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Hamilton–Jacobi theory and integrability for autonomous and non-autonomous contact systems

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ABSTRACT

In this paper, we study the integrability of contact Hamiltonian systems, both time-dependent and independent. In order to do so, we construct a Hamilton–Jacobi theory for these systems following two approaches, obtaining two different Hamilton–Jacobi equations. Compared to conservative Hamiltonian systems, contact Hamiltonian systems depend of one additional parameter. The fact of obtaining two equations reflects whether we are looking for solutions depending on this additional parameter or not. In order to illustrate the theory developed in this paper, we study three examples: the free particle with a linear external force, the freely falling particle with linear dissipation and the damped and forced harmonic oscillator.

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1. Introduction

Recently there has been a renewed interest in using contact geometry [46,60] to describe mechanical systems. These systems, unlike symplectic Hamiltonian systems, lead to dissipated rather than conserved quantities [28,30,43,45]. These systems are also relevant to describe mechanical systems with certain types of damping [6,8,28,62], quantum mechanics [9,15], Lie systems [38], circuit theory [49], thermodynamics [2,7], control theory [27,73,79] and black holes [47], among many others [5,51]. The underlying variational principle is the so-called Herglotz principle [29,57,70], a generalization of the well-known Hamilton principle, which gives rise to action-dependent Lagrangian systems. These Lagrangians are becoming popular in theoretical physics [63–65]. Recently, contact mechanics have been generalized to deal with classical field theories with damping [22,42,44,52,76]

Hamilton–Jacobi theory provides a remarkably powerful method to integrate the dynamics of many Hamiltonian systems. In particular, for a completely integrable system, if one knows a complete solution of the Hamilton–Jacobi problem, the dynamics of the system can be reduced to quadratures [3,48,53–55]. Geometrically, the Hamilton–Jacobi problem consists on finding a section γ of $\pi_Q : \mathrm{T}^*Q \to Q$ which transforms integral curves of a projected vector field X_H^γ on Q into integral curves of the dynamical vector field X_H on T^*Q [1,12]. This idea can be naturally extended to other vector bundles. As a matter of fact, it has been applied in many other different contexts, such as nonholonomic systems [13,31,58,74], singular Lagrangian systems [32,33,66], higher-order systems [18], field theories [10,36,37,80,81] or systems with external forces [24,25]. A unifying Hamilton–Jacobi theory for almost-Poisson manifolds was developed in [34]. Hamilton–Jacobi theory has also been extended to Hamiltonian systems with non-canonical symplectic structures [72], non-Hamiltonian systems [75], locally conformally symplectic manifolds [40], Nambu–Poisson [19] and Nambu–Jacobi [20] manifolds, Lie algebroids [67] and implicit differential systems [39,41]. The applications of Hamilton–Jacobi theory include the relation between classical and quantum mechanics [9,14,71], information geometry [16,17], control theory [78] and the study of phase transitions [61]. Hamilton–Jacobi theory for autonomous contact Hamiltonian systems has been studied in [26,35,50].

We have recently initiated the study of time-dependent contact Hamiltonian systems [21,45,77], and the underlying geometric structures, which we call cocontact manifolds since they are a combination of cosymplectic (the setting for studying time-dependent Hamiltonian systems) and contact structures. Such structures consist of two one-forms, τ and η, where τ is closed and $\tau \wedge \eta \wedge (\mathrm{d}\eta)^n$ is a volume form, in a $(2n+2)$-dimensional manifold. The local model for cocontact manifolds is the product bundle $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ with a cocontact structure induced by the canonical symplectic structure of the cotangent bundle. In fact, in [21] we have been able to identify that a cocontact structure gives rise to a Jacobi structure whose characteristic foliation is formed by contact leaves.

The aim of the present paper is to develop a Hamilton–Jacobi theory for time-dependent contact Hamiltonian systems. This will also allow us to construct time-dependent solutions of the Hamilton–Jacobi problem for autonomous contact systems, which, unlike time-independent solutions, cover nonzero energy levels. We follow the line undertaken in previous papers [26,35], considering sections of the canonical fibrations $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$ and $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}$, which allows us to project the Hamiltonian vector field to the base and, by comparing the values on the section, we obtain the corresponding Hamilton–Jacobi equations. This study is particularly useful since it allows us to study the symmetries, conserved quantities and integrability of the system.

In the first of the approaches, where sections of $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$ are considered, complete solutions depend on $n+1$ parameters (instead of the usual $n = \dim Q$ parameters in the classical Hamilton–Jacobi theory). We also make use of this approach to construct complete solutions, depending on n parameters, for autonomous contact Hamiltonian systems. In the second approach we consider sections of $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}$, and complete solutions depend of n parameters (roughly speaking, the additional parameter is absorbed by the extra $\mathbb{R}$-component of the base). Furthermore, this second approach motivates a new definition of integrable contact Hamiltonian system.

The paper is structured as follows. Section 2 is devoted to review time-dependent contact Hamiltonian systems introducing the basic elements needed. In Section 3 we study symmetries and dissipated quantities in cocontact Hamiltonian systems. In Section 4 we develop the action-independent approach to the Hamilton–Jacobi problem, study complete solutions and apply our results for integrating time-independent contact Hamiltonian systems. We also present an example: a free particle with linear friction. In Section 5 we deal with the action-dependent approach to the Hamilton–Jacobi problem, study complete solutions and introduce a new definition of integrable contact system. We also discuss two examples as applications of this approach: the freely falling particle with linear dissipation and the damped and forced harmonic oscillator.

From now on, all the manifolds and mappings are assumed to be smooth, connected and second-countable. Sum over crossed repeated indices is understood.

2. Review on time-dependent contact systems

In this section we are going to review some fundamentals on cocontact geometry and time-dependent contact Hamiltonian systems (for more details see [21]).

2.1. Cocontact manifolds

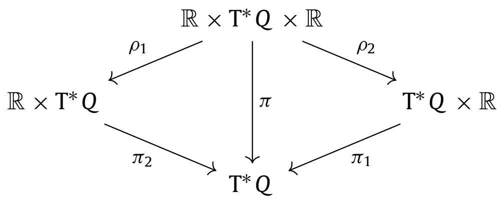
Definition 2.1. A **cocontact structure** on a $(2n+2)$-dimensional manifold M is a couple (τ, η), where $\tau, \eta \in \Omega^1(M)$ and $\mathrm{d}\tau = 0$, such that $\tau \wedge \eta \wedge (\mathrm{d}\eta)^n$ is a volume form on M. In this case, (M, τ, η) is called a **cocontact manifold**.

Given a cocontact manifold (M, τ, η), the distribution $\mathcal{H} = \ker \eta$ is called the **horizontal** or **contact distribution**. Notice that this distribution has corank one and is maximally non-integrable.

Example 2.2. Let (P, η_0) be a contact manifold[1] and consider the product manifold $M = \mathbb{R} \times P$. Denoting by $\mathrm{d}t$ the pullback to M of the volume form in $\mathbb{R}$ and by η the pullback of η_0 to M, we have that $(\mathrm{d}t, \eta)$ is a cocontact structure on M.

Example 2.3. Let $(P, \tau, -\mathrm{d}\theta)$ be an exact cosymplectic manifold [11] and consider the product manifold $M = P \times \mathbb{R}$. Denoting by z the coordinate in $\mathbb{R}$ we define the one-form $\eta = \mathrm{d}z - \theta$. Then, (τ, η) is a cocontact structure on $M = P \times \mathbb{R}$.

Example 2.4 *(Canonical cocontact manifold).* Let Q be an n-dimensional smooth manifold with local coordinates (q^i) and consider its cotangent bundle T^*Q with induced natural coordinates (q^i, p_i). Consider the product manifolds $\mathbb{R} \times \mathrm{T}^*Q$ with coordinates (t, q^i, p_i), $\mathrm{T}^*Q \times \mathbb{R}$ with coordinates (q^i, p_i, z) and $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ with coordinates (t, q^i, p_i, z) and the canonical projections



Let $\theta_0 \in \Omega^1(\mathrm{T}^*Q)$ be the Liouville one-form of the cotangent bundle, which has local expression $\theta_0 = p_i \mathrm{d}q^i$. Then, $(\mathrm{d}t, \theta_2)$, where $\theta_2 = \pi_2^*\theta_0$, is a cosymplectic structure in $\mathbb{R} \times \mathrm{T}^*Q$. On the other hand, if $\theta_1 = \pi_1^*\theta_0$, we have that $\eta_1 = \mathrm{d}z - \theta_1$ is a contact form in $\mathrm{T}^*Q \times \mathbb{R}$.

Finally, consider the 1-form $\theta = \rho_1^*\theta_2 = \rho_2^*\theta_1 = \pi^*\theta_0 \in \Omega^1(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R})$ and let $\eta = \mathrm{d}z - \theta$. Then, $(\mathrm{d}t, \eta)$ is a cocontact structure in $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$. The local expression of the one-form η is

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i .$$

Given a cocontact manifold (M, τ, η), we have the **flat isomorphism**.

$$\flat : v \in \mathrm{T}M \longmapsto (\iota_v \tau)\tau + \iota_v \mathrm{d}\eta + (\iota_v \eta)\, \eta \in \mathrm{T}^*M .$$

This isomorphism can be trivially extended to an isomorphism of $\mathscr{C}^\infty(M)$-modules $\flat : \mathfrak{X}(M) \to \Omega^1(M)$. The inverse of the flat isomorphism is denoted by $\sharp = \flat^{-1} : \Omega^1(M) \to \mathfrak{X}(M)$ and called the **sharp isomorphism**.

Moreover, we have the following results, whose proofs can be found in [21].

[1] A **contact structure** on an odd-dimensional manifold M is a one-codimensional maximally non-integrable distribution C on M. In this case, (M, C) is a **contact manifold**. A **contact form** on M is a one-form $\eta \in \Omega^1(M)$ such that $\ker \eta$ becomes a contact structure on M. In this case, (M, η) is called a **co-oriented contact manifold** [46]. However, since we are only interested in local aspects of contact manifolds, we will consider that all our contact manifolds are co-oriented.

Proposition 2.5. *On every cocontact manifold (M,τ,η) there exist two distinguished vector fields R_t, R_z on M such that*

$$\iota_{R_t}\tau = 1\,, \qquad \iota_{R_t}\eta = 0\,, \qquad \iota_{R_t}\mathrm{d}\eta = 0\,,$$
$$\iota_{R_z}\tau = 0\,, \qquad \iota_{R_z}\eta = 1\,, \qquad \iota_{R_z}\mathrm{d}\eta = 0\,,$$

*or, equivalently, $R_t = \flat^{-1}(\tau)$ and $R_z = \flat^{-1}(\eta)$. These vector fields R_t and R_z are called **time and contact Reeb vector fields** respectively.*

Theorem 2.6 *(Cocontact Darboux theorem). Given a cocontact manifold (M,τ,η), around every point $x\in M$ there exist local coordinates (t,q^i,p_i,z) such that*

$$\tau = \mathrm{d}t\,, \quad \eta = \mathrm{d}z - p_i\mathrm{d}q^i\,.$$

*These coordinates are called **canonical** or **Darboux** coordinates. In addition, in Darboux coordinates, the Reeb vector fields read*

$$R_t = \frac{\partial}{\partial t}\,, \quad R_z = \frac{\partial}{\partial z}\,.$$

Proposition 2.7. *Let (M,τ,η) be a cocontact manifold. Then, (M,Λ,E) is a Jacobi manifold, where*

$$\Lambda(\alpha,\beta) = -\mathrm{d}\eta(\sharp\alpha,\sharp\beta)\,, \quad E = -R_z\,.$$

The bivector Λ induces a $\mathscr{C}^\infty(M)$-module morphism $\hat\Lambda\colon \Omega^1(M)\to\mathfrak{X}(M)$ given by

$$\hat\Lambda(\alpha) = \Lambda(\alpha,\cdot) = \sharp\alpha - \alpha(R_z)R_z - \alpha(R_t)R_t\,. \tag{1}$$

It can be seen that $\ker\hat\Lambda = \langle\tau,\eta\rangle$. The morphism $\hat\Lambda$ is also denoted by $\sharp_\Lambda$ in the literature [23,28].

Taking Darboux coordinates (t,q^i,p_i,z), the bivector Λ has local expression

$$\Lambda = \frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i} - p_i\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial z}\,,$$

and the Jacobi bracket reads

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} - \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial z}\right) - f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\,.$$

In particular, one has

$$\{q^i,q^j\} = \{p_i,p_j\} = 0\,, \qquad \{q^i,p_j\} = \delta^i_j\,, \qquad \{q^i,z\} = -q^i\,, \qquad \{p_i,z\} = -2p_i\,.$$

2.2. Cocontact Hamiltonian systems

Definition 2.8. A **cocontact Hamiltonian system** is family (M,τ,η,H) where (τ,η) is a cocontact structure on M and $H\colon M\to\mathbb{R}$ is a Hamiltonian function. The **cocontact Hamilton equations** for a curve $\psi\colon I\subset\mathbb{R}\to M$ are

$$\begin{cases}\iota_{\psi'}\mathrm{d}\eta = \Big(\mathrm{d}H - (\mathscr{L}_{R_s}H)\eta - (\mathscr{L}_{R_t}H)\tau\Big)\circ\psi\,,\\ \iota_{\psi'}\eta = -H\circ\psi\,,\\ \iota_{\psi'}\tau = 1\,,\end{cases} \tag{2}$$

where $\psi'\colon I\subset\mathbb{R}\to\mathrm{T}M$ is the canonical lift of ψ to the tangent bundle $\mathrm{T}M$. The **cocontact Hamiltonian equations** for a vector field $X_H\in\mathfrak{X}(M)$ are:

$$\begin{cases}\iota_{X_H}\mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_{R_s}H)\eta - (\mathscr{L}_{R_t}H)\tau\,,\\ \iota_{X_H}\eta = -H\,,\\ \iota_{X_H}\tau = 1\,,\end{cases}$$

or equivalently, $\flat(X_H) = \mathrm{d}H - \left(\mathscr{L}_{R_s}H + H\right)\eta + \left(1 - \mathscr{L}_{R_t}H\right)\tau$. The unique solution to these equations is called the **cocontact Hamiltonian vector field**.

Given a curve ψ with local expression $\psi(r) = (f(r), q^i(r), p_i(r), z(r))$, the third equation in (2) imposes that $f(r) = r + c$, where c is some constant, thus we will denote $r \equiv t$, while the other equations read:

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i}\,, \\ \dot{p}_i = -\left(\dfrac{\partial H}{\partial q^i} + p_i \dfrac{\partial H}{\partial z}\right)\,, \\ \dot{z} = p_i \dfrac{\partial H}{\partial p_i} - H\,. \end{cases} \tag{3}$$

On the other hand, the local expression of the cocontact Hamiltonian vector field is

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}\,.$$

The integral curves of the cocontact Hamiltonian vector field satisfy the following variational principle [69], which is a Hamiltonian version of the Herglotz principle [57].

Theorem 2.9 *(Hamiltonian formulation of the Herglotz principle). Given a Hamiltonian $H : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R}$, a curve $c = (\mathrm{Id}_{\mathbb{R}}, q, p, z) : [0, T] \to \mathrm{T}^*Q \times \mathbb{R}$ is an integral curve of the Hamiltonian vector field X_H if and only if it is a critical point of the action map*

$$\mathcal{A}(c) = \int_0^T \big(p(t)\dot{q}(t) - H(t, q(t), p(t), z(t))\big)\, \mathrm{d}t \tag{4}$$

among all curves satisfying $c(0) = c_0$, $c(T) = c_T$ and $\dot{z} = p(t)\dot{q}(t) - H(t, q(t), p(t), z(t))$.

3. Symmetries and dissipated quantities in cocontact systems

There are several notions of symmetries in contact mechanics depending on the structures they preserve [30,43]. However, in the present paper we will restrict ourselves to what we call generalized dynamical symmetries (see [45] for other notions of symmetry). In some cases we will restrict ourselves to the case of cocontact manifolds of the form $M = \mathbb{R} \times N$ where N is a contact manifold (see Example 2.2). In this case, the natural projection $\mathbb{R} \times N \to \mathbb{R}$ defines a global canonical coordinate t.

Definition 3.1. Let (M, τ, η, H) be a cocontact Hamiltonian system and let X_H be its cocontact Hamiltonian vector field.

- If $M = \mathbb{R} \times N$ with N a contact manifold, a **generalized dynamical symmetry** is a diffeomorphism $\Phi \colon M \to M$ such that $\eta(\Phi_* X_H) = \eta(X_H)$ and $\Phi^* t = t$.
- An **infinitesimal generalized dynamical symmetry** is a vector field $Y \in \mathfrak{X}(M)$ such that $\eta([Y, X_H]) = 0$ and $\iota_Y \tau = 0$. In particular, if $M = \mathbb{R} \times N$ with N a contact manifold, the flow of Y is made of generalized dynamical symmetries.

Definition 3.2. Let (M, τ, η, H) be a cocontact Hamiltonian system. A **dissipated quantity** is a function $f \in \mathscr{C}^\infty(M)$ such that

$$X_H f = -(R_z H) f\,.$$

It is worth pointing out that, unlike in the contact case, the Hamiltonian function is not, in general, a dissipated quantity. Indeed, using that

$$X_H H = -(R_z H) H + R_t H\,,$$

it is clear that H is a dissipated quantity if and only if it is time-independent, i.e. $R_t H = 0$. This resembles the cosymplectic case, where the Hamiltonian function is conserved if, and only if, it is time-independent (see [11]).

Proposition 3.3. *A function $f \in \mathscr{C}^\infty(M)$ is a dissipated quantity if and only if $\{f, H\} = R_t(f)$, where $\{\cdot, \cdot\}$ is the Jacobi bracket associated to the cocontact structure (τ, η).*

Proof. The Jacobi bracket of f and H is given by

$$\{f, H\} = \Lambda(\mathrm{d}f, \mathrm{d}H) + f E(H) - H E(f) = -\mathrm{d}\eta\,(\sharp \mathrm{d}f, \sharp \mathrm{d}H) - f R_z(H) + H R_z(f)\,,$$

but

$$\sharp\mathrm{d}f = X_f + (R_z(f) + f)\, R_z - (1 - R_t(f))\, R_t\,,$$

so

$$\iota_{\sharp\mathrm{d}f}\mathrm{d}\eta = \iota_{X_f}\mathrm{d}\eta = \mathrm{d}f - R_z(f)\eta - R_t(f)\tau\,,$$

and thus

$$\mathrm{d}\eta(\sharp\mathrm{d}f, \sharp\mathrm{d}H) = X_H(f) + R_z(f)H - R_t(f)\,.$$

Hence,

$$\{H, f\} + R_t(f) = X_H(f) + R_z(H)f\,.$$

In particular, the right-hand side vanishes if and only if f is a dissipated quantity. □

Theorem 3.4 *(Noether's theorem). Let Y be an infinitesimal generalized dynamical symmetry of the cocontact Hamiltonian system (M, τ, η, H). Then, $f = -\iota_Y\eta$ is a dissipated quantity of the system. Conversely, given a dissipated quantity $f \in \mathscr{C}^\infty(M)$, the vector field $Y = X_f - R_t$, where X_f is the Hamiltonian vector field associated to f, is an infinitesimal generalized dynamical symmetry and $f = -\iota_Y\eta$.*

Proof. Let $f = -\iota_Y\eta$, where Y is an infinitesimal generalized dynamical symmetry. Then,

$$\begin{aligned}\mathscr{L}_{X_H}f &= -\mathscr{L}_{X_H}\iota_Y\eta = -\iota_Y\mathscr{L}_{X_H}\eta - \iota_{[X_H,Y]}\eta = \\ &\iota_Y\left(R_z(H)\eta + R_t(H)\tau\right) = R_z(H)\iota_Y\eta = -R_z(H)f\,,\end{aligned}$$

and thus f is a dissipated quantity.

On the other hand, given a dissipated quantity f, let $Y = X_f - R_t$. Then, it is clear that $f = -\iota_Y\eta$. In addition, $\iota_Y\tau = 0$, and

$$\begin{aligned}\iota_{[X_H,Y]}\eta &= \mathscr{L}_{X_H}\iota_Y\eta - \iota_Y\mathscr{L}_{X_H}\eta = -\mathscr{L}_{X_H}f + \iota_Y\left(R_z(H)\eta + R_t(H)\tau\right) \\ &= R_z(H)f - R_z(H)\iota_Y\eta = 0\,. \quad \square\end{aligned}$$

The symmetries presented yield dissipated quantities. However, we are also interested in finding conserved quantities. The latter are important due to their elation with complete solutions of the Hamilton–Jacobi problem (see Section 5).

Definition 3.5. A **conserved quantity** of a cocontact Hamiltonian system (M, τ, η, H) is a function $g \in \mathscr{C}^\infty(M)$ such that

$$X_H g = 0\,.$$

Taking into account that every dissipated quantity changes with the same rate $R_z(H)$, we have the following result, whose proof is straightforward.

Proposition 3.6. *Consider a cocontact Hamiltonian system (M, τ, η, H).*

- *If f_1, f_2 are two dissipated quantities and $f_2 \neq 0$, then f_1/f_2 is a conserved quantity.*
- *If f is a dissipated quantity and g is a conserved quantity, then fg is a dissipated quantity.*

4. The action-independent approach

4.1. Hamilton–Jacobi theory. The action-independent approach

Let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ be a cocontact Hamiltonian system, where $\tau = \mathrm{d}t$, $\eta = \mathrm{d}z - \theta_0$ and $\theta_0 = p_i\mathrm{d}q^i$ is the Liouville one-form of the cotangent bundle. Consider a section γ of the bundle $\pi^t_Q : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$, locally given by

$$\begin{aligned}\gamma : \mathbb{R} \times Q &\longrightarrow \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \\ \left(t, q^i\right) &\longmapsto \left(t, q^i, \gamma_i(t,q), S(t,q)\right)\,.\end{aligned}$$

Let us introduce the vector field X_H^γ on $\mathbb{R} \times Q$ given by

$$X_H^\gamma = \mathrm{T}\pi_Q^t \circ X_H \circ \gamma \,,$$

where X_H is the Hamiltonian vector field of $(\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}, \tau, \eta, H)$. Suppose that X_H^γ and X_H are γ-related, i.e.,

$$X_H \circ \gamma = \mathrm{T}\gamma \circ X_H^\gamma \,, \tag{5}$$

so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R} & \xrightarrow{\;X_H\;} & \mathrm{T}(\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}) \\ \gamma\uparrow\downarrow\pi_Q^t & & \mathrm{T}\pi_Q^t\downarrow\uparrow\mathrm{T}\gamma \\ \mathbb{R}\times Q & \xrightarrow{\;X_H^\gamma\;} & \mathrm{T}(\mathbb{R}\times Q) \end{array}$$

Locally,

$$X_H\circ\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + \gamma_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(\gamma_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}\,,$$

and

$$\mathrm{T}\gamma\circ X_H^\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \left(\frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial \gamma_j}{\partial q^i}\right)\frac{\partial}{\partial p_i} + \left(\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i}\right)\frac{\partial}{\partial z}\,,$$

so equation (5) holds if and only if

$$\begin{aligned} -\left(\frac{\partial H}{\partial q^i} + \gamma_i\frac{\partial H}{\partial z}\right) &= \frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial \gamma_j}{\partial q^i}\,, \\ \gamma_i\frac{\partial H}{\partial p_i} - H &= \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i}\,. \end{aligned} \tag{6}$$

Definition 4.1. Given a section $\alpha : \mathbb{R}\times Q \to \mathbb{R}\times \bigwedge^k \mathrm{T}^*Q$ and $t\in\mathbb{R}$, let

$$\begin{aligned} \alpha_{(t)} : Q &\longrightarrow \bigwedge^k \mathrm{T}^*Q \\ x &\longmapsto \mathrm{pr}_{\bigwedge^k \mathrm{T}^*Q}(\alpha(t,x))\,, \end{aligned}$$

where $\mathrm{pr}_{\bigwedge^k \mathrm{T}^*Q} : \mathbb{R}\times\bigwedge^k \mathrm{T}^*Q \to \bigwedge^k \mathrm{T}^*Q$ is the canonical projection. The **exterior derivative of** α **at fixed** t is the section of $\mathbb{R}\times\bigwedge^{k+1}\mathrm{T}^*Q \to \mathbb{R}\times Q$ given by

$$\mathrm{d}_Q\alpha(t,x) = (t, \mathrm{d}\alpha_{(t)}(x))\,.$$

In coordinates, for $f\in\mathscr{C}^\infty(\mathbb{R}\times Q)$ and $\alpha(t,x) = (t, \alpha_i \mathrm{d}_x q^i)$ a section of the bundle $\mathbb{R}\times Q\to\mathbb{R}\times\bigwedge^k \mathrm{T}^*Q$, the local expressions are

$$\begin{aligned} \mathrm{d}_Q f &= \left(t, \frac{\partial f}{\partial q^i}\mathrm{d}_x q^i\right)\,, \\ \mathrm{d}_Q\alpha &= \left(t, \frac{\partial \alpha_j}{\partial q^i}\mathrm{d}_x q^i\wedge \mathrm{d}_x q^j\right)\,. \end{aligned}$$

Since we shall be considering fixed t, we will often make the abuse of notation

$$\mathrm{d}_Q f = \frac{\partial f}{\partial q^i}\mathrm{d}_x q^i\,.$$

Definition 4.2. Given $f\in\mathscr{C}^\infty(\mathbb{R}\times Q)$, the 1**-jet of** f **at fixed** t is the section $j_t^1 f : \mathbb{R}\times Q\to\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}$ given by

$$j_t^1 f(t,x) = (\mathrm{d}_Q f, f)\,.$$

Let us recall that a **Legendrian submanifold** $N\hookrightarrow M$ of a $(2n+1)$-dimensional contact manifold (M,η) is an n-dimensional submanifold such that $\eta|_N = 0$ (see [23]).

Proposition 4.3. *Let γ be a section of $\pi_Q^t : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$. Then, for every $t \in \mathbb{R}$, $\operatorname{Im}\gamma(t,\cdot)$ is a Legendrian submanifold of $(\mathrm{T}^*Q \times \mathbb{R}, \eta)$ if and only if it is the image of the 1-jet at fixed t of a function, namely,*

$$\gamma(t,x) = j_t^1 f(t,x) = (\mathrm{d}_Q f, f)\,.$$

Proof. Let $t \in \mathbb{R}$ and let $\gamma : \mathbb{R} \times Q \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ such that $\gamma(t,q) = (t, \alpha(t,q), f(t,q))$. Clearly, $\gamma^*\tau = 0$, hence $\operatorname{Im}\gamma$ is Legendrian if and only if $\gamma^*\eta = 0$. Thus,

$$\gamma^*\eta = f^*\mathrm{d}z - \alpha^*\theta_Q = \mathrm{d}_Q f - \alpha\,,$$

so $\gamma^*\eta$ vanishes precisely when $\alpha = \mathrm{d}_Q f$. □

Now, suppose that $\operatorname{Im}\gamma$ is a Legendrian submanifold. By Proposition 4.3, we have that

$$\gamma_i = \frac{\partial S}{\partial q^i}\,,$$

so equations (6) can be written as

$$-\left(\frac{\partial H}{\partial q^i} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial z}\right) = \frac{\partial^2 S}{\partial t \partial q^i} + \frac{\partial H}{\partial p_j}\frac{\partial S}{\partial q^i \partial q^j}\,, \tag{7a}$$

$$\frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i} - H = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i}\,, \tag{7b}$$

equation (7a) implies that

$$\mathrm{d}_Q(H \circ j_t^1 S) + \mathrm{d}_Q(R_t S) = 0\,, \tag{8}$$

while equation (7b) yields

$$H = -\frac{\partial S}{\partial t}\,,$$

that is,

$$H \circ j_t^1 S + \frac{\partial S}{\partial t} = 0\,. \tag{9}$$

Clearly, equation (8) is implied by equation (9). We have thus proven the following.

Theorem 4.4 *(Action-independent Hamilton–Jacobi theorem). Let γ be a section of $\pi_Q^t : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$ such that, for every $t \in \mathbb{R}$, $\operatorname{Im}\gamma(t,\cdot)$ is a Legendrian submanifold of $(\mathrm{T}^*Q \times \mathbb{R}, \eta)$. Then, X_H^γ and X_H are γ-related if and only if equation (9) holds. This equation will be called the* ***action-independent Hamilton–Jacobi*** *equation for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$. The function S such that $\gamma = j_t^1 S$ is called a* ***generating function*** *for H.*

In order to study the integrability of cocontact Hamiltonian systems, it is of interest to introduce the following.

Definition 4.5. Let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ be a cocontact Hamiltonian system. A **complete solution of the action-independent Hamilton–Jacobi problem** for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ is a local diffeomorphism $\Phi\colon \mathbb{R} \times Q \times \mathbb{R}^{n+1} \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ such that, for each $\lambda \in \mathbb{R}^{n+1}$,

$$\begin{aligned}\Phi_\lambda\colon \mathbb{R} \times Q &\to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}\\ \left(t, q^i\right) &\mapsto \Phi\left(t, q^i, \lambda\right)\end{aligned}$$

is a solution of the action-independent Hamilton–Jacobi problem for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$.

It is worth noting that complete solutions depend on $n+1$ real parameters, one extra parameter in comparison with the (co)symplectic case. In order to consider complete solutions depending on just n parameters, we shall introduce a different approach to the Hamilton–Jacobi problem for (co)contact Hamiltonian systems (see Section 5).

Let $\alpha\colon \mathbb{R} \times Q \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, and $\pi_i\colon \mathbb{R}^{n+1} \to \mathbb{R}$ denote the canonical projections. One can define the $n+1$ functions $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$, so that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{R}\times Q\times\mathbb{R}^{n+1} & \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} & \mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R} \\
\downarrow \alpha & & \downarrow f_i \\
\mathbb{R}^{n+1} & \xrightarrow{\pi_i} & \mathbb{R}
\end{array}
$$

Theorem 4.6. *Let $\Phi\colon \mathbb{R}\times Q\times\mathbb{R}^{n+1}\to\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}$ be a complete solution of the action-independent Hamilton–Jacobi problem for $(\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R},\tau,\eta,H)$. Then,*

(i) *For each $i\in\{1,\dots,n+1\}$, the function $f_i=\pi_i\circ\alpha\circ\Phi^{-1}$ is a constant of the motion. However, these functions are not necessarily in involution, i.e., $\{f_i,f_j\}\neq 0$.*
(ii) *For each $i\in\{1,\dots,n+1\}$, the function $\hat{f}_i=gf_i$, where g is a dissipated quantity, is also a dissipated quantity. Moreover, if $R_t(H)=0$ and taking $g=H$, these functions are in involution, i.e., $\{\hat{f}_i,\hat{f}_j\}=0$.*

Proof. We can write

$$
\operatorname{Im}\Phi_\lambda=\{x\in\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}\mid f_i(x)=\lambda_i,\ i=1,\dots,n\}=\bigcap_{i=1}^{n} f_i^{-1}(\lambda_i)\,,
$$

where $\lambda=(\lambda,\dots,\lambda_n)\in\mathbb{R}^n$. Since X_H is tangent to any of the submanifolds $\operatorname{Im}\Phi_\lambda$, we deduce that

$$
X_H f_i=0\,,
$$

so each of the functions f_i, for $i=1,\dots,n$, is a constant of the motion.

On the other hand, we can compute

$$
\{f_i,f_j\}=X_{f_j}(f_i)-R_t(f_i)-f_iR_z(f_j)\,,
$$

which does not vanish in general. By Proposition 3.6, the product of a conserved quantity and a dissipated quantity is a dissipated quantity. Let f_i and f_j be conserved quantities and take $g=H$. Then,

$$
\begin{aligned}
\{\hat{f}_i,\hat{f}_j\}&=\{Hf_i,Hf_j\}=f_j\{Hf_i,H\}+H\{Hf_i,f_j\}-f_jHR_z(Hf_i)\\
&=-f_jH\{H,f_i\}+f_if_jHR_z(H)-f_iH\{f_j,H\}-H^2\{f_j,f_i\}+f_iH^2R_z(f_j)-f_jHR_z(Hf_i)\quad\square\\
&=0\,.
\end{aligned}
\tag{10}
$$

Remark 4.7. Making use of a symplectization, one can study a time-independent contact Hamiltonian system as an exact symplectic Hamiltonian system with one additional dimension. Dissipated quantities in involution with respect to the Jacobi bracket of the contact system lead to conserved quantities in involution with respect to the Poisson bracket of the associated symplectic system. On the other hand, the celebrated Liouville–Arnold Theorem [3] permits to construct action-angle coordinates of a $2n$-dimensional symplectic Hamiltonian system with n conserved quantities in involution, leading to integrability by quadratures. Therefore, dissipated quantities in involution could lead to integrability by quadratures of contact Hamiltonian systems. However, this relation is highly non-trivial and it is subject of further research. An alternative approach to Hamilton–Jacobi theory and integrability by quadratures for contact Hamiltonian systems can be found in [56].

Complete solutions of the Hamilton–Jacobi problem may be used to integrate the dynamics of the system as follows:

(i) Solve the Hamilton–Jacobi equation

$$
H\circ j^1_t S_\lambda+\frac{\partial S_\lambda}{\partial t}=0
$$

for arbitrary values of $\lambda\in\mathbb{R}^{n+1}$. Let $\Phi_\lambda=j^1_tS_\lambda$.

(ii) Compute the integral curves $\sigma\colon\mathbb{R}\to\mathbb{R}\times Q,\ \sigma(t)=(t,q^i(t))$ of X_H^γ, which are given by

$$
\frac{\mathrm{d}q^i}{\mathrm{d}t}=\left.\frac{\partial H}{\partial p_i}\right|_{\operatorname{Im}\Phi_\lambda}\,,
\tag{11}
$$

where the restriction to $\operatorname{Im}\Phi_\lambda$ means that one has to write $p_i=\partial S_\lambda/\partial q^i$ and $z=S_\lambda$.

(iii) The integral curves $\tilde{\sigma}$ of X_H on $\operatorname{Im}\Phi_\lambda$ are given by $\Phi_\lambda\circ\sigma$, namely,

$$
\tilde{\sigma}(t)=\Phi_\lambda\circ\sigma(t)=\left(\sigma(t),\frac{\partial S_\lambda}{\partial q^i}(\sigma(t)),S_\lambda(\sigma(t))\right)\,.
$$

It is worth noting that computing the integral curves of X_H^γ is not always straightforward. However, there are some relevant cases in which it is particularly easy.

Example 4.8. Suppose that $Q = \mathbb{R}^n$ with the Euclidean norm. If the generating function is **separable**, i.e., $S(t, q^1, q^2, \dots, q^n) = S_0(t) + S_1(q^1) + \cdots + S_n(q^n)$, and the Hamiltonian is mechanical, namely, $H = \frac{\|p\|^2}{2m(t)} + V(t, q, z)$, then equations (11) simplify to

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{1}{m(t)} S_i'(q^i)\,.$$

4.2. Example: the free particle with time-dependent mass and a linear external force

Consider the cocontact Hamiltonian system $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \mathrm{d}t, \eta, H)$, where

$$H = \frac{p^2}{2m(t)} - \frac{\kappa}{m(t)} z,$$

with m a function depending only on t, expressing the mass of the particle, and κ a positive constant. Then, the action-independent Hamilton–Jacobi equation for H is given by

$$\frac{1}{2m(t)}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{\kappa}{m(t)} S(t,q) + \frac{\partial S}{\partial t} = 0\,,$$

that is,

$$\left(\frac{\partial S}{\partial q}\right)^2 - 2\kappa S(t,q) + 2m(t)\frac{\partial S}{\partial t} = 0\,. \tag{12}$$

Suppose that the generating function S is separable, namely, $S(t,q) = \alpha(t) + \beta(q)$. Then, equation (12) can be written as

$$\left(\frac{\mathrm{d}\beta}{\mathrm{d}q}\right)^2 - 2\kappa\alpha(t) - 2\kappa\beta(q) + 2m(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t} = 0\,,$$

so

$$2m(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t} - 2\gamma\alpha(t) = 0\,,$$
$$\left(\frac{\mathrm{d}\beta}{\mathrm{d}q}\right)^2 - 2\kappa\beta(q) = 0.$$

Then,

$$\alpha_{\lambda_1}(t) = \lambda_1 e^{\kappa \int_0^t \frac{1}{m(s)}\mathrm{d}s} \qquad \beta_{\lambda_2}(q) = \left(\sqrt{\frac{\kappa}{2}}q + \lambda_2\right)^2,$$

that is,

$$S_\lambda(t,q) = \lambda_1 e^{\kappa \int_0^t \frac{1}{m(s)}\mathrm{d}s} + \left(\sqrt{\frac{\kappa}{2}}q + \lambda_2\right)^2,$$

and

$$\Phi(t,q,\lambda) = j_t^1 S_\lambda(t,q) = \left(t, q, \sqrt{2\kappa}\left(\sqrt{\frac{\kappa}{2}}q + \lambda_2\right), \lambda_1 e^{\kappa \int_0^t \frac{1}{m(s)}\mathrm{d}s} + \left(\sqrt{\frac{\kappa}{2}}q + \lambda_2\right)^2\right)$$

is a complete solution. Its inverse is given by

$$\Phi^{-1} : (t,q,p,z) \mapsto \left(t, q, e^{-\kappa \int_0^t \frac{1}{m(s)}\mathrm{d}s}\left(z - \frac{p^2}{2\kappa}\right), \frac{p - \kappa q}{\sqrt{2\kappa}}\right).$$

Hence,

$$f_1(t,q,p,z) = e^{-\kappa \int_0^t \frac{1}{m(s)}\mathrm{d}s}\left(z - \frac{p^2}{2\kappa}\right),$$

and

$$f_2(t,q,p,z) = \frac{p - \kappa q}{\sqrt{2\kappa}}$$

are conserved quantities.

The Hamiltonian vector field of H is given by

$$X_H = \frac{\partial}{\partial t} + \frac{p}{m(t)}\frac{\partial}{\partial q} + \frac{\kappa p}{m(t)}\frac{\partial}{\partial p} + \left(\frac{p^2}{2m(t)} + \frac{\kappa}{m(t)} z\right)\frac{\partial}{\partial z}\,.$$

One can check that $X_H(f_1) = X_H(f_2) = 0$. Moreover,

$$X_H^\gamma = \frac{\partial}{\partial t} + \frac{p}{m(t)}\frac{\partial}{\partial q}\bigg|_{\operatorname{Im}\Phi_\lambda} = \frac{\partial}{\partial t} + \frac{\sqrt{2\kappa}\left(\sqrt{\frac{\kappa}{2}}q + \lambda_2\right)}{m(t)}\frac{\partial}{\partial q}\,,$$

whose integral curves $\sigma(t) = (t, q(t))$ are given by

$$q(t) = e^{\int_1^t \frac{\kappa}{m(s)}\mathrm{d}s}\left(\int_1^t \frac{\sqrt{2\kappa}e^{-\int_1^u \frac{\kappa}{m(s)}\mathrm{d}s}\lambda}{m(u)}\mathrm{d}u + c\right)\,,$$

where c is a constant. Then, the integral curves of X_H along $\operatorname{Im}\Phi_\lambda$ are given by $\Phi_\lambda \circ \sigma(t) = (t, q(t), p(t), z(t))$, where

$$p(t) = \sqrt{2\kappa}\left(\sqrt{\frac{\kappa}{2}}q(t) + \lambda_2\right)\,,$$

and

$$z(t) = \lambda_1 e^{\kappa\int_0^t \frac{1}{m(s)}\mathrm{d}s} + \left(\sqrt{\frac{\kappa}{2}}q(t) + \lambda_2\right)^2\,.$$

4.3. The variational interpretation of the solution to Hamilton–Jacobi equation

Suppose that $\sigma\colon [0,T] \to Q$ is a trajectory given by the cocontact Hamilton equations (3) for the Hamiltonian function $H\colon \mathbb{R}\times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R}$. If $\gamma = j_t^1 S$ is a solution to the Hamilton–Jacobi problem for H, the generating function S can be interpreted as the action of the lifted curve $j_t^1 S \circ \sigma$ up to a constant.

Theorem 4.9. *Suppose that $S \in \mathscr{C}^\infty(\mathbb{R}\times Q)$ is a generating function for H. Let $\sigma\colon [0,T] \to Q$ be a curve with local expression $\sigma(t) = (q^i(t))$ such that $c = (\mathrm{Id}, \sigma)\colon t \in \mathbb{R} \mapsto (t, \sigma(t)) \in \mathbb{R}\times Q$ is an integral curve of X_H^γ. Then,*

$$(S\circ c)(t) = \mathcal{A}(j_t^1 S \circ \sigma)(t) + S_0\,,$$

for some $S_0 \in \mathbb{R}$, where $\mathcal{A}$ denotes the action map (4).

Proof. Assume that $S \in \mathscr{C}^\infty(\mathbb{R}\times Q)$ is a generating function for H. Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}S(t,q(t)) &= \frac{\partial S(t,\sigma(t))}{\partial t} + \frac{\partial S(t,\sigma(t))}{\partial q^i}\dot{q}^i(t) \\
&= \frac{\partial S(t,\sigma(t))}{\partial q^i}\dot{q}^i(t) - H\circ j_t^1 S\circ\sigma(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(j_t^1 S\circ\sigma)(t)\,,
\end{aligned}$$

where we have used the Hamilton–Jacobi equation (9) on the second step, and the definition of the action map (4) on the last step. Hence,

$$S(t,q(t)) = \mathcal{A}(j_t^1 S\circ q)(t) + S_0\,,$$

for some constant S_0. □

4.4. A new approach for the Hamilton–Jacobi problem in time-independent contact Hamiltonian systems

Let us recall that a contact Hamiltonian system (M, η, H) is a contact manifold (M, η) together with a Hamiltonian function $H : M \to \mathbb{R}$ (see [28,43]). The Hamiltonian vector field of H is locally given by

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}.$$

The analogous of Theorem 4.4 for autonomous contact Hamiltonian systems was developed in [35] (see also [26]):

Theorem 4.10 *(Hamilton–Jacobi theorem for autonomous systems). Let $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ be a contact Hamiltonian system with contact Hamiltonian vector field X_H. Consider a section γ of $\pi_Q : \mathrm{T}^*Q \times \mathbb{R} \to Q$ such that $\operatorname{Im}\gamma$ is a Legendrian submanifold of $(\mathrm{T}^*Q \times \mathbb{R}, \eta)$. Then, X_H^γ and X_H are γ-related if and only if*

$$H \circ \gamma = 0. \tag{13}$$

The problem with this approach is that it cannot be used to completely integrate the system. Indeed, equation (13) implies that every integral curve of $X_H \circ \gamma$ is contained in $H^{-1}(0)$. This can be solved by regarding the contact Hamiltonian system $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ as the cocontact Hamiltonian system $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \mathrm{d}t, \eta, \widehat{H})$, where $\widehat{H} = H \circ \rho_2$ (i.e. $\widehat{H}(t, q, p, z) = H(q, p, z)$), such that $R_t(\widehat{H}) = 0$ and making use of Theorem 4.4. Suppose that S is of the form $S(t, q) = \alpha(q) + \beta(t)$. Then, equation (9) yields

$$H \circ j^1\alpha + \frac{\partial \beta}{\partial t} = 0,$$

that is,

$$H\left(q^i, \frac{\partial \alpha}{\partial q^i}, z\right) + \dot{\beta}(t) = 0.$$

With a suitable choice of α and β, one can cover energy levels distinct from $H = 0$.

Definition 4.11. Let $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ be a contact Hamiltonian system, and let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, \widehat{H} = H \circ \rho_2)$ be its associated cocontact Hamiltonian system. A **complete solution of the action-independent Hamilton–Jacobi problem** for $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ is a map $\widehat{\Phi} : \mathbb{R} \times Q \times \mathbb{R}^n \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ such that $\Phi = \rho_2 \circ \widehat{\Phi}$ is a local diffeomorphism and, for each $\lambda \in \mathbb{R}^n$,

$$\begin{aligned} \widehat{\Phi}_\lambda : \mathbb{R} \times Q &\to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \\ \left(t, q^i\right) &\mapsto \widehat{\Phi}\left(t, q^i, \lambda\right) \end{aligned}$$

is a solution of the action-independent Hamilton–Jacobi problem for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, \widehat{H})$.

Let $\alpha : \mathbb{R} \times Q \times \mathbb{R}^n \to \mathbb{R}^n$, and $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections. One can define the n functions $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$, so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} \times Q \times \mathbb{R}^n & \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} & \mathrm{T}^*Q \times \mathbb{R} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f_i} \\ \mathbb{R}^n & \xrightarrow{\pi_i} & \mathbb{R} \end{array}$$

Then,

$$\operatorname{Im}\Phi_\lambda = \bigcap_{i=1}^n f_i^{-1}(\lambda_i),$$

where $\Phi_\lambda(t, q) = \Phi(t, q, \lambda)$, and

$$\operatorname{Im}\widehat{\Phi}_\lambda = \bigcap_{i=1}^n (f_i \circ \rho_2)^{-1}(\lambda_i),$$

so the functions $f_i \circ \rho_2$ are constants of the motion for $\widehat{H}$, and thus the functions f_i are constants of the motion for H.

Example 4.12 *(The free particle with a linear external force).* Consider the cocontact Hamiltonian system $(\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}, \mathrm{d}t, \eta, H)$, where

$$H=\frac{p^2}{2}-\kappa z,$$

with κ a positive constant. Let $\widehat{H}=H\circ\rho_2$ be the associated time-dependent Hamiltonian. Then, the action-independent Hamilton–Jacobi equation for $\widehat{H}$ is given by

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2-\kappa S(t,q)+\frac{\partial S}{\partial t}=0\,,$$

that is,

$$\left(\frac{\partial S}{\partial q}\right)^2-2\kappa S(t,q)+2\frac{\partial S}{\partial t}=0\,. \tag{14}$$

Suppose that the generating function S is separable, namely, $S(t,q)=\alpha(t)+\beta(q)$. Then, equation (14) can be written as

$$\left(\frac{\mathrm{d}\beta}{\mathrm{d}q}\right)^2-2\kappa\alpha(t)-2\kappa\beta(q)+2\frac{\mathrm{d}\alpha}{\mathrm{d}t}=0\,,$$

so

$$2\frac{\mathrm{d}\alpha}{\mathrm{d}t}-2\kappa\alpha(t)=0\,,$$
$$\left(\frac{\mathrm{d}\beta}{\mathrm{d}q}\right)^2-2\kappa\beta(q)=0.$$

Thus,

$$\alpha(t)=e^{\kappa t}\,,\qquad \beta_\lambda(q)=\left(\sqrt{\frac{\kappa}{2}}q+\lambda\right)^2,$$

that is,

$$S_\lambda(t,q)=e^{\kappa t}+\left(\sqrt{\frac{\kappa}{2}}q+\lambda\right)^2,$$

and

$$\widehat{\Phi}(t,q,\lambda)=j^1_t S_\lambda(t,q)=\left(t,q,\sqrt{2\kappa}\left(\sqrt{\frac{\kappa}{2}}q+\lambda\right),e^{\kappa t}+\left(\sqrt{\frac{\kappa}{2}}q+\lambda\right)^2\right)$$

is a complete solution. Then,

$$\Phi\colon (t,q,\lambda)\mapsto\left(q,\sqrt{2\kappa}\left(\sqrt{\frac{\kappa}{2}}q+\lambda\right),e^{\kappa t}+\left(\sqrt{\frac{\kappa}{2}}q+\lambda\right)^2\right),$$

whose inverse is given by

$$\Phi^{-1}\colon (q,p,z)\mapsto\left(\frac{1}{\kappa}\log\left|z-\frac{p^2}{2\kappa}\right|,q,\frac{p-\kappa q}{\sqrt{2\kappa}}\right).$$

Hence,

$$f_1(t,q,p,z)=\frac{p-\kappa q}{\sqrt{2\kappa}}$$

is a conserved quantity.

The Hamiltonian vector field of H is given by

$$X_H=p\frac{\partial}{\partial q}+\kappa p\frac{\partial}{\partial p}+\left(\frac{p^2}{2}+\kappa z\right)\frac{\partial}{\partial z}\,.$$

One can check that $X_H(f_1)=0$. Moreover,

$$X_H^\gamma = p\frac{\partial}{\partial q}\bigg|_{\operatorname{Im}\Phi_\lambda} = \sqrt{2\kappa}\left(\sqrt{\frac{\kappa}{2}}q+\lambda\right)\frac{\partial}{\partial q}\,,$$

whose integral curves $\sigma(t)=(t,q(t))$ are given by

$$q(t) = ce^{\kappa t} - \sqrt{\frac{2}{\kappa}}\lambda\,,$$

where c is a constant. Then, the integral curves of X_H along $\operatorname{Im}\Phi_\lambda$ are given by $\Phi_\lambda\circ\sigma(t)=(q(t),p(t),z(t))$, where

$$p(t) = \sqrt{2\kappa}\left(\sqrt{\frac{\kappa}{2}}q(t)+\lambda\right) = \kappa\, ce^{\kappa t}\,,$$

and

$$z(t) = e^{\kappa t} + \left(\sqrt{\frac{\kappa}{2}}q(t)+\lambda\right)^2 = e^{\kappa t} + \frac{\kappa}{2}c^2 e^{2\kappa t}\,.$$

5. The action-dependent approach

5.1. Hamilton–Jacobi theory. The action-dependent approach

In Section 4 we have introduced a Hamilton–Jacobi theory for time-dependent contact Hamiltonian systems. In particular, this approach was shown to be useful to deal with time-independent contact Hamiltonian systems, where time is used as a free parameter. Nevertheless, this approach has a couple of drawbacks. First, complete solutions depend on $n+1$ parameters, instead of the n parameters that are required for symplectic Hamiltonian systems [12]. Additionally, time-independent solutions only cover the zero-energy level.

In order to solve these problems, in this section we propose an alternative approach, considering solutions of the Hamilton–Jacobi problem depending on the action variable z. Let us consider a section γ of the bundle $\pi_Q^{t,z}:\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R}\to\mathbb{R}\times Q\times\mathbb{R}$, locally given by

$$\begin{aligned}\gamma:\mathbb{R}\times Q\times\mathbb{R} &\longrightarrow \mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R}\\ (t,x,z) &\longmapsto (t,x,\gamma_i(t,x,z),z)\,.\end{aligned}$$

As in the previous approach, assume that X_H^γ and X_H are γ-related, so that the following diagram commutes:

$$\begin{array}{ccc}\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R} & \xrightarrow{X_H} & \mathrm{T}(\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R})\\ \gamma\uparrow\downarrow\pi_Q^{t,z} & & \mathrm{T}\pi_Q^{t,z}\downarrow\uparrow\mathrm{T}\gamma\\ \mathbb{R}\times Q\times\mathbb{R} & \xrightarrow{X_H^\gamma} & \mathrm{T}(\mathbb{R}\times Q\times\mathbb{R})\end{array}$$

Locally,

$$X_H\circ\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i}+\gamma_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(\gamma_i\frac{\partial H}{\partial p_i}-H\right)\frac{\partial}{\partial z}\,,$$

and

$$\mathrm{T}\gamma\circ X_H^\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \left(\frac{\partial\gamma_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial\gamma_i}{\partial q^j} + \left(\gamma_j\frac{\partial H}{\partial p_j}-H\right)\frac{\partial\gamma_i}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(\gamma_i\frac{\partial H}{\partial p_i}-H\right)\frac{\partial}{\partial z}\,,$$

so X_H^γ and X_H are γ-related if and only if

$$-\left(\frac{\partial H}{\partial q^i}+\gamma_i\frac{\partial H}{\partial z}\right) = \frac{\partial\gamma_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial\gamma_i}{\partial q^j} + \frac{\partial\gamma_i}{\partial z}\left(\gamma_j\frac{\partial H}{\partial p_j}-H\right)\,. \tag{15}$$

Note that $\operatorname{Im}\gamma$ is $(n+2)$-dimensional, so it no longer makes sense to require it to be Legendrian [21]. We will require it to be coisotropic instead.

Let us recall that, given a Jacobi manifold (M,Λ,E) and a distribution $\mathcal{D}\subseteq\mathrm{T}M$, the **orthogonal complement** $\mathcal{D}^\perp$ of $\mathcal{D}$ is given by [23,68]

$$\mathcal{D}_x^\perp = \hat{\Lambda}\left(\mathcal{D}_x^\circ\right)\,,$$

where $\mathcal{D}_x^\circ = \left\{\alpha_x \in \mathrm{T}_x^*M \mid \alpha_x(v) = 0,\ \forall\ v \in \mathcal{D}_x\right\}$ denotes the annihilator. In particular, a cocontact manifold (M, τ, η) is a Jacobi manifold (see Proposition 2.7) and its morphism $\hat{\Lambda}$ is given by equation (1). A submanifold $N \hookrightarrow M$ is said to be **coisotropic** if $\mathrm{T}N^\perp \subseteq \mathrm{T}N$.

Definition 5.1. Given a section $\alpha : \mathbb{R} \times Q \times \mathbb{R} \to \mathbb{R} \times \bigwedge^k \mathrm{T}^*Q \times \mathbb{R}$ and $t, z \in \mathbb{R}$, let

$$\begin{aligned} \alpha_{(t,z)} : Q &\longrightarrow \bigwedge^k \mathrm{T}^*Q \\ x &\longmapsto \mathrm{pr}_{\bigwedge^k \mathrm{T}^*Q}\left(\alpha(t, x, z)\right), \end{aligned}$$

where $\mathrm{pr}_{\bigwedge^k \mathrm{T}^*Q} : \mathbb{R} \times \bigwedge^k \mathrm{T}^*Q \times \mathbb{R} \to \bigwedge^k \mathrm{T}^*Q$ is the canonical projection. The **exterior derivative of** α **at fixed** t **and** z is the section of $\mathbb{R} \times \bigwedge^{k+1} \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}$ given by

$$\mathrm{d}_Q\, \alpha(t, x, z) = (t, \mathrm{d}\alpha_{(t,z)}(x), z)\,.$$

The coisotropic condition can be written in local coordinates as follows.

Lemma 5.2. *Assume that an $(n+2)$-dimensional submanifold N of a $(2n+2)$-dimensional cocontact manifold (M, τ, η) is locally the zero set of the constraint functions $\{\phi_a\}_{a=1,\ldots,n}$. Then, N is coisotropic if and only if the following equation holds in Darboux coordinates:*

$$\left(\frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z}\right)\frac{\partial \phi_b}{\partial p_i} - \left(\frac{\partial \phi_b}{\partial q^i} + p_i \frac{\partial \phi_b}{\partial z}\right)\frac{\partial \phi_a}{\partial p_i} = 0\,. \tag{16}$$

Proof. Assume that (M, τ, η) is a $(2n+2)$-dimensional cocontact manifold. Let $N \hookrightarrow M$ be a k-dimensional submanifold locally given as the zero set of functions $\phi_a : U \to \mathbb{R}$, with $a \in \{1, \ldots, 2n+2-k\}$. We have that

$$\mathrm{T}N^\perp = \left\langle \{Z_a\}_{a=1,\ldots,2n+2-k}\right\rangle\,,$$

where

$$Z_a = \hat{\Lambda}(\mathrm{d}\phi_a) = \left(\frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z}\right)\frac{\partial}{\partial p_i} - \frac{\partial \phi_a}{\partial p_i}\left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}\right).$$

Therefore, N is coisotropic if and only if $Z_a(\phi_b) = 0$ for all a, b, which in Darboux coordinates yields equation (16). □

Proposition 5.3. *Let γ be a section of $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ over $\mathbb{R} \times Q \times \mathbb{R}$. Then $\operatorname{Im}\gamma$ is a coisotropic submanifold if and only if*

$$\frac{\partial \gamma_i}{\partial q^j} + \gamma_j \frac{\partial \gamma_i}{\partial z} = \frac{\partial \gamma_j}{\partial q^i} + \gamma_i \frac{\partial \gamma_j}{\partial z}. \tag{17}$$

Proof. Equation (17) is obtained by applying the previous result to the submanifold $N = \operatorname{Im}\gamma$, which is locally defined by the constraints $\phi_i = p_i - \gamma_i$. □

Now suppose that the γ appearing in equation (15) is such that $\operatorname{Im}\gamma$ is coisotropic. Then, by means of equation (17) we obtain

$$\frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j}\frac{\partial \gamma_j}{\partial q^i} + \gamma_i\left(\frac{\partial H}{\partial p_j}\frac{\partial \gamma_j}{\partial z} + \frac{\partial H}{\partial z}\right) + \frac{\partial \gamma_i}{\partial t} = H \frac{\partial \gamma_i}{\partial z}\,,$$

or, globally,

$$\mathrm{d}_Q\,(H \circ \gamma) + \frac{\partial}{\partial z}(H \circ \gamma)\gamma + \mathscr{L}_{R_t}\gamma = (H \circ \gamma)\mathscr{L}_{\frac{\partial}{\partial z}}\gamma\,. \tag{18}$$

Theorem 5.4 *(Action-dependent Hamilton–Jacobi theorem). Let γ be a section of $\pi_Q^{t,z} : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}$ such that $\operatorname{Im}\gamma$ is a coisotropic submanifold of $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta)$. Then, X_H^γ and X_H are γ-related if and only if equation (18) holds. This equation will be called the **action-dependent Hamilton–Jacobi** equation for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$.*

Definition 5.5. Let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ be a cocontact Hamiltonian system. A **complete solution of the action-dependent Hamilton–Jacobi problem** for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ is a local diffeomorphism $\Phi\colon \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ such that, for each $\lambda \in \mathbb{R}^n$,

$$
\begin{aligned}
\Phi_\lambda \colon \mathbb{R} \times Q \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \\
\left(t, q^i, z\right) &\longmapsto \Phi\left(t, q^i, \lambda, z\right)
\end{aligned}
$$

is a solution of the action-dependent Hamilton–Jacobi problem for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$.

Let $\alpha \colon \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $\pi_i \colon \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections. Let us define the functions $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$, so that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} & \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} & \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \\
\downarrow \alpha & & \downarrow f_i \\
\mathbb{R}^n & \xrightarrow{\quad \pi_i \quad} & \mathbb{R}
\end{array}
$$

Theorem 5.6. *Let $\Phi \colon \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ be a complete solution of the action-dependent Hamilton–Jacobi problem for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$. Then,*

(i) *For each $i \in \{1, \dots, n\}$, the function $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ is a constant of the motion. However, these functions are not necessarily in involution, i.e., $\{f_i, f_j\} \neq 0$.*
(ii) *For each $i \in \{1, \dots, n\}$, the function $\hat{f}_i = g f_i$, where g is a dissipated quantity, is also a dissipated quantity. Moreover, if $R_t H = 0$ and taking $g = H$, these functions are in involution, i.e., $\{\hat{f}_i, \hat{f}_j\} = 0$.*

Proof. Observe that

$$
\operatorname{Im} \Phi_\lambda = \bigcap_{i=1}^{n} f_i^{-1}(\lambda_i)\,,
$$

where $\lambda = (\lambda, \dots, \lambda_n) \in \mathbb{R}^n$. In other words,

$$
\operatorname{Im} \Phi_\lambda = \{x \in \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \mid f_i(x) = \lambda_i, i = 1, \dots, n\}\,.
$$

Therefore, since X_H is tangent to any of the submanifolds $\operatorname{Im} \Phi_\lambda$, we deduce that

$$
X_H(f_i) = 0\,.
$$

Moreover, we can compute

$$
\{f_i, f_j\} = \Lambda(\mathrm{d}f_i, \mathrm{d}f_j) - f_i R_z(f_j) + f_j R_z(f_i)\,,
$$

but

$$
\Lambda(\mathrm{d}f_i, \mathrm{d}f_j) = \hat{\Lambda}(\mathrm{d}f_i)(f_j) = 0\,,
$$

since $(\mathrm{T}\operatorname{Im} \Phi_\lambda)^\perp = \hat{\Lambda}((\mathrm{T}\operatorname{Im} \Phi_\lambda)^\circ) \subset \mathrm{T}\operatorname{Im} \Phi_\lambda$, so

$$
\{f_i, f_j\} = -f_i R_z(f_j) + f_j R_z(f_i)\,.
$$

By Proposition 3.6, we already know that the product of a conserved quantity and a dissipated quantity is a dissipated quantity. Let f_i and f_j be conserved quantities and take $g = H$. Then, by equation (10), $\{\hat{f}_i, \hat{f}_j\}$ vanishes. □

From a complete solution of the Hamilton–Jacobi problem one can reconstruct the dynamics of the system. If σ is an integral curve of the vector field X_H^γ, then $\Phi_\lambda \circ \sigma$ is an integral curve of X_H, thus recovering the dynamics of the original system.

5.2. Integrable contact Hamiltonian systems

Let $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ be a contact Hamiltonian system. Recall that the action-dependent Hamilton–Jacobi equation for $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ is given by [26]

$$
\mathrm{d}_Q\,(H \circ \gamma) + \frac{\partial}{\partial z}(H \circ \gamma)\gamma = (H \circ \gamma)\mathscr{L}_{\frac{\partial}{\partial z}}\gamma\,.
$$

A **complete solution of the action-dependent Hamilton–Jacobi problem** for $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ is a local diffeomorphism $\Phi \colon Q \times \mathbb{R}^n \times \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ such that, for each $\lambda \in \mathbb{R}^n$,

$$\begin{aligned}\Phi_\lambda \colon\ Q \times \mathbb{R} &\longrightarrow \mathrm{T}^* Q \times \mathbb{R}\\ \left(q^i, z\right) &\longmapsto \Phi\left(q^i, \lambda, z\right)\end{aligned}$$

is a solution of the action-dependent Hamilton–Jacobi problem for $(\mathrm{T}^* Q \times \mathbb{R}, \eta, H)$.

Let $\Phi \colon Q \times \mathbb{R}^n \times \mathbb{R} \to \mathrm{T}^* Q \times \mathbb{R}$ be a complete solution of the Hamilton–Jacobi problem for $(\mathrm{T}^* Q \times \mathbb{R}, \eta, H)$. Then,

$$\mathcal{F} = \left\{\mathcal{F}_\lambda = \operatorname{Im}\Phi_\lambda \mid \lambda \in \mathbb{R}^n\right\} \subseteq \mathrm{T}^* Q \times \mathbb{R}$$

is a foliation in coisotropic submanifolds.

In the symplectic case, since solutions of the Hamilton–Jacobi equation are closed one-forms on Q, the images of a complete solution for each choice of parameters λ form a Lagrangian foliation invariant under the action of the Hamiltonian flow. This structure is called an integrable system. In analogy, we introduce the following definition:

Definition 5.7. Let (M, η, H) be a contact Hamiltonian system and let $\mathcal{F}$ be a foliation consisting of $(n+1)$-dimensional coisotropic (with respect to the Jacobi structure of the contact manifold) leaves invariant under the flow of the Hamiltonian vector field X_H. Then we call $(M, \eta, H, \mathcal{F})$ an **integrable system**.

Remark 5.8. Each of the leaves $\mathcal{F}_\lambda$ is invariant under the flow of X_H and X_{f_i}. Since $\mathcal{F}_\lambda$ is an $(n+1)$-dimensional manifold with $n+1$ independent and commuting tangent vector fields, if the vector fields are complete, by [3, Ch. 10, Lem. 2] it is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$, where $\mathbb{T}^k$ is the k-dimensional torus.

The definition above can be compared to the ones given in [4,59]:

- In [4], Boyer proposes a concept of completely integrable system for the so-called good Hamiltonians, that is, the Hamiltonian function is preserved along the flow of the Reeb vector field. This is a particular case of our definition, in which both the Hamiltonian and the constants of the motion do not depend on z.
- In [59], Khesin and Tabachnikov call a foliation **co-Legendrian** when it is transverse to $\mathcal{H}$ and $\mathrm{T}\mathcal{F} \cap \mathcal{H}$ is integrable. Then they define an integrable system as a particular case of a co-Legendrian foliation with some extra regularity conditions. In the case that the dimension of the leaves is $n+1$, the following proposition shows that co-Legendrian foliations are particular cases of coisotropic foliations.

Proposition 5.9. *Let $i\colon N \hookrightarrow M$ be a submanifold of a $(2n+1)$-dimensional contact manifold (M, η). If N is an $(n+1)$-dimensional co-Legendrian submanifold, then it is also a coisotropic submanifold.*

Proof. Let us write $\mathrm{T}N = \mathcal{D}_\mathcal{H} \oplus \mathcal{E}$, where $\mathcal{D}_\mathcal{H} = \mathrm{T}N \cap \mathcal{H}$. Then, $\mathrm{T}N^\perp = \mathcal{D}_\mathcal{H}^\perp \cap \mathcal{E}^\perp$. Obviously, η vanishes in $\mathrm{T}N \cap \mathcal{H}$. Moreover, since $\mathcal{D}_\mathcal{H}$ is integrable,

$$0 = \eta([v, w]) = \iota_{[v,w]}\eta = \mathscr{L}_v \iota_w \eta - \iota_w \mathscr{L}_v \eta = -\iota_w \iota_v \mathrm{d}\eta - \iota_w \mathrm{d}\iota_v \eta = -\iota_w \iota_v \mathrm{d}\eta\,,$$

for any $v, w \in \mathcal{D}_\mathcal{H}$, so $\mathrm{d}\eta_{|\mathcal{D}_\mathcal{H}} = 0$. Observe that $\hat{\Lambda}_{|\mathcal{H}} = \sharp_{|\mathcal{H}}$, and $\sharp_{|\mathcal{H}} \colon \mathcal{H} \to \langle R\rangle^\circ$, $\sharp_{|\mathcal{H}}^{-1}(v) = \iota_v \mathrm{d}\eta$ is an isomorphism. Since $\mathrm{d}\eta_{|\mathcal{D}_\mathcal{H}} = 0$, $\sharp_{|\mathcal{H}}^{-1}(\mathcal{D}_\mathcal{H}) \subseteq \mathcal{D}_\mathcal{H}^\circ$. Thus, $\mathcal{D}_\mathcal{H} \subseteq \hat{\Lambda}(\mathcal{D}_\mathcal{H}^\circ) = \mathcal{D}_\mathcal{H}^\perp$. By a dimension counting argument, we can see that both spaces are equal and, thus, $\mathcal{D}_\mathcal{H} = \mathcal{D}_\mathcal{H}^\perp$. □

We also note that a foliation $\tilde{\mathcal{F}}$ by Legendrian submanifolds can never be invariant by the Hamiltonian flow. Indeed, let $\tilde{F} \in \tilde{\mathcal{F}}$. The leaves of $\tilde{\mathcal{F}}$ are Lagrangian, thus $\mathrm{T}\tilde{F}_0 \subseteq \ker \eta$. Since $\eta(X_H) = -H$, X_H can only be tangent to the leaves at the zero set of H, hence its flow cannot leave invariant the whole foliation.

Observe that Definition 5.7 can be naturally extended to cocontact Hamiltonian systems.

Definition 5.10. Let (M, τ, η, H) be a cocontact Hamiltonian system and let $\mathcal{F}$ be a foliation consisting of $(n+2)$-dimensional coisotropic leaves (with respect to the Jacobi structure of the cocontact manifold) invariant under the flow of the cocontact Hamiltonian vector field X_H. Then we call $(M, \tau, \eta, H, \mathcal{F})$ an **integrable cocontact system**.

5.3. Example 1: freely falling particle with linear dissipation

Consider a particle of time-dependent mass $m(t)$ which is freely falling and subject to a dissipation linear in the velocity with proportionality constant γ. The Hamiltonian function $H : \mathbb{R} \times \mathrm{T}^*\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$H(t, q, p, z) = \frac{p^2}{2m(t)} + m(t) g q + \frac{\gamma}{m(t)} z\,,$$

where g is the gravity. The Hamiltonian vector field corresponding to this Hamiltonian function is

$$X_H = \frac{\partial}{\partial t} + \frac{p}{m(t)}\frac{\partial}{\partial q} - \left(m(t)g + \frac{\gamma}{m(t)}p\right)\frac{\partial}{\partial p} + \left(\frac{p^2}{2m(t)} - m(t)gq - \frac{\gamma}{m(t)}z\right)\frac{\partial}{\partial z}\,.$$

Its integral curves $(t(r), q(r), p(r), z(r))$ satisfy the system of differential equations

$$\dot t = 1\,, \qquad \dot q = \frac{p}{m(t)}\,, \qquad \dot p = -m(t)g - \frac{\gamma}{m(t)}p\,, \qquad \dot z = \frac{p^2}{2m(t)} - m(t)gq - \frac{\gamma}{m(t)}z\,.$$

Combining the second and third equations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(m(t)\dot q) = -m(t)g - \gamma\dot q\,.$$

In order to solve the Hamilton–Jacobi problem, we look for a conserved quantity linearly independent from the Hamiltonian, i.e., a function f on $\mathrm{T}\mathbb{R}\times\mathbb{R}$ such that $X_H f = 0$. For the sake of simplicity, one can assume that f does not depend on q or z. Indeed, one can verify that

$$f(t,q,p,z) = e^{\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\left(p + g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u\right)$$

is a conserved quantity. We can thus express the momentum p as a function of t and a real parameter λ, namely,

$$P(t,\lambda) = e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\left(\lambda - g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u\right)\,,$$

and obtain a complete solution of the Hamilton–Jacobi problem for H:

$$\phi_\lambda : (t,q,z) \longmapsto \left(t, q, p \equiv e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\left(\lambda - g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u\right), z\right)\,.$$

In this case, equation (17) holds trivially, so $\operatorname{Im}\Phi_\lambda$ is coisotropic.

In addition, one can verify that

$$k(t,q,p,z) = p + g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u = e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s} f(t,q,p,z)$$

is a dissipated quantity, that is, $\{k, H\} - R_t k = 0$.

5.4. Example 2: damped forced harmonic oscillator

Consider the product manifold $\mathbb{R}\times\mathrm{T}^*\mathbb{R}\times\mathbb{R}$ with natural coordinates (t,q,p,z). The Hamiltonian function

$$H(t,q,p,z) = \frac{p^2}{2m} + \frac{k}{2}q^2 - qF(t) + \frac{\gamma}{m}z$$

describes a harmonic oscillator with elastic constant k, friction coefficient γ and subjected to an external time-dependent force $F(t)$ [21].

The Hamiltonian vector field is

$$X_H = \frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial q} + \left(-kq + F(t) - \frac{p}{m}\gamma\right)\frac{\partial}{\partial p} + \left(\frac{p^2}{2m} - \frac{k}{2}q^2 + qF(t) - \frac{\gamma}{m}z\right)\frac{\partial}{\partial z}\,,$$

and its integral curves $(t(r), q(r), p(r), z(r))$ satisfy

$$\dot t = 1\,, \qquad \dot q = \frac{p}{m}\,, \qquad \dot p = -kq + F(t) - \frac{p}{m}\gamma\,, \qquad \dot z = \frac{p^2}{2m} - \frac{k}{2}q^2 + qF(t) - \frac{\gamma}{m}z\,.$$

Combining the second and the third equations above, we obtain the second-order differential equation

$$m\ddot q + \gamma\dot q + kq = F(t)\,,$$

which corresponds to a damped forced harmonic oscillator. One can check that the function

$$g(t,q,p,z)=e^{\frac{\gamma t}{2m}}\left(\frac{\sinh\left(\frac{\kappa t}{2m}\right)(2kmq+\gamma p)}{\kappa}+p\cosh\left(\frac{\kappa t}{2m}\right)\right)-\int_1^t F(s)e^{\frac{\gamma s}{2m}}\left(\cosh\left(\frac{\kappa s}{2m}\right)+\frac{\gamma\sinh\left(\frac{\kappa s}{2m}\right)}{\kappa}\right)\mathrm{d}s\,,$$

where $\kappa=\sqrt{\gamma^2-4km}$, is a conserved quantity. It is worth noting that, since $\sinh = x+\mathcal{O}(x^3)$ and $\cosh x = 1+\mathcal{O}(x^2)$ near $x=0$, $\sinh(ix)=i\sin x$ and $\cosh(ix)=\cos x$, the equation above is well-defined and real-valued for any of $\kappa\in\mathbb{C}$. Thus, we can write p in terms of t,q,z and a real parameter λ as

$$P(t,q,\lambda,z)=\frac{e^{-\frac{\gamma t}{2m}}\left(\kappa\int_1^t e^{\frac{s\gamma}{2m}}F(s)\left(\cosh\left(\frac{\kappa s}{2m}\right)+\frac{\gamma\sinh\left(\frac{\kappa s}{2m}\right)}{\kappa}\right)\mathrm{d}s-2kmqe^{\frac{\gamma t}{2m}}\sinh\left(\frac{\kappa t}{2m}\right)+\kappa\lambda\right)}{\gamma\sinh\left(\frac{\kappa t}{2m}\right)+\kappa\cosh\left(\frac{\kappa t}{2m}\right)}\,,$$

and obtain a complete solution of the Hamilton–Jacobi problem:

$$\Phi_\lambda:\ (t,q,\lambda,z)\mapsto(t,q,p\equiv P(t,q,\lambda,z),z)\ .$$

Obviously equation (17) is satisfied, hence $\operatorname{Im}\Phi_\lambda$ is coisotropic. In addition,

$$f(t,q,p,z)=e^{-\frac{\gamma t}{m}}\left[e^{\frac{\gamma t}{2m}}\left(\frac{\sinh\left(\frac{\kappa t}{2m}\right)(2kmq+\gamma p)}{\kappa}+p\cosh\left(\frac{\kappa t}{2m}\right)\right)\right.$$
$$\left.-\int_1^t e^{\frac{s\gamma}{2m}}F(s)\left(\cosh\left(\frac{\kappa s}{2m}\right)+\frac{\gamma\sinh\left(\frac{\kappa s}{2m}\right)}{\kappa}\right)\mathrm{d}s\right],$$

is a dissipated quantity.

6. Conclusions and outlook

The main contributions of the present paper are the following:

- We have obtained two different Hamilton–Jacobi equations for time-dependent contact Hamiltonian systems: the so-called action independent and action-dependent approaches. In particular, the action-independent approach is useful for time-independent contact Hamiltonian systems, where the use of time as a free parameter allows to integrate the system at non-zero energy levels. In addition, we have introduced a notion of complete solution in the action-independent approach. Each of these complete solutions is associated with a family of $n+1$ independent dissipated quantities in involution (where n is the number of degrees of freedom of the system).
- The action-dependent approach also permits to introduce a natural notion of complete solution to the Hamilton–Jacobi problem. Each of these complete solutions is associated with a family of n independent dissipated quantities in involution. Moreover, the image of a complete solution is a coisotropic submanifold.
- We introduce a new notion of integrable system in a contact manifold, taking into account the dynamics given by the Hamiltonian vector field, and extending the concept of complete solution. This allows us to study the dynamics outside the zero-energy level.

As we have pointed out in Remarks 4.7 and 5.8, there is a relationship between solutions of Hamilton–Jacobi equations and several notions of integrability. Namely, the existence of foliations by coisotropic tori, integrability by quadratures and the construction of action-angle coordinates. Further research is needed to clarify these notions and their relationships in contact Hamiltonian systems.

Other topics for future research include the reduction problem, the Hamilton–Jacobi equations for the evolution vector field and its possible applications to thermodynamics as well as the extension to higher order systems. The study of the discrete Hamilton–Jacobi equations and applications to the construction of geometric integrators is also on the agenda.

Declaration of competing interest

The authors have no conflicts to declare.

Data availability

No data was used for the research described in the article.

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