partial q^i} + p_i^\alpha \dfrac{\partial h}{\partial z^\alpha} \right), \\ D_\alpha^\alpha = p_i^\alpha \dfrac{\partial h}{\partial p_i^\alpha} - h. \end{cases}$$

Proposition 4.4. Let $\mathbf{X} \in \mathfrak{X}^k(M)$ be an integrable k-vector field. Then $\mathbf{X}$ is a solution to (6) if and only if every integral section of $\mathbf{X}$ satisfies the k-cocontact Hamilton–De Donder–Weyl equations (4).

Proof. Recall that since $\mathbf{X}$ is integrable, every point of M is in the image of an integral section of $\mathbf{X}$. The proposition is a direct consequence of this fact and of Eqs. (4) and (6). □

It is worth noting that, as in the k-symplectic and k-contact cases, Eqs. (4) and (6) are not completely equivalent since a solution to (4) may not be an integral section of an integrable k-vector field $\mathbf{X}$ solution to Eq. (6).

The following proposition provides an alternative way of writing the k-cocontact Hamilton–De Donder–Weyl equations for k-vector fields:

Proposition 4.5. The k-cocontact Hamilton–De Donder–Weyl equations (6) are equivalent to

$$\begin{cases} \mathscr{L}_{X_\alpha} \eta^\alpha = -(\mathscr{L}_{R_\alpha^t} h)\tau^\alpha - (\mathscr{L}_{R_\alpha^z} h)\eta^\alpha, \\ i(X_\alpha)\eta^\alpha = -h, \\ i(X_\alpha)\tau^\beta = \delta_\alpha^\beta. \end{cases}$$

V. LAGRANGIAN FORMALISM

In this section, we devise the Lagrangian counterpart of the formulations introduced in Sec. IV. We begin by introducing the geometric structures of the phase bundle and defining the notion of a second-order partial differential equation. In second place, we develop the Lagrangian formalism and introduce the k-cocontact Euler–Lagrange equations as the Hamilton–De Donder–Weyl of a k-cocontact Lagrangian system.

A. Geometry of the phase bundle

The phase space for the Lagrangian counterpart of the k-cocontact formalism will be the product bundle $M = \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ endowed with natural coordinates $(t^\alpha, q^i, v_\alpha^i, z^\alpha)$. We have the natural projections

$$
\begin{aligned}
&\tau_1^\alpha : M \to \mathbb{R}, && \tau_1^\alpha(t^1,\dots,t^k,v_{q_1},\dots,v_{q_k},z^1,\dots,z^k) = t^\alpha,\\
&\tau_2 : M \to \oplus^k \mathrm{T}Q, && \tau_2(t^1,\dots,t^k,v_{q_1},\dots,v_{q_k},z^1,\dots,z^k) = (v_{q_1},\dots,v_{q_k}),\\
&\tau_2^\alpha : M \to \mathrm{T}Q, && \tau_2^\alpha(t^1,\dots,t^k,v_{q_1},\dots,v_{q_k},z^1,\dots,z^k) = v_{q_\alpha},\\
&\tau^\alpha : \oplus^k \mathrm{T}Q \to \mathrm{T}Q, && \tau^\alpha(t^1,\dots,t^k,v_{q_1},\dots,v_{q_k},z^1,\dots,z^k) = v_{q_\alpha},\\
&\tau_3^\alpha : M \to \mathbb{R}, && \tau_3^\alpha(t^1,\dots,t^k,v_{q_1},\dots,v_{q_k},z^1,\dots,z^k) = z^\alpha,\\
&\tau_0 : M \to \mathbb{R}^k \times Q \times \mathbb{R}^k, && \tau_0(t^1,\dots,t^k,v_{q_1},\dots,v_{q_k},z^1,\dots,z^k) = (t^1,\dots,t^k,q,z^1,\dots,z^k),
\end{aligned}
$$

which can be summarized in the following diagram:

$$
\begin{tikzcd}
\mathbb{R} & \mathbb{R}^k \times \bigoplus^k \mathrm{T}Q \times \mathbb{R}^k \arrow[l, "\tau_1^\alpha"'] \arrow[r, "\tau_3^\alpha"] \arrow[d, "\tau_2"] \arrow[dr, "\tau_2^\alpha"] \arrow[dd, bend right, "\tau_0"'] & \mathbb{R} \\
& \bigoplus^k \mathrm{T}Q \arrow[r, "\tau^\alpha"'] & \mathrm{T}Q \\
& \mathbb{R}^k \times Q \times \mathbb{R}^k &
\end{tikzcd}
.
$$

Since the bundle $\tau_2 : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}Q$ is trivial, the canonical structures in $\oplus^k\mathrm{T}Q$, namely, the canonical k-tangent structure (J^α) and the Liouville vector field Δ, can be extended to $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ in a natural way. Their local expressions remain the same,

$$
J^\alpha = \frac{\partial}{\partial v_\alpha^i} \otimes \mathrm{d}q^i, \qquad \Delta = v_\alpha^i \frac{\partial}{\partial v_\alpha^i}.
$$

These canonical structures can be used to extend the notion of SOPDE (second-order partial differential equation) to the bundle $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$:

Definition 5.1. A k-vector field $\mathbf{\Gamma} = (\Gamma_\alpha) \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ is a *second-order partial differential equation* or SOPDE if $J^\alpha(\Gamma_\alpha) = \Delta$.

A straightforward computation shows that the local expression of a SOPDE reads

$$
\Gamma_\alpha = A_\alpha^\beta \frac{\partial}{\partial t^\beta} + v_\alpha^i \frac{\partial}{\partial q^i} + C_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i} + D_\alpha^\beta \frac{\partial}{\partial z^\beta}.
$$

Definition 5.2. Consider a map $\psi : \mathbb{R}^k \to \mathbb{R}^k \times Q \times \mathbb{R}^k$ with $\psi = (t^\alpha, \phi, z^\alpha)$, where $\phi : \mathbb{R}^k \to Q$. The *first prolongation* of ψ to $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ is the map $\psi' : \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ given by $\psi' = (t^\alpha, \phi', z^\alpha)$, where ϕ' is the first prolongation of ϕ to $\oplus^k\mathrm{T}Q$. The map ψ' is said to be *holonomic*.

Let $\psi : \mathbb{R}^k \to \mathbb{R}^k \times Q \times \mathbb{R}^k$ be a map with local expression $\psi(r) = (t^\alpha(r), q^i(r), z^\alpha(r))$, where $r \in \mathbb{R}^k$. Then, its first prolongation has local expression

$$
\psi'(r) = \left(t^\alpha(r), q^i(r), \frac{\partial q^i}{\partial r^\alpha}(r), z^\alpha(r)\right).
$$

Proposition 5.3. An integrable k-vector field $\mathbf{\Gamma} \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ is a SOPDE *if and only if its integral sections areholonomic .*

It is important to point out that the product manifold $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ does not have a canonical k-cocontact structure, in contrast to what happens to the manifold $\mathbb{R}^k \times \oplus^k\mathrm{T}^*Q \times \mathbb{R}^k$, where we do have a natural k-cocontact structure as seen in Example 3.6. In what follows, we will show that, in favorable cases, given a Lagrangian function L defined on $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ one can build up a k-cocontact structure.

Definition 5.4. A *Lagrangian function* on $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ is a function $L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}$.

- The *Lagrangian energy* associated with the Lagrangian function L is the function $E_L \in \mathscr{C}^\infty(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ given by $E_L = \Delta(L) - L$.
- The *Cartan forms* associated with the Lagrangian L are

$$\theta_L^\alpha = {}^tJ^\alpha \circ \mathrm{d}L \in \Omega^1\left(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k\right), \qquad \omega_L^\alpha = -\mathrm{d}\theta_L^\alpha \in \Omega^2\left(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k\right),$$

where ${}^tJ^\alpha$ denotes the transpose of J^α.

- The *contact forms* associated with the Lagrangian L are

$$\eta_L^\alpha = \mathrm{d}z^\alpha - \theta_L^\alpha \in \Omega^1\left(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k\right).$$

- The couple $(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k, L)$ is a *k-cocontact Lagrangian system.*

It is clear that $\mathrm{d}\eta_L^\alpha = \omega_L^\alpha$. The local expressions in natural coordinates $(t^\alpha, q^i, v_\alpha^i, z^\alpha)$ of the objects introduced in the previous definition are

$$E_L = v_\alpha^i \frac{\partial L}{\partial v_\alpha^i} - L,$$
$$\theta_L^\alpha = \frac{\partial L}{\partial v_\alpha^i}\mathrm{d}q^i,$$
$$\eta_L^\alpha = \mathrm{d}z^\alpha - \frac{\partial L}{\partial v_\alpha^i}\mathrm{d}q^i,$$
$$\mathrm{d}\eta_L^\alpha = \frac{\partial^2 L}{\partial t^\beta \partial v_\alpha^i}\mathrm{d}q^i \wedge \mathrm{d}t^\beta + \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i}\mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i}\mathrm{d}q^i \wedge \mathrm{d}v_\beta^j + \frac{\partial^2 L}{\partial z^\beta \partial v_\alpha^i}\mathrm{d}q^i \wedge \mathrm{d}z^\beta.$$

Before introducing the Legendre map associated with a Lagrangian function, let us recall the notion of a fiber derivative. Given two vector bundles E, F over the same base manifold B and a bundle map $f : E \to F$, the *fiber derivative* of f is the map $\mathcal{F}f : E \to \mathrm{Hom}(E, F) \cong F \otimes E^*$ obtained by restricting the map f to the fibers $f_b : E_b \to F_b$ and computing the usual derivative: $\mathcal{F}f(e_b) = Df_b(e_b)$. If the second vector bundle is trivial and has rank 1, namely, for a function $f : E \to \mathbb{R}$, then $\mathcal{F}f : E \to E^*$. This fiber derivative has a fiber derivative $\mathcal{F}(\mathcal{F}f) = \mathcal{F}^2 f : E \to E^* \otimes E^*$, called the *fiber Hessian* of f. For every $e_b \in E_b \subset E$, $\mathcal{F}^2 f(e_b)$ is a symmetric bilinear form on E_b. The fiber derivative $\mathcal{F}f$ is a local diffeomorphism at a point $e \in E$ if and only if the Hessian $\mathcal{F}^2 f(e)$ is non-degenerate (see Ref. 43 for more details).

Definition 5.5. Given a Lagrangian function $L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}$, the *Legendre map* of L is its fiber derivative as a function on the vector bundle $\tau_0 : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}^k \times Q \times \mathbb{R}^k$. Namely, the Legendre map of a Lagrangian function $L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}$ is the map

$$\mathcal{F}L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k,$$

given by

$$\mathcal{F}L(t, v_{q_1}, \ldots, v_{q_k}, z) = (t, \mathcal{F}L(t, \cdot, z)(v_{q_1}, \ldots, v_{q_k}), z),$$

where $\mathcal{F}L(t, \cdot, z)$ denotes the Legendre map of the Lagrangian function with t and z freezed.

In natural coordinates $(t^\alpha, q^i, v_\alpha^i, z^\alpha)$, the Legendre map has local expression

$$\mathcal{F}L(t^\alpha, q^i, v_\alpha^i, z^\alpha) = \left(t^\alpha, q^i, \frac{\partial L}{\partial v_\alpha^i}, z^\alpha\right).$$

Proposition 5.6. The Cartan forms satisfy

$$\theta_L^\alpha = (\pi_2^\alpha \circ \mathcal{F}L)^*\theta, \qquad \omega_L^\alpha = (\pi_2^\alpha \circ \mathcal{F}L)^*\omega,$$

where $\theta \in \Omega^1(\mathrm{T}^*Q)$ and $\omega = -d\theta \in \Omega^2(\mathrm{T}^*Q)$ are the Liouville and symplectic canonical forms of the cotangent bundle T^*Q.

The regularity of the Legendre map characterizes the Lagrangian functions that yield k-cocontact structures on the phase bundle $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$.

Proposition 5.7. Consider a Lagrangian function $L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}$. The following are equivalent:

1. *The Legendre map $\mathcal{F}L$ is a local diffeomorphism.*
2. *The fiber Hessian of the Lagrangian L, namely, the map*

$$\mathcal{F}^2 L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \left(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k\right) \otimes \left(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k\right),$$

is everywhere nondegenerate, where the tensor product is of vector bundles over $\mathbb{R}^k \times Q \times \mathbb{R}^k$.

(3) *The family $(\tau^\alpha = \mathrm{d}t^\alpha, \eta_L^\alpha)$ is a k-cocontact structure on $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$.*

Proof. Taking natural coordinates $(t^\alpha, q^i, v_\alpha^i, z^\alpha)$, we have

$$\mathcal{F}^2 L(t^\alpha, q^i, v_\alpha^i, z^\alpha) = \left(t^\alpha, q^i, W_{ij}^{\alpha\beta}, z^\alpha\right), \quad \text{where} \quad W_{ij}^{\alpha\beta} = \left(\frac{\partial^2 L}{\partial v_\alpha^i \partial v_\beta^j}\right).$$

The conditions in the proposition mean that the matrix $W = (W_{ij}^{\alpha\beta})$ is everywhere nonsingular. □

Definition 5.8. A Lagrangian function $L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}$ is said to be *regular* if the equivalent statements in Proposition 5.7 hold. Otherwise L is said to be *singular*. In addition, if the Legendre map $\mathcal{F}L$ is a global diffeomorphism, L is a *hyperregular* Lagrangian.

Let $(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k, L)$ be a regular k-cocontact Lagrangian system. By Theorem 3.4, the Reeb vector fields $(R_L^t)_\alpha, (R_L^z)_\alpha \in \mathfrak{X}(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ are uniquely given by the relations

$$\begin{aligned} &i\big((R_L^t)_\alpha\big)\,\mathrm{d}\eta_L^\beta = 0, \qquad &&i\big((R_L^t)_\alpha\big)\,\eta_L^\beta = 0, \qquad &&i\big((R_L^t)_\alpha\big)\,\mathrm{d}t^\beta = \delta_\alpha^\beta, \\ &i\big((R_L^z)_\alpha\big)\,\mathrm{d}\eta_L^\beta = 0, \qquad &&i\big((R_L^z)_\alpha\big)\,\eta_L^\beta = \delta_\alpha^\beta, \qquad &&i\big((R_L^z)_\alpha\big)\,\mathrm{d}t^\beta = 0. \end{aligned}$$

The local expressions of the Reeb vector fields are

$$\begin{aligned} (R_L^t)_\alpha &= \frac{\partial}{\partial t^\alpha} - W_{\gamma\beta}^{ji}\frac{\partial^2 L}{\partial t^\alpha \partial v_\gamma^j}\frac{\partial}{\partial v_\beta^i}, \\ (R_L^z)_\alpha &= \frac{\partial}{\partial z^\alpha} - W_{\gamma\beta}^{ji}\frac{\partial^2 L}{\partial z^\alpha \partial v_\gamma^j}\frac{\partial}{\partial v_\beta^i}, \end{aligned}$$

where $W_{\alpha\beta}^{ij}$ is inverse of the Hessian matrix $W_{ij}^{\alpha\beta} = \left(\frac{\partial^2 L}{\partial v_\alpha^i \partial v_\beta^j}\right)$, namely,

$$W_{\alpha\beta}^{ij}\frac{\partial^2 L}{\partial v_\beta^j \partial v_\gamma^k} = \delta_k^i \delta_\alpha^\gamma.$$

B. k-cocontact Euler–Lagrange equations

We have proved in Sec. V A that every regular k-cocontact Lagrangian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k, L)$ yields the k-cocontact Hamiltonian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k, \tau^\alpha = \mathrm{d}t^\alpha, \eta^\alpha, E_L)$. Taking this into account, we can define,

Definition 5.9. Let $(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k, L)$ be a k-cocontact Lagrangian system. The *k-cocontact Euler–Lagrange equations for a holonomic map* $\psi : \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ are

$$\begin{cases} i(\psi'_\alpha)\mathrm{d}\eta_L^\alpha = \left(\mathrm{d}E_L - (\mathscr{L}_{(R_L^t)_\alpha} E_L)\mathrm{d}t^\alpha - (\mathscr{L}_{(R_L^z)_\alpha} E_L)\eta_L^\alpha\right) \circ \psi, \\ i(\psi'_\alpha)\eta_L^\alpha = -E_L \circ \psi, \\ i(\psi'_\alpha)\mathrm{d}t^\beta = \delta_\alpha^\beta. \end{cases} \tag{7}$$

The *k-cocontact Lagrangian equations for a k-vector field* $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ are

$$\begin{cases} i(X_\alpha)\mathrm{d}\eta_L^\alpha = \mathrm{d}E_L - (\mathscr{L}_{(R_L^t)_\alpha} E_L)\mathrm{d}t^\alpha - (\mathscr{L}_{(R_L^z)_\alpha} E_L)\eta_L^\alpha, \\ i(X_\alpha)\eta_L^\alpha = -E_L, \\ i(X_\alpha)\mathrm{d}t^\beta = \delta_\alpha^\beta. \end{cases} \tag{8}$$

A k-vector field $\mathbf{X}$ solution to Eq. (8) is said to be a *k-cocontact Lagrangian vector field*.

The next proposition states that, if the Lagrangian L is regular, the Lagrangian Eq. (8) always has solutions, although they are not unique in general. It is a direct translation of Proposition 2.7 into the Lagrangian language.

Proposition 5.10. Consider a regular k-cocontact Lagrangian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k, L)$*. Then, the k-cocontact Lagrangian Eq. (8) admits solutions. They are not unique if* $k > 1$.

Consider a map $\psi : \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ with local expression in natural coordinates $\psi(r) = (t^\alpha(r), q^i(r), v^i_\alpha(r), z^\alpha(r))$, where $r = (r^1, \dots, r^k) \in \mathbb{R}^k$. Then, Eq. (7) for the map ψ read

$$\begin{cases} \dfrac{\partial t^\beta}{\partial r^\alpha} = \delta^\beta_\alpha, \\ \dfrac{\partial}{\partial r^\alpha}\left(\dfrac{\partial L}{\partial v^i_\alpha} \circ \psi\right) = \left(\dfrac{\partial L}{\partial q^i} + \dfrac{\partial L}{\partial z^\alpha}\dfrac{\partial L}{\partial v^i_\alpha}\right) \circ \psi, \\ \dfrac{\partial (z^\alpha)}{\partial r^\alpha} = L \circ \psi. \end{cases} \tag{9}$$

For a k-vector field $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$, with local expression in natural coordinates

$$X_\alpha = A^\beta_\alpha \frac{\partial}{\partial t^\beta} + B^i_\alpha \frac{\partial}{\partial q^i} + C^i_{\alpha\beta} \frac{\partial}{\partial v^i_\beta} + D^\beta_\alpha \frac{\partial}{\partial z^\beta}.$$

Equation (8) read

$$0 = A^\beta_\alpha - \delta^\beta_\alpha, \tag{10}$$

$$0 = \left(B^j_\alpha - v^j_\alpha\right)\frac{\partial^2 L}{\partial v^j_\alpha \partial z^\beta}, \tag{11}$$

$$0 = \left(B^j_\alpha - v^j_\alpha\right)\frac{\partial^2 L}{\partial v^j_\alpha \partial t^\beta}, \tag{12}$$

$$0 = \left(B^j_\alpha - v^j_\alpha\right)\frac{\partial^2 L}{\partial v^i_\beta \partial v^j_\alpha}, \tag{13}$$

$$\begin{aligned} 0 = \left(B^j_\alpha - v^j_\alpha\right)\frac{\partial^2 L}{\partial q^i \partial v^j_\alpha} + \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial t^\alpha \partial v^i_\alpha} - \frac{\partial^2 L}{\partial q^j \partial v^i_\alpha} B^j_\alpha \\ - \frac{\partial^2 L}{\partial v^j_\beta \partial v^i_\alpha} C^j_{\alpha\beta} - \frac{\partial^2 L}{\partial z^\beta \partial v^i_\alpha} D^\beta_\alpha + \frac{\partial L}{\partial z^\alpha}\frac{\partial L}{\partial v^i_\alpha}, \end{aligned} \tag{14}$$

$$0 = L + \frac{\partial L}{\partial v^i_\alpha}\left(B^i_\alpha - v^i_\alpha\right) - D^\alpha_\alpha. \tag{15}$$

If the Lagrangian function L is regular, Eq. (13) yield the conditions $B^i_\alpha = v^i_\alpha$, namely, the k-vector field $\mathbf{X}$ has to be a SOPDE. In this case, Eqs. (11) and (12) hold identically and Eqs. (10), (14), and (15) yield

$$A^\beta_\alpha = \delta^\beta_\alpha, \tag{16}$$

$$\frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial z^\alpha}\frac{\partial L}{\partial v^i_\alpha} = \frac{\partial^2 L}{\partial t^\alpha \partial v^i_\alpha} + \frac{\partial^2 L}{\partial q^j \partial v^i_\alpha} v^j_\alpha + \frac{\partial^2 L}{\partial v^j_\beta \partial v^i_\alpha} C^j_{\alpha\beta} + \frac{\partial^2 L}{\partial z^\beta \partial v^i_\alpha} D^\beta_\alpha, \tag{17}$$

$$D^\alpha_\alpha = \mathcal{L}. \tag{18}$$

If the SOPDE $\mathbf{X}$ is integrable, Eqs. (16)–(18) are the Euler–Lagrange Eq. (9) for its integral maps. Therefore, we have proven the following:

Proposition 5.11. Let $L : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}$ *be a regular Lagrangian and consider a Lagrangian k-vector field* $\mathbf{X}$*, namely, a solution to Eq. (8). Then* $\mathbf{X}$ *is a* SOPDE *and if, in addition,* $\mathbf{X}$ *is integrable, its integral sections are solutions to the k-cocontact Euler–Lagrange equations (7).*

The SOPDE $\mathbf{X}$ *is called an Euler–Lagrange k-vector field associated with the Lagrangian function L.*

Remark 5.12. If the Lagrangian function L is regular or hyperregular, the Legendre map $\mathcal{F}L$ is a (local) diffeomorphism between $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ and $\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ such that $\mathcal{F}L^*\eta^\alpha = \eta^\alpha_L$. In addition, there exists, at least locally, a function $h \in \mathscr{C}^\infty(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k)$

such that $h \circ \mathcal{F}L = E_L$. Then, we have the k-cocontact Hamiltonian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k, \eta^\alpha, h)$, for which $\mathcal{F}L_*(R_L^t)_\alpha = R_\alpha^t$ and $\mathcal{F}L_*(R_L^z)_\alpha = R_\alpha^z$. If $\mathbf{\Gamma}$ is an Euler–Lagrange k-vector field associated to the Lagrangian function L in $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$, we have that the k-vector field $\mathbf{X} = \mathcal{F}L_* \mathbf{\Gamma}$ is a k-cocontact Hamiltonian k-vector field associated to h in $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$, and conversely.

Remark 5.13. In the case $k = 1$, we recover the cocontact Lagrangian formalism presented in the recent paper[31] for time-dependent contact Lagrangian systems.

Remark 5.14. It is important to point out that the field equations obtained in this work from both the Hamiltonian and Lagrangian formalisms coincide with those obtained by means of the so-called *multicontact formalism* introduced in Ref. 44 as a generalization of the multisymplectic setting.

C. Lagrangian functions with holonomic damping term

In this section, a particular type of Lagrangian function is studied in full detail: the so-called Lagrangians with holonomic damping term.[18] This family of Lagrangians is particularly interesting since it appears in many physical examples.

Definition 5.15. A *Lagrangian function with holonomic damping term* in $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ is a function $\mathcal{L} = L + \phi \in \mathscr{C}^\infty(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$, where $L = \bar{\tau}_2^* L_\circ$, where $\bar{\tau}_2 : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}Q$ for some Lagrangian function $L_\circ \in \mathscr{C}^\infty(\mathbb{R}^k \times \oplus^k \mathrm{T}Q)$ and $\phi = \tau_0^* \phi_\circ$, for $\phi_\circ \in \mathscr{C}^\infty(\mathbb{R}^k \times Q \times \mathbb{R}^k)$.

Taking natural coordinates $(t^\alpha, q^i, v_\alpha^i, z^\alpha)$ in $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$, a Lagrangian with holonomic damping term has the expression

$$\mathcal{L}(t^\alpha, q^i, v_\alpha^i, z^\alpha) = L(t^\alpha, q^i, v_\alpha^i) + \phi(t^\alpha, q^i, z^\alpha). \tag{19}$$

It is clear that the momenta $p_i^\alpha = \partial\mathcal{L}/\partial v_\alpha^i$ defined by the Legendre map are independent of the coordinates z^α, namely, one has that $\frac{\partial^2 \mathcal{L}}{\partial z^\alpha \partial v_\beta^i} = 0$ for Lagrangian functions with holonomic damping term.

Proposition 5.16. Consider the Lagrangian function with holonomic damping term $\mathcal{L} = L + \phi$. Then, its Cartan forms, contact forms, Lagrangian energy, and Reeb vector fields read

$$\theta_{\mathcal{L}}^\alpha = \theta_L^\alpha, \quad \eta_{\mathcal{L}}^\alpha = \mathrm{d}z^\alpha - \theta_L^\alpha, \quad E_{\mathcal{L}} = E_L - \phi, \quad (R_{\mathcal{L}}^t)_\alpha = \frac{\partial}{\partial t^\alpha}, \quad (R_{\mathcal{L}}^z)_\alpha = \frac{\partial}{\partial z^\alpha}.$$

where θ_L^α are the Cartan one-forms of L considered (via pull-back) as one-forms on $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$, and E_L is the energy of L as a function on $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$.

The Legendre map of $\mathcal{L}$, namely, $\mathcal{F}\mathcal{L} : \mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$, can be expressed as $\mathcal{F}\mathcal{L} = \mathcal{F}L \times \mathrm{Id}_{\mathbb{R}}^k$, where $\mathcal{F}L$ is the Legendre map of L. The fibered Hessians are related by $\mathcal{F}^2\mathcal{L}(t^\alpha, v_{q_\alpha}, z^\alpha) = \mathcal{F}^2 L(t^\alpha, v_{q_\alpha})$. Moreover, $\mathcal{L}$ is regular if, and only if, L is regular.

The proof of this proposition is straightforward by taking local coordinates. It is also clear that $\mathcal{L}$ is hyperregular if and only if L is hyperregular. In this case, the Legendre map $\mathcal{F}\mathcal{L}$ is a diffeomorphism and one can state the canonical Hamiltonian formulation for the Lagrangian with holonomic damping term $\mathcal{L} = L + \phi$ via the Legendre map.

Consider the k-cocontact Lagrangian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k, \mathcal{L})$, where $\mathcal{L} = L + \phi$ is a Lagrangian function with holonomic damping term as in (19). Recall that the dynamical equations for k-vector fields of this system are

$$\begin{cases} i(X_\alpha)\mathrm{d}\eta_{\mathcal{L}}^\alpha = \mathrm{d}E_{\mathcal{L}} - (\mathscr{L}_{(R_{\mathcal{L}}^t)_\alpha} E_{\mathcal{L}})\mathrm{d}t^\alpha - (\mathscr{L}_{(R_{\mathcal{L}}^z)_\alpha} E_{\mathcal{L}})\eta_{\mathcal{L}}^\alpha, \\ i(X_\alpha)\eta_{\mathcal{L}}^\alpha = -E_{\mathcal{L}}, \\ i(X_\alpha)\mathrm{d}t^\beta = \delta_\alpha^\beta. \end{cases}$$

Take natural coordinates $(t^\alpha, q^i, v_\alpha^i, z^\alpha)$ in $\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k$ and consider a k-vector field $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \mathrm{T}Q \times \mathbb{R}^k)$ with local expression

$$X_\alpha = A_\alpha^\beta \frac{\partial}{\partial t^\beta} + B_\alpha^i \frac{\partial}{\partial q^i} + C_{\alpha\beta}^i \frac{\partial}{\partial v_\beta^i} + D_\alpha^\beta \frac{\partial}{\partial z^\beta}.$$

Then, the second and third Lagrangian equations for the k-vector field $\mathbf{X}$ read

$$A_\alpha^\beta = \delta_\alpha^\beta, \qquad 0 = \mathcal{L} + \frac{\partial L}{\partial v_\alpha^i}\left(B_\alpha^i - v_\alpha^i\right) - D_\alpha^\alpha,$$

and this is Eq. (15) for the Lagrangian function $\mathcal{L} = L + \phi$. The first Lagrangian equation for k-vector fields yields

$$\left(B_\alpha^j - v_\alpha^j\right)\frac{\partial^2 L}{\partial v_\beta^i \partial v_\alpha^j} = 0, \tag{20}$$
$$\left(\frac{\partial^2 L}{\partial q^i \partial v_\alpha^j} - \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i}\right)B_\alpha^j - \frac{\partial^2 L}{\partial t^\alpha \partial v_\alpha^i} - \frac{\partial^2 L}{\partial q^i \partial v_\alpha^j}v_\alpha^j - \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i}C_{\alpha\beta}^j = -\frac{\partial L}{\partial q^i} - \frac{\partial \phi}{\partial q^i} - \frac{\partial \phi}{\partial z^\alpha}\frac{\partial L}{\partial v_\alpha^i},$$

which correspond to Eq. (14) for the Lagrangian $\mathcal{L}$. Notice that Eq. (11) are identities since $\frac{\partial^2 L}{\partial v_\alpha^j \partial z^\beta} = 0$.

Finally, as in Proposition 5.11, if the Lagrangian function $\mathcal{L}$ is regular, namely, if L is regular, Eq. (20) implies that $B_\alpha^j = v_\alpha^j$. Therefore, the k-vector field is a SOPDE and the dynamical equations become

$$\frac{\partial t^\alpha}{\partial r^\beta} = \delta_\alpha^\beta,$$
$$\frac{\partial z^\alpha}{\partial r^\alpha} = \mathcal{L},$$
$$\frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i}\frac{\partial^2 q^j}{\partial r^\alpha \partial r^\beta} + \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i}\frac{\partial q^j}{\partial r^\alpha} + \frac{\partial^2 L}{\partial t^\alpha \partial v_\alpha^i} - \frac{\partial L}{\partial q^i} = \frac{\partial}{\partial r^\alpha}\left(\frac{\partial L}{\partial v_\alpha^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial \phi}{\partial q^i} + \frac{\partial \phi}{\partial z^\alpha}\frac{\partial L}{\partial v_\alpha^i}.$$

These are the expressions in natural coordinates of the Euler–Lagrange Eq. (9), for the Lagrangian with holonomic damping term $\mathcal{L} = L + \phi$.

VI. k-CONTACT SYSTEMS VERSUS AUTONOMOUS k-COCONTACT SYSTEMS

In this section, we are going to compare the k-contact and k-cocontact formulations of field theories. We will work with the canonical manifolds $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ and $\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$. However, due to the Darboux theorems, the results can easily be extended to the case M and $\mathbb{R}^k \times M$ being M a general k-contact manifold. These two canonical manifolds are related by the canonical projection $\bar{\pi}_2 : \mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$. We will denote by $\bar{\eta}^\alpha$ and η^α the canonical contact one-forms of $\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ and $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$, respectively. They are related by the relations $\bar{\eta}^\alpha = \bar{\pi}_2^*\eta^\alpha$ and have the same local expression $\eta^\alpha = \mathrm{d}z^\alpha - p_i^\alpha \mathrm{d}q^i$. The Reeb vector fields will be denoted by $\bar{R}_\alpha^z$ and R_α^z and have local expression $\partial/\partial z^\alpha$.

Definition 6.1. A k-cocontact Hamiltonian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, \mathrm{d}t^\alpha, \eta^\alpha, h)$ is said to be *autonomous* if $R_\alpha^t(h) = \partial h/\partial t^\alpha = 0$ for every $\alpha = 1, \ldots, k$.

Notice that if a Hamiltonian function h does not depend on the variables t^α, there exists a function $h_\circ \in \mathscr{C}^\infty(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k)$ such that $h = \bar{\pi}_2^* h_\circ$.

For an autonomous k-cocontact Hamiltonian system, Eq. (6) read

$$\begin{cases} i(X_\alpha)\mathrm{d}\eta^\alpha = \mathrm{d}h - (\mathscr{L}_{R_\alpha^z}h)\eta^\alpha, \\ i(X_\alpha)\eta^\alpha = -h, \\ i(X_\alpha)\tau^\beta = \delta_\alpha^\beta. \end{cases} \tag{21}$$

Proposition 6.2. Every autonomous k-cocontact Hamiltonian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, h)$ defines a k-contact Hamiltonian system $(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, h_\circ)$, where $h = \bar{\pi}_2^* H_\circ$, and conversely.

Theorem 6.3. *Consider an autonomous k-cocontact Hamiltonian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, h)$ and let $(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, h_\circ)$ be its associated k-contact Hamiltonian system. Then, every section $\bar{\psi} : \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ solution to the Hamilton–De Donder–Weyl Eq. (5) for the system $(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, h)$ defines a map $\psi : \mathbb{R}^k \to \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ solution to the Hamilton–De Donder–Weyl Eq. (2) for the k-contact Hamiltonian system $(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, h_\circ)$, and conversely.*

Proof. Since $h = \bar{\pi}_2^* h_\circ$, one has

$$\frac{\partial h}{\partial q^i} = \frac{\partial h_\circ}{\partial q^i}, \qquad \frac{\partial h}{\partial p_i^\alpha} = \frac{\partial h_\circ}{\partial p_i^\alpha}, \qquad \frac{\partial h}{\partial z^\alpha} = \frac{\partial h_\circ}{\partial z^\alpha}. \tag{22}$$

Let $\bar{\psi} : \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ be a section of the projection $\bar{\pi}_1 : \mathbb{R}^k \times \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k \to \mathbb{R}^k$, which in coordinates reads $\bar{\psi}(t) = (t, \bar{\psi}^i(t), \bar{\psi}_i^\alpha(t), \bar{\psi}^\alpha(t))$ with $t \in \mathbb{R}^k$. We can construct the map $\psi = \bar{\pi}_2 \circ \bar{\psi} : \mathbb{R}^k \to \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$, which in coordinates reads $\psi(t)$

$= (\psi^i(t), \psi_i^\alpha(t), \psi^\alpha(t)) = (\bar{\psi}^i(t), \bar{\psi}_i^\alpha(t), \bar{\psi}^\alpha(t))$. Then, if $\bar{\psi}$ is a solution to the Hamilton–De Donder–Weyl Eq. (5), from (22) one obtains that ψ is a solution to the k-contact Hamilton–De Donder–Weyl Eq. (2).

Conversely, consider a map $\psi : \mathbb{R}^k \to \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$. Define $\bar{\psi} = (\mathrm{Id}_{\mathbb{R}^k}, \psi) : \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$. If $\psi(t) = (\psi^i(t), \psi_i^\alpha(t), \psi^\alpha(t))$, then $\bar{\psi}(t) = (t, \bar{\psi}^i(t), \bar{\psi}_i^\alpha(t), \bar{\psi}^\alpha(t))$ with $\bar{\psi}^i(t) = \psi^i(t)$, $\bar{\psi}_i^\alpha(t) = \psi_i^\alpha(t)$ and $\bar{\psi}^\alpha(t) = \psi^\alpha(t)$. Note that $\mathrm{Im}\bar{\psi} =$ graph ψ. Therefore, if ψ is a solution the k-contact Hamilton–De Donder–Weyl equations (2), we have that $\bar{\psi}$ is a solution to the Hamilton–De Donder–Weyl equations (5). □

The following result relates the k-vector field solution to Eqs. (3) and (21). First, we have to introduce the notion of the suspension of a vector field (see Ref. 1, p. 374 for the definition of suspension in the context of mechanics).

Let $\mathbf{X} = (X_1, \ldots, X_k)$ be a k-vector field on $\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$. For every $\alpha = 1, \ldots, k$, let $\bar{X}_\alpha \in \mathfrak{X}(\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ be the *suspension* of the corresponding vector field X_α in $\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ defined as follows: for every $\mathrm{p} \in \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$, let $\gamma_\mathrm{p}^\alpha : \mathbb{R} \to \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ be the integral curve of X_α passing through p. Then, if $x_0 = (x_0^1, \ldots, x_0^k) \in \mathbb{R}^k$, we can construct the curve $\bar{\gamma}_\mathrm{p}^\alpha : \mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ passing through the point $\bar{\mathrm{p}} = (x_0, \mathrm{p}) \in \mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ given by $\bar{\gamma}_{\bar{\mathrm{p}}}^\alpha(x) = (x_0^1, \ldots, x_0^\alpha + x, \ldots, x_0^k; \gamma_\mathrm{p}(x))$. Then, $\bar{X} \in \mathfrak{X}(\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ is the vector field tangent to $\bar{\gamma}_{\bar{\mathrm{p}}}^\alpha$ at (x_0, p).

In natural coordinates, if X_α has local expression

$$X_\alpha = A_\alpha^i \frac{\partial}{\partial q^i} + B_{\alpha i}^\beta \frac{\partial}{\partial p_i^\beta} + C_\alpha^\beta \frac{\partial}{\partial z^\beta},$$

one has that $\bar{X}_\alpha$ is locally given by

$$\bar{X}_\alpha = \frac{\partial}{\partial t^\alpha} + \bar{A}_\alpha^i \frac{\partial}{\partial q^i} + \bar{B}_{\alpha i}^\beta \frac{\partial}{\partial p_i^\beta} + \bar{C}_\alpha^\beta \frac{\partial}{\partial z^\beta} = \frac{\partial}{\partial t^\alpha} + \bar{\pi}_2^*(A_\alpha^i) \frac{\partial}{\partial q^i} + \bar{\pi}_2^*(B_{\alpha i}^\beta) \frac{\partial}{\partial p_i^\beta} + \bar{\pi}_2^*(C_\alpha^\beta) \frac{\partial}{\partial z^\beta}.$$

Theorem 6.4. *Consider an autonomous k-cocontact Hamiltonian system $(\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k, h)$ and let $(\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k, h_\circ)$ be its associated k-contact Hamiltonian system. Then, every k-vector field $\mathbf{X} \in \mathfrak{X}^k(\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ solution to Eq. (3) defines a k-vector field $\bar{\mathbf{X}} \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ solution to Eq. (21).*

In addition, $\mathbf{X}$ is integrable if and only if its associated $\bar{\mathbf{X}}$ is also integrable.

Proof. Let $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ be a solution to Eq. (3). Define $\bar{X}_\alpha \in \mathfrak{X}(\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ as the suspension of the corresponding vector field $X_\alpha \in \mathfrak{X}(\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$.

Notice that the vector fields $\bar{X}_\alpha$ are $\bar{\pi}_2$-projectable, and $(\bar{\pi}_2)_* \bar{X}_\alpha = X_\alpha$. Therefore, we have defined a k-vector field $\bar{\mathbf{X}}$ in $\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$. Therefore, we have

$$\begin{aligned} i_{\bar{X}_\alpha} \mathrm{d}\bar{\eta}^\alpha - \mathrm{d}h - (\mathscr{L}_{\bar{R}_\alpha^z} h)\bar{\eta}^\alpha &= i_{\bar{X}_\alpha} \mathrm{d}(\bar{\pi}_2^* \eta^\alpha) - \mathrm{d}(\bar{\pi}_2^* h_\circ) - (\mathscr{L}_{R_\alpha^z} h_\circ)(\bar{\pi}_2^* \eta^\alpha), \\ &= \pi_2^* \big(i_{(\bar{\pi}_2)_* \bar{X}_\alpha} \mathrm{d}\eta^\alpha - \mathrm{d}h_\circ - (\mathscr{L}_{R_\alpha^z} h)\eta^\alpha \big), \\ &= \pi_2^* \big(i_{X_\alpha} \mathrm{d}\eta^\alpha - \mathrm{d}h_\circ - (\mathscr{L}_{R_\alpha^z} h)\eta^\alpha \big), \\ &= 0, \end{aligned}$$

since $\mathbf{X} = (X_\alpha)$ satisfies Eq. (3). It is easy to check that the other equations also hold. Therefore, $\bar{\mathbf{X}} = (\bar{X}_\alpha)$ satisfies Eq. (21).

In addition, if $\psi : \mathbb{R}^k \to \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ is an integral section of $\mathbf{X}$, one has that $\bar{\psi} : \mathbb{R}^k \to \mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ such that $\bar{\psi} = (\mathrm{Id}_{\mathbb{R}^k}, \psi)$ (see Theorem 6.3) is an integral section of $\bar{\mathbf{X}}$.

On the other hand, if $\bar{\psi}$ is an integral section of $\bar{\mathbf{X}}$, Eq. (21) hold for the map $\bar{\psi}(t) = (t, \bar{\psi}^i(t), \bar{\psi}_i^\alpha(t), \bar{\psi}^\alpha(t))$. Since $\bar{A}_\alpha^i = \bar{\pi}_2^*(A_\alpha^i)$, $\bar{B}_{\alpha i}^\beta = \pi_2^*(B_{\alpha i}^\beta)$, and $\bar{C}_\alpha^\beta = \bar{\pi}_2^*(C_\alpha^\beta)$, this is equivalent to say that Eq. (1) hold for the map $\psi(t) = (\psi^i(t), \psi_i^\alpha(t), \psi^\alpha(t))$ or, equivalently, ψ is an integral section of $\mathbf{X}$. □

Notice that the converse statement of the previous theorem is not true. Actually, the k-vector fields that are solutions to the geometric field Eq. (21) are not completely determined, and then there are k-vector fields in $\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ that are not $\bar{\pi}_2$-projectable, for instance, taking their undetermined components to be not $\bar{\pi}_2$-projectable. However, if we only consider those solutions that are integral sections of k-vector fields solution to the geometric field equations, one can prove that every integrable k-vector field $\bar{\mathbf{X}} \in \mathfrak{X}^k(\mathbb{R}^k \times \oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ solution to the k-cocontact Hamilton–De Donder–Weyl equations is associated with an integrable k-vector field $\mathbf{X} \in \mathfrak{X}^k(\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ solution to the k-contact Hamilton–De Donder–Weyl equations.

The results presented in this section can be translated to the Lagrangian formalism when considering regular autonomous Lagrangians ($\partial L/\partial t^\alpha = 0$, or equivalently, $\partial E_L/\partial t^\alpha = 0$).

VII. AN EXAMPLE: ONE-DIMENSIONAL NONLINEAR WAVE EQUATION WITH DAMPING

A one-dimensional nonlinear wave with an external time-dependent forcing can be modeled by the equation

$$u_{tt} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial u_x}(t,u_x)\right) - \frac{\partial g}{\partial u}(t,u), \tag{23}$$

where $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ and $u(t,x)$, $f(t,u_x)$, and $g(t,u)$ are smooth functions. Notice that if $g(t,u) = 0$ and $f(t,u_x) = c^2 u_x^2/2$ with $c \in \mathbb{R}$, we recover the usual wave equation $u_{tt} = c^2 u_{xx}$. This equation can be obtained from the Lagrangian function $L : \mathbb{R}^2 \times \oplus^2 \mathrm{T}\mathbb{R} \to \mathbb{R}$[45] given by

$$L(t,x;u,u_t,u_x) = \frac{1}{2}u_t^2 - f(t,u_x) - g(t,u),$$

where we will assume the regularity condition $\frac{\partial^2 f}{\partial u_x^2} \neq 0$. We are going to modify this Lagrangian function in order to add a damping term proportional to u_t to Eq. (23).

A. Lagrangian formalism

Consider the Lagrangian function with holonomic damping term $\mathcal{L} : \mathbb{R}^2 \times \oplus^2 \mathrm{T}\mathbb{R} \times \mathbb{R}^2$ given by $\mathcal{L}(t,x;u,u_t,u_x;z^t,z^x) = L(t,x;u,u_t,u_x) + \phi(x,z^t)$, where $\phi(x,z^t) = -\gamma(x)z^t$. Then, we have

$$\mathcal{L}(t,x;u,u_t,u_x;z^t,z^x) = \frac{1}{2}u_t^2 - f(t,u_x) - g(t,u) - \gamma(x)z^t. \tag{24}$$

For this Lagrangian, we have

$$\begin{gathered}
\mathrm{d}\mathcal{L} = -\left(\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t}\right)\mathrm{d}t - z^t\frac{\partial \gamma}{\partial x} - \frac{\partial g}{\partial u}\mathrm{d}u + u_t\mathrm{d}u_t - \frac{\partial f}{\partial u_x}\mathrm{d}u_x - \gamma(x)\mathrm{d}z^t,\\
E_{\mathcal{L}} = \frac{1}{2}u_t^2 - u_x\frac{\partial f}{\partial u_x} + f(t,u_x) + g(t,u) + \gamma(x)z^t,\\
\mathrm{d}E_{\mathcal{L}} = \left(-u_x\frac{\partial^2 f}{\partial t\partial u_x} + \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t}\right)\mathrm{d}t + \frac{\partial \gamma}{\partial x}z^t\mathrm{d}x + \frac{\partial g}{\partial u}\mathrm{d}u + u_t\mathrm{d}u_t - u_x\frac{\partial^2 f}{\partial u_x^2}\mathrm{d}u_x + \gamma(x)\mathrm{d}z^t,\\
\eta^1_{\mathcal{L}} = \mathrm{d}z^t - u_t\mathrm{d}u, \qquad \mathrm{d}\eta^1_{\mathcal{L}} = \mathrm{d}u\wedge\mathrm{d}u_t,\\
\eta^2_{\mathcal{L}} = \mathrm{d}z^x + \frac{\partial f}{\partial u_x}\mathrm{d}u, \qquad \mathrm{d}\eta^2_{\mathcal{L}} = \frac{\partial^2 f}{\partial t\partial u_x}\mathrm{d}t\wedge\mathrm{d}u + \frac{\partial^2 f}{\partial u_x^2}\mathrm{d}u_x\wedge\mathrm{d}u,\\
(R^t_{\mathcal{L}})_1 = \frac{\partial}{\partial t} - \left(\frac{\partial^2 f}{\partial u_x^2}\right)^{-1}\frac{\partial^2 f}{\partial t\partial u_x}\frac{\partial}{\partial u_x}, \qquad (R^t_{\mathcal{L}})_2 = \frac{\partial}{\partial x}, \qquad (R^z_{\mathcal{L}})_1 = \frac{\partial}{\partial z^t}, \qquad (R^z_{\mathcal{L}})_2 = \frac{\partial}{\partial z^x}.
\end{gathered}$$

Now, consider a 2-vector field $\mathbf{X} = (X_1, X_2) \in \mathfrak{X}^2(\mathbb{R}^2 \times \oplus^2\mathrm{T}\mathbb{R} \times \mathbb{R}^2)$ with local expression

$$X_\alpha = A^t_\alpha\frac{\partial}{\partial t} + A^x_\alpha\frac{\partial}{\partial x} + B_\alpha\frac{\partial}{\partial u} + C_{\alpha t}\frac{\partial}{\partial u_t} + C_{\alpha x}\frac{\partial}{\partial u_x} + D^t_\alpha\frac{\partial}{\partial z^t} + D^x_\alpha\frac{\partial}{\partial z^x}.$$

For this 2-vector field, the third equation in (8) gives the conditions $A^t_1 = 1$, $A^x_1 = 0$, $A^t_2 = 0$, and $A^x_2 = 1$. We have

$$i(X_\alpha)\mathrm{d}\eta^\alpha_{\mathcal{L}} = -B_2\frac{\partial^2 f}{\partial t\partial u_x}\mathrm{d}t + \left(-C_{1t} + A^t_2\frac{\partial^2 f}{\partial t\partial u_x} + C_{2x}\frac{\partial^2 f}{\partial u_x^2}\right)\mathrm{d}u + B_1\mathrm{d}u_t - \frac{\partial^2 f}{\partial u_x^2}B_2\mathrm{d}u_x,$$

and

$$\mathrm{d}E_{\mathcal{L}} - (\mathscr{L}_{(R^t_{\mathcal{L}})_\alpha}E_{\mathcal{L}})\mathrm{d}t^\alpha - (\mathscr{L}_{(R^z_{\mathcal{L}})_\alpha}E_{\mathcal{L}})\eta^\alpha_{\mathcal{L}} = -u_x\frac{\partial^2 f}{\partial t\partial u_x} + \left(\frac{\partial g}{\partial u} + \gamma(x)u_t\right)\mathrm{d}u + u_t\mathrm{d}u_t - u_x\frac{\partial^2 f}{\partial u_x^2}\mathrm{d}u_x,$$

and then the first equation in (8) gives the conditions

$$(B_2 - u_x)\frac{\partial^2 f}{\partial t \partial u_x} = 0, \tag{25}$$

$$C_{1t} - C_{2x}\frac{\partial^2 f}{\partial u_x^2} + \frac{\partial g}{\partial u} + \gamma(x)u_t = 0, \tag{26}$$

$$B_1 = u_t, \tag{27}$$

$$B_2 = u_x. \tag{28}$$

Finally, the second equation in (8) yields $D_1^t + D_2^x = \mathcal{L}$.

Notice that conditions (27) and (28) are the holonomy conditions, while (25) holds identically. Consider now an integral section $\psi(r) = (t(r), x(r);\ u(r), u_t(r), u_x(r);\ z^t(r), z^x(r))$ of the 2-vector field $\mathbf{X}$. Then, combining Eqs. (27) and (28) into (26), we obtain the damped nonlinear wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial u_x}(t, u_x)\right) + \frac{\partial g}{\partial u}(t, u) + \gamma(x)\frac{\partial u}{\partial t} = 0.$$

In the particular case $f(t, u_x) = c^2 u_x/2$, we get

$$u_{tt} - c^2 u_{xx} + \frac{\partial g}{\partial u}(t, u) + \gamma(x)u_t = 0.$$

B. Hamiltonian formalism

In order to give a Hamiltonian description of the system introduced earlier, let us consider the Legendre map associated with the Lagrangian function $\mathcal{L}$ given in (24). The Legendre map associated with $\mathcal{L}$ is the map $\mathcal{FL} : \mathbb{R}^2 \times \oplus^2 \mathrm{T}\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \times \oplus^2 \mathrm{T}^*\mathbb{R} \times \mathbb{R}^2$ given by

$$\mathcal{FL}(t, x; u, u_t, u_x; z^t, z^x) = \left(t, x; u, p^t \equiv u_t, p^x \equiv -\frac{\partial f}{\partial u_x}; z^t, z^x\right).$$

Notice that the regularity condition $\frac{\partial^2 f}{\partial u_x^2}$ assumed implies that the Legendre map is a local diffeomorphism and thus the Lagrangian $\mathcal{L}$ is regular. In order to simplify the computations, from now on we will consider the particular case $f(t, u_x) = u_x^2/2$.

Consider then the product manifold $\mathbb{R}^2 \times \oplus^2 \mathrm{T}^*\mathbb{R} \times \mathbb{R}^2$ equipped with local coordinates $(t, x; u, p^t, p^x; z^t, z^x)$. This manifold has a canonical 2-cocontact structure given by

$$\tau^1 = \mathrm{d}t, \qquad \tau^2 = \mathrm{d}x, \qquad \eta^1 = \mathrm{d}z^t - p^t \mathrm{d}u, \qquad \eta^2 = \mathrm{d}z^x - p^x \mathrm{d}u.$$

It is clear that $\mathrm{d}\eta^1 = \mathrm{d}u \wedge \mathrm{d}p^t$ and $\mathrm{d}\eta^2 = \mathrm{d}u \wedge \mathrm{d}p^x$. In this case, the Reeb vector fields are

$$R_1^t = \frac{\partial}{\partial t}, \qquad R_2^t = \frac{\partial}{\partial x}, \qquad R_1^z = \frac{\partial}{\partial z^t}, \qquad R_2^z = \frac{\partial}{\partial z^x}.$$

The Hamiltonian function h such that $\mathcal{FL}^* h = E_{\mathcal{L}}$ is

$$h(t, x; u, p^t, p^x; z^t, z^x) = \frac{1}{2}(p^t)^2 - \frac{1}{2}(p^x)^2 + g(t, u) + \gamma(x)z^t.$$

Consider a 2-vector field $\mathbf{Y} = (Y_1, Y_2) \in \mathfrak{X}^2(\mathbb{R}^2 \times \oplus^2 \mathrm{T}^*\mathbb{R} \times \mathbb{R}^2)$ with local expression

$$Y_\alpha = A_\alpha^t \frac{\partial}{\partial t} + A_\alpha^x \frac{\partial}{\partial x} + B_\alpha \frac{\partial}{\partial u} + C_\alpha^t \frac{\partial}{\partial p^t} + C_\alpha^x \frac{\partial}{\partial p^x} + D_\alpha^t \frac{\partial}{\partial z^t} + D_\alpha^x \frac{\partial}{\partial z^x}.$$

The Hamilton–De Donder–Weyl Eq. (6) for the 2-vector field $\mathbf{Y}$ yield the conditions

$$\begin{cases} A_1^t = 1, \quad A_1^x = 0, \quad A_2^t = 0, \quad A_2^x = 1, \\ B_1 = p^t, \quad B_2 = -p^x, \\ C_1^t + C_2^x = -\dfrac{\partial g}{\partial u} - \gamma(x)p^t, \\ D_1^t + D_2^x = \dfrac{1}{2}(p^t)^2 - \dfrac{1}{2}(p^x)^2 - g(t, u) - \gamma(x)z^t. \end{cases}$$

Consider now an integral section $\psi(r) = (t(r), x(r);\, u(r), p^t(r), p^x(r);\, z^t(r), z^x(r))$ of the 2-vector field $\mathbf{Y}$. As in the Lagrangian case, it is clear that ψ satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial g}{\partial u}(t,u) + \gamma(x)\frac{\partial u}{\partial t} = 0,$$

which corresponds to the equation of a damped vibrating string with external forcing.

VIII. CONCLUSIONS AND FURTHER RESEARCH

In this paper, we have introduced a new geometric framework to describe non-autonomous non-conservative field theories: k-cocontact structures. This geometric structure combines the notions of k-contact and k-cosymplectic manifolds and permits the development of Hamiltonian and Lagrangian formulations of non-autonomous non-conservative field theories.

In more detail, in Definition 3.1, we introduced the notion of k-cocontact structure as a couple of families of k differential one-forms satisfying certain properties. We have studied the geometry of these manifolds and, in particular, we have proved the existence of Darboux-type coordinates.

Using this geometric framework, the notion of a k-cocontact Hamiltonian system is presented, along with its corresponding field equations, generalizing the Hamilton–De Donder–Weyl equations of Hamiltonian field theory. We have also compared this formulation with the k-contact formalism introduced in Ref. 34 and shown that they are partially equivalent for autonomous field theories.

Moreover, we have developed a Lagrangian formulation for non-autonomous non-conservative field theories. In particular, we have given the conditions determining whether a Lagrangian function yields a k-cocontact structure, and we have introduced the corresponding field equations generalizing the well-known Euler–Lagrange equations.

In order to illustrate the formalisms introduced in this paper, we have studied in full detail the example of a nonlinear damped wave equation with an external time-dependent forcing, both in the Lagrangian and Hamiltonian formulations.

The formalisms introduced in this work open some lines of future research. The first would be to compare the k-cocontact formulation introduced in this paper and the k-contact formalism[34,35] with the so-called multicontact formalism[44] recently introduced. In this work, we have only considered regular Lagrangian functions. The singular case would require the weakening of the notion of k-cocontact structure and the definition of the notion of *k-precocontact structure*. Another very interesting line of research would be to study the symmetries of k-cocontact systems, obtaining conservation and dissipation laws.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Xavier Rivas: Conceptualization (lead); Formal analysis (lead); Funding acquisition (equal); Investigation (lead); Methodology (lead); Validation (lead); Writing – original draft (lead); Writing – review & editing (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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